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Progress of Theoretical Physics, Vol. 19, No. 1, January 1958

Many-Body Pseudopotential for Hard-Sphere Interaction

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A many-body pseudopotential for a hard-sphere system which is mathematically equivalent to the boundary conditions imposed on the wave functions is obtained as a generalization of Huang and Yang's binary pseudopotential, and the lowest order term of the ground state energy resulting from the triple collision is calculated for the Bose system. Instead of the direct perturbational treatment of the pseudopotential, the use of the ρ_k-representation for the Bose system is proposed and the ground state energy is calculated in this representation. It is shown that the energy thus calculated is of the form of an expansion in powers of $(\rho a^3)^{1/2}$, where ρ is the average density and a the hard-sphere diameter, which is in agreement with Lee and Yang's conclusion drawn from the binary collision expansion method. Some discussions associated with the roton spectrum of liquid He^4 are given.

§ 1. Introduction

In order to treat the hard-sphere interaction, Huang and Yang[1] have proposed the method of the pseudopotential and calculated the energy eigenvalue and the wave function of the many-body system in a form of the power series of the hard-sphere diameter a. The same method has been applied by Huang, Yang and Luttinger[2] to the theory of the virial expansion and of the Bose-Einstein condensation of an imperfect Bose gas. The great merit of such a method consists in being able to transform the problem for the hard-sphere system into a form where the usual perturbational calculation can be applied of which use is impossible for the original form of the potential function. In spite of these successes, however, there can be pointed out two difficult points in Huang and Yang's original treatment of the pseudopotential. One of them is found in their neglect of the triple and higher order collision terms. Since the triple collision term is estimated to be of the order a^4 by a dimensional analysis, it is not possible to calculate exactly the terms of order a^4 by their pseudopotential. Another difficulty lies in the direct perturbational treatment of the pseudopotential. According to Huang and Yang the ground state energy E_0 of N Bose particles is given by

$$E_0/N=\frac{4\pi a(N-1)}{V}\left\{1+\frac{Ca}{L}+\frac{a^2}{L^2}\left[C^2+\frac{\xi}{\pi^2}(2N-5)\right]\right\}$$

up to the terms of order a^3, where $V(=L^3)$ is the volume of the system, C and ξ are numerical constants. If one keeps the density $\rho=N/V$ constant and makes N tend to infinity, the above expression diverges as $N^{1/3}$. Of course, such a behaviour of the energy

is inconsistent with the usual property of the macroscopic system that the energy per particle does not depend on the particle number explicitly, provided that the surface effects are neglected.

The divergence difficulty mentioned above was noticed by Lee, Huang and Yang[3] themselves and an expression for E_0/N having the correct volume dependence was obtained with an explicit consideration of the eigenvalue problem for the pseudopotential. A somewhat different approach to the problem was undertaken by Brueckner and Sawada,[1] which is essentially identical with that given by Brueckner and Levinson[2] in their general theory of many-body systems with strong interactions, and the peculiar features of the Bose system giving rise to the perturbation divergence were clarified. Brueckner and Sawada's method is powerful not only in avoiding the divergence difficulty but also in dealing with the condensed system where the "binary" pseudopotential introduced by Huang and Yang seems to be no longer valid. It may be desirable, however, to be able to deal with the system at high densities by means of the pseudopotential method. For this purpose, it is necessary to find a many-body pseudopotential including the many-body collision terms besides the binary one, since the triple or higher order collision terms are expected to be important for the condensed system. In this paper we shall undertake to find such a many-body pseudopotential.

In § 2 we shall consider a generalized pseudopotential for the two body system and in § 3 the pseudopotential for the many-body system which is mathematically equivalent to the boundary conditions imposed on the wave functions will be given. In § 4 the expressions correct up to the terms of order a^4 will be given and the lowest order term of the ground state energy resulting from the triple collision will be calculated for the Bose system. In § 5 we shall show that the f_k-representation for the Bose system developed by Bogolyubov and Zubarev[6] is appropriate for dealing with the pseudopotential, because this representation leads to the result that E_0/N has the correct volume dependence. The phonon spectrum will be obtained and compared with one deduced by Landau. It will turn out that the theoretical spectrum cannot account for the roton spectrum. Some comments on the pseudopotential method associated with this problem will be given.

§ 2. Generalized pseudopotential for the two body system

The explanation of the physical and mathematical concepts of the pseudopotentials has been thoroughly given by Huang and Yang and will not be repeated here. We mention here, however, that the investigation of the following equation

$$\left.\begin{aligned}(\nabla^2+k_r^2)\,\varphi &= f(\boldsymbol{x})\\ \varphi\,(r=a) &= 0\end{aligned}\right\} \qquad (2\cdot 1)$$

and of the singular properties at $r=0$ of the solution φ provides a key to find an exact pseudopotential for the many-body system.

The physical meaning of eq. $(2\cdot 1)$ will best be visualized if we consider the corresponding boundary value problem in the electrostatics. The appearance of $f(\boldsymbol{x})$ on the

right in eq. (2·1) means that there is a charge distribution $-f(\boldsymbol{x})/4\pi$ in the vacuum space and the boundary condition at $r=a$ is equivalent to the presence of a perfect conducting spherical shell of radius a. As is easily recognized new multipole induced by the charge distribution will appear at $r=0$ in addition to the multipole already present in the vacuum space, in order to make the potential vanish at $r=a$. Since the perfect conducting shell has the shielding properties that the electric field cannot enter into this shell, the induced multipole will be determined only by the situations inside the shell. It is easy to find this multipole, because the solution of the inhomogeneous equation (2·1) is easily found.

Next consider the following two body equation

$$(\nabla_1^2+\nabla_2^2+2k^2)\,\Psi=f(\boldsymbol{x}_1,\ \boldsymbol{x}_2) \tag{2·2}$$

$$\Psi=0 \quad \text{at} \quad |\boldsymbol{x}_1-\boldsymbol{x}_2|=a. \tag{2·3}$$

If we eliminate the coordinates of the center of mass, eq. (2·2) is reduced to the form of eq. (2·1). The general solution of the differential equation (2·2) is a sum of the solution, Ψ_1, of the corresponding homogeneous equation ($f(\boldsymbol{x}_1, \boldsymbol{x}_2)=0$) and the particular solution, Ψ_2, of eq. (2·2). The solution satisfying the boundary condition eq. (2·3) is given by

$$\Psi=\Psi_1+\Psi_2 \tag{2·4}$$

$$\Psi_1=\sum_{K,l,m} A_{K,l,m}e^{i\boldsymbol{K}\cdot\boldsymbol{X}}\{\cos\delta_l j_l(k_r r)-\sin\delta_l n_l(k_r r)\}\,Y_{l,m}(\theta,\ \varphi) \tag{2·5}$$

$$\Psi_2=\sum_{K,l,m} e^{i\boldsymbol{K}\cdot\boldsymbol{X}}Y_{l,m}(\theta,\ \varphi)\int_a^r dr'\,k_r r'^2\{j_l(k_r r')n_l(k_r r)-n_l(k_r r')j_l(k_r r)\}\cdot f_{K,l,m}(r'), \tag{2·6}$$

where

$$\tan\delta_l=j_l(k_r a)/n_l(k_r a), \tag{2·7}$$

$$k_r^2=k^2-(\boldsymbol{K}^2/4), \tag{2·8}$$

and

$$f_{K,l,m}(r)=\frac{1}{2V}\int d\boldsymbol{X}'\,d\Omega'\,e^{-i\boldsymbol{K}\cdot\boldsymbol{X}'}f\left(\boldsymbol{X}'+\frac{\boldsymbol{x}'}{2},\ \boldsymbol{X}'-\frac{\boldsymbol{x}'}{2}\right)Y_{l,m}^*(\theta',\ \varphi'). \tag{2·9}$$

In these expressions $\boldsymbol{X}$ is the coordinate of the center of mass $(=\boldsymbol{x}_1+\boldsymbol{x}_2/2)$, $(r,\ \theta,\ \varphi)$ the polar coordinate of the relative coordinate $\boldsymbol{x}=\boldsymbol{x}_1-\boldsymbol{x}_2$, δ_l the phase shift for the hard-sphere potential, $Y_{l,m}(\theta,\ \varphi)$ the spherical harmonics, j_l spherical Bessel function, n_l spherical Neumann function, and $A_{K,l,m}$'s are arbitrary constants to be determined by the boundary conditions besides eq. (2·3). The integration $d\Omega'$ in eq. (2·9) extends over the entire solid angles keeping the magnitude of $\boldsymbol{x}'$ at r.

The sums $\sum_{\boldsymbol{K}}$ in eqs. (2·5) and (2·6) extend over all values of $(n_1,\ n_2,\ n_3)$ by which $\boldsymbol{K}$ is defined as

$$\boldsymbol{K}=(2\pi/L)\ \ (n_1,\ n_2,\ n_3) \tag{2·10}$$

where n_1, n_2, n_3 are $\pm$integers. In eq. (2·8) k_r represents the magnitude of the wave vector for the relative motion.

Now consider the limiting case $r=0$. As is seen from eqs. (2·5) and (2·6) $r=0$ is the singular point of the functions Ψ_1 and Ψ_2. This corresponds to the presence of the multipole at $r=0$. Before entering into the detailed investigation of the singular properties, we mention some remarks about the dependence of $f_{K,l,m}(r)$ on r. In § 3, we shall put

$$f(\boldsymbol{x}_1,\ \boldsymbol{x}_2) = -\sum_{n\neq 1,2} (\nabla_n^2+k^2)\,\Psi(1,\,2,\cdots N) \tag{2·11}$$

where $\Psi(1,\,2,\cdots N)$ is the wave function of the N-body system. Regarding Ψ as a function of coordinates 1 and 2 alone, we expand it in the sets $e^{iK\cdot X}Y_{l,m}(\theta,\,\varphi)$, i.e.,

$$\Psi = \sum_{K,l,m} e^{iK\cdot X}Y_{l,m}(\theta,\,\varphi)\,R_{K,l,m}(r)\,A(3,\,4,\cdots N) \tag{2·12}$$

where $A(3,\,4,\cdots N)$ is some function of the coordinates except 1 and 2. We assume here that the $R_{K,l,m}(r)$ is a sum of a term regular at $r=0$ and a term singular at the same point and that their dependences on r are characterized by

$$\textit{regular term} = r^l \times (\textit{power series in even powers of } r)$$

$$\textit{singular term} = r^{-l-1} \times (\textit{power series in even powers of } r).$$

We shall write these properties symbolically as

$$R_{K,l,m}(r) = r^{-l-1}(1+r^2+r^4+\cdots)+r^l(1+r^2+r^4+\cdots). \tag{2·13}$$

Then with the help of eqs. (2·9) and (2·11) we have

$$n_l(k_r r)\int_a^r dr' r'^2 j_l(k_r r')\,f_{K,l,m}(r') = r^{-l-1}(1+r^2+r^4+\cdots)+r^{l+2}(1+r^2+r^4+\cdots) \tag{2·14}$$

and

$$j^l(k_r r)\int_a^r dr' r'^2 n_l(k_l r')\,f_{K,l,m}(r') = r^{-l+1}(1+r^2+r^4+\cdots)+r^l(1+r^2+r^4\cdots). \tag{2·15}$$

These, together with (2·4), (2·5) and (2·6), justify our assumption, eq. (2·13).

From eqs. (2·14) and (2·15) we have

$$\Psi_1 \to -\sum_{K,l,m} A_{K,l,m}\,e^{iK\cdot X}\sin\delta_l\,n_l(k_r r)\,Y_{l,m}(\theta,\,\varphi) \tag{2·16}$$

$$\Psi_2 \to -\sum_{K,l,m} e^{iK\cdot X}Y_{l,m}(\theta,\,\varphi)\,n_l(k_r r)\int_0^a dr' k_r r'^2 j_l(k_r r')\,f_{K,l,m}(r') \tag{2·17}$$

as $r\to 0$.

Therefore as an equation including multipoles at $r=0$ we have

$$(\nabla_1^2+\nabla_2^2+2k^2)\Psi=f(\boldsymbol{x}_1,\boldsymbol{x}_2)-4\delta(r)\sum_{K,l,m}A_{K,l,m}e^{i\boldsymbol{K}\cdot\boldsymbol{X}}\sin\delta_l\frac{(2l+1)!!}{k_r^{l+1}r^{l+2}}Y_{l,m}(\theta,\varphi)$$

$$-4\delta(r)\sum_{K,l,m}e^{i\boldsymbol{K}\cdot\boldsymbol{X}}\frac{(2l+1)!!}{k_r^{l}r^{l+2}}Y_{l,m}(\theta,\varphi)\int_0^a dr'r'^2j_l(k_rr')f_{K,l,m}(r') \qquad (2\cdot18)$$

where use has been made of the following equation

$$(\nabla^2+k_r^2)\{n_l(k_rr)Y_{l,m}(\theta,\varphi)\}=2\delta(r)\frac{(2l+1)!!}{k_r^{l+1}r^{l+2}}Y_{l,m}(\theta,\varphi) \qquad (2\cdot19)$$

which was proved by Huang and Yang.[1)]

The elimination of the arbitrary constants $A_{K,l,m}$'s from eq. (2·18) leads to the pseudopotential for the generalized two body equation (2·2). For this purpose it is convenient to define $\Psi_{K,l,m}$ in the following manner

$$\Psi_{K,l,m}(r)=\frac{1}{V}\iint d\boldsymbol{X}d\Omega\, e^{-i\boldsymbol{K}\cdot\boldsymbol{X}}\Psi\left(\boldsymbol{X}+\frac{\boldsymbol{x}}{2},\ \boldsymbol{X}-\frac{\boldsymbol{x}}{2}\right)Y_{l,m}^*(\theta,\varphi). \qquad (2\cdot20)$$

From eqs. (2·14), (2·15) and (2·20) we have

$$A_{K,l,m}\delta(r)=\frac{1}{\cos\delta_l}\frac{(2l+1)!!}{(2l+1)!\,k_r^{l}}\delta(r)\frac{\partial^{2l+1}}{\partial r^{2l+1}}(r^{l+1}\Psi_{K,l,m})$$

$$+\frac{k_r}{\cos\delta_l}\frac{\delta(r)}{(2l-1)!}\frac{\partial^{2l-1}}{\partial r^{2l-1}}\left[r^{2l-1}\int_a^r dr'r'^2n_l(k_rr')f_{K,l,m}(r')\right] \qquad (2\cdot21)$$

where for $l=0$ we should put

$$\frac{1}{(2l-1)!}\frac{\partial^{2l-1}}{\partial r^{2l-1}}r^{2l-1}=1 \quad (l=0). \qquad (2\cdot22)$$

Substitution of eq. (2·21) in (2·18) leads to

$$(\nabla_1^2+\nabla_2^2+2k^2)\Psi=f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)+\delta(r)P_{12}\Psi+\delta(r)K_{12}f \qquad (2\cdot23)$$

where

$$r=|\boldsymbol{x}_1-\boldsymbol{x}_2|,$$

$$P_{12}=-4\sum_{K,l}\frac{\tan\delta_l}{r^{l+2}k_r^{2l+1}}\cdot\frac{[(2l+1)!!]^2}{(2l+1)!}\frac{\partial^{2l+1}}{\partial r^{2l+1}}(r^{l+1}\Omega_{K,l}(1,2)), \qquad (2\cdot24)$$

$$K_{12}=-2\sum_{K,l}\frac{(2l+1)!!}{r^{l+2}k_r^{l}}\int_0^a dr\,r^2j_l(k_rr)\Omega_{K,l}(1,2)-$$

$$-2\sum_{K,l}\frac{\tan\delta_l}{r^{l+2}k_r^{l}}\frac{(2l+1)!!}{(2l-1)!}\frac{\partial^{2l-1}}{\partial r^{2l-1}}\left[r^{2l-1}\int_a^r dr\,r^2n_l(k_rr)\Omega_{K,l}(1,2)\right] \qquad (2\cdot25)$$

and

$$\Omega_{K,l}(1,2)\Psi(1,2)=\sum_{m=-l}^{l}\frac{e^{i\boldsymbol{K}\cdot\boldsymbol{X}}Y_{l,m}(\theta,\varphi)}{V}\iint d\boldsymbol{X}'\,d\Omega'\,e^{-i\boldsymbol{K}\cdot\boldsymbol{X}'}$$

$$Y^{*}_{l,m}(\theta',\varphi')\Psi\left(\boldsymbol{X}'+\frac{\boldsymbol{x}'}{2},\ \boldsymbol{X}'-\frac{\boldsymbol{x}'}{2}\right). \tag{2·26}$$

This is an equation for the pseudopotential for eqs. (2·2) and (2·3). The pseudopotential thus obtained is also shown to be the sufficient condition for the function to vanish at $r=a$.

In eq. (2·23) P_{12} is the pseudopotential for the binary collision obtained by Huang and Yang. K_{12} is an integral operator defined by eq. (2·25) and $K_{12}f$ gives the multipole induced by the charge distribution f. As is seen from eq. (2·25) $K_{12}f$ is essentially determined by the charge distribution inside the sphere of radius a, indicating the shielding properties of the perfect conducting shell mentioned at the beginning of this section.

From eq. (2·26) it follows that

$$\Omega_{K,l}(1,2)\,e^{i\boldsymbol{K}'\cdot\boldsymbol{X}}Y_{l',m}(\theta,\varphi)=\delta_{K,K'}\delta_{l,l'}\,e^{i\boldsymbol{K}\cdot\boldsymbol{X}}Y_{l,m}(\theta,\varphi).$$

This implies that $\Omega_{K,l}$ is a projection operator for the motion of the center of mass and for the angular momentum. The sums $\sum_{m=-l}^{l}$ in eq. (2·26) may be transformed with the use of the additional theorem of the spherical harmonics. Such transformations, however, will not be needed in the following discussions.

In closing this section we wish to consider the dependence of K_{12} on a. Such a consideration will be necessary in § 4. If $f_{K,l,m}$ in eq. (2·23) is constant, the l-th term of K_{12} is proved to be of the order a^{l+3} by noting that $\tan\delta_l\approx(k_r a)^{2l+1}$, i. e.,

$$K_{12}\sim a^{l+3}\quad(f_{K,l,m}=\text{const.}). \tag{2·27}$$

On the other hand if $f_{K,l,m}$ is of the form expressed in eq. (2·13) we have

$$K_{12}\sim a^2+a^4+\cdots\quad(f_{K,l,m}\sim r^{-l-1}\ \text{as}\ r\to 0). \tag{2·28}$$

Hence, in general, we can consider K_{12} is of the order a^2.

§ 3. Pseudopotential for the many-body system

Consider N hard-spheres in a box of volume V. The Schrödinger wave equation is

$$\sum_{i=1}^{N}(\nabla_i^2+k^2)\Psi=0 \tag{3·1}$$

where $\hbar^2k^2/2m$ corresponds to the energy eigenvalue per particle. eq. (3·1) must be solved under the boundary condition

$$\Psi=0\ \text{ at }\ |\boldsymbol{x}_i-\boldsymbol{x}_j|=a$$

in addition to the ordinary one (e. g. the periodic boundary condition). If we extend the definition of the wave function to the entire $3N$-dimensional configuration space by a requirement that eq. (3·1) is also satisfied inside the hard-spheres, then the terms

corresponding to the multipole at $r_{ij}=|\boldsymbol{x}_i-\boldsymbol{x}_j|=0$ must be added to the right side of eq. (3·1). Hence we have

$$\sum_{i=1}^{N}(\nabla_i^2+k^2)\Psi=\sum_{i<j}\delta(r_{ij})P_{ij}\Psi+\sum_{i<j}\delta(r_{ij})Q_{ij}\Psi \tag{3·2}$$

where P_{ij} is the pseudopotential for the binary collision defined by eq. (2·24) and Q_{ij} is one corresponding to the higher order collisions which will be determined below.

Write eq. (3·2) in the following form

$$\begin{aligned}(\nabla_1^2+\nabla_2^2+2k^2)\Psi=&-\sum_{n\neq 1,2}(\nabla_n^2+k^2)\Psi+\sum_{k<l}^{(1,2)\prime}\delta(r_{kl})(P_{kl}+Q_{kl})\Psi\\&+\delta(r_{12})P_{12}\Psi+\delta(r_{12})Q_{12}\Psi,\end{aligned} \tag{3·3}$$

where $\sum^{(1,2)\prime}$ means that the pair (12) is excluded from the summation. The comparison of this equation with eq. (2·23) leads to

$$f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=-\sum_{n\neq 1,2}(\nabla_n^2+k^2)\Psi+\sum_{k<l}^{(1,2)\prime}\delta(r_{kl})(P_{kl}+Q_{kl})\Psi. \tag{3·4}$$

Then it follows

$$Q_{12}\Psi=K_{12}f=K_{12}\{-\sum_{n\neq 1,2}(\nabla_n^2+k^2)+\sum_{k<l}^{(1,2)\prime}\delta(r_{kl})(P_{kl}+Q_{kl})\}\Psi$$

from the condition that the wave function must vanish at $|\boldsymbol{x}_1-\boldsymbol{x}_2|=a$. Or, taking 1, 2 for general i, j we have

$$Q_{ij}=-K_{ij}\sum_{n\neq i,j}(\nabla_n^2+k^2)+K_{ij}\sum_{k<l}^{(i,j)\prime}\delta(r_{kl})P_{kl}+K_{ij}\sum_{k<l}^{(i,j)\prime}\delta(r_{kl})Q_{kl}, \tag{3·5}$$

which is an equation to determine Q_{ij}. As mentioned in § 2, K_{ij} is an integral operator and therefore eq. (3·5) leads to an integral equation for Q_{ij} which is solved by an iteration procedure. As will be shown in § 4 the inhomogeneous part (the first and the second terms) in eq. (3·5) is of the order a^4, so that the iterated terms are of the higher orders (at least of the order a^6 by eq. (2·28)). Hence the third term in eq. (3·5) can be ignored as long as the terms up to a^4 are concerned.

Here we encounter a question whether the second term in eq. (3·4) changes the dependence of $f_{K,l,m}(r)$ on r, because only the first one was taken into account in eq. (2·11). The answer to this question is as follows.

From eqs. (3·2) and (3·5) we have

$$Q_{ij}=K_{ij}(\nabla_i^2+\nabla_j^2+2k^2)-K_{ij}\delta(r_{ij})(P_{ij}+Q_{ij}). \tag{3·6}$$

Therefore the operator Q_{ij} refers only to the pair (ij) like P_{ij}. Hence it follows that all operations in the second term in eq. (3·4) do not change the dependence of $f_{K,l,m}(r)$ on r, since the pair (12) is excluded from the summation.

Above expression for Q_{ij} is simpler than in eq. (3·5) at a glance, yet the calculations can be carried out rather easily by means of the latter. For this reason we shall employ Q_{ij} expressed in eq. (3·5) in the following discussions. However, if one wishes to carry out the calculations in the second quantization formalism, one is needed to employ eq. (3·6) rather than eq. (3·5).

§ 4. Pseudopotential correct up to the order a^4 and the ground state energy resulting from the triple collision for the Bose system

In the previous section we have obtained the pseudopotential for the higher order collisions. If it were possible to solve the eigenvalue problem for eq. (3·2) under the ordinary boundary condition only, the solution should yield an exact energy eigenvalue and an exact wave function in the region having the physical meaning, i. e., outside each hard-sphere.

Here we shall proceed with Huang and Yang, treating the pseudopotential as a small perturbation and expanding the physical quantities in powers of a. The pseudopotential up to the order a (correct up to the order a^3 for the energy) has already been obtained by Huang and Yang. The same could be calculated to any desired orders according to our method. However, we shall confine ourselves to the terms up to a^4 and principally deal with the Bose system.

We shall first consider the binary pseudopotential. The phase shift appearing in eq. (2·24) is given by $\delta_0=-k_r a$ for the S-state and hence we have

$$\tan\delta_0/k_r=-a-(k_r^2/3)a^3+O(a^5)=-a-(a^3/3)\{k^2-(\boldsymbol{K}^2/4)\}+O(a^5). \qquad (4\cdot1)$$

The contributions from the P-state vanish, since the P-wave scattering is forbidden for the Bose system, which is also seen from the definition of $\Omega_{K,l}$ in eq. (2·26). The terms corresponding to larger l-values can be neglected because their contributions are at least of the order a^5. Hence the consideration of the term $l=0$ is sufficient. For the S-state we have

$$\delta(r_{ij})/r_{ij}^2=2\pi\delta(\boldsymbol{x}_i-\boldsymbol{x}_j).$$

Therefore, from eq. (2·24) it follows

$$\begin{aligned}\delta(r_{ij})P_{ij}=&8\pi\delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\{a+(a^3k^2/3)\}(\partial/\partial r_{ij})\left(r_{ij}\sum_K\Omega_{K,0}(i,j)\right)\\&-(2\pi a^3/3)\delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\sum_K\boldsymbol{K}^2(\partial/\partial r_{ij})\left(r_{ij}\Omega_{K,0}(i,j)\right)\end{aligned} \qquad (4\cdot2)$$

which is correct up to the order a^4. In this expression the operator $A(i,j)=\sum_K\Omega_{K,0}(i,j)$ can be taken as unit operator. This will be proved as follows: from eq. (2·26) we have

$$A(i,j)\Psi=\frac{1}{4\pi}\int d\Omega'\,\Psi\left(\cdots,\boldsymbol{X}^{(i)}+\frac{\boldsymbol{x}'}{2},\cdots,\boldsymbol{X}^{(j)}-\frac{\boldsymbol{x}'}{2},\cdots\right). \qquad (4\cdot3)$$

This equation implies that the operation of $A(i,j)$ on Ψ is nothing but to take an angular average keeping the center of mass at rest. Hence the relation $A(i,j)=1$ is trivial for the case when Ψ does not include the pair (ij) or includes the pair (ij) only, since we are considering the S-state. If Ψ is a function of the coordinate i and $l(\neq j)$, we have

$$A(i,j)\Psi(i,l)=\sum a(\boldsymbol{k}_i,\boldsymbol{k}_l)\,e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}+i\boldsymbol{k}_l\cdot\boldsymbol{x}_l}(\sin k_i r_{ij}/2)/(k_i r_{ij}/2)$$

expanding $\Psi(i, l)$ into the Fourier series, i. e., $\Psi(i, l) = \sum a(\boldsymbol{k}_i, \boldsymbol{k}_l) \exp i(\boldsymbol{k}_i \cdot \boldsymbol{x}_i + \boldsymbol{k}_l \cdot \boldsymbol{x}_l)$. From this equation it follows that

$$\delta(\boldsymbol{x}_i - \boldsymbol{x}_j)\,(\partial/\partial r_{ij})\,(r_{ij} A(i, j)\,\Psi(i, l)) = \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \sum a(\boldsymbol{k}_i, \boldsymbol{k}_l)\, e^{i\boldsymbol{k}_i \cdot \boldsymbol{x}_i + i\boldsymbol{k}_l \cdot \boldsymbol{x}_l}$$

$$= \delta(\boldsymbol{x}_i - \boldsymbol{x}_j)\,(\partial/\partial r_{ij})\,(r_{ij}\Psi(i, l))$$

which means $A(i, j)$ is equivalent to unit operator.

Next we shall consider Q_{ij}. Neglecting the third term in eq. (3·5), we have

$$Q_{ij} = -K_{ij} \sum_{n \neq i, j} (\nabla_n^2 + k^2) + K_{ij} \sum_{k<l}^{(i,j)\prime} \delta(r_{kl}) P_{kl}.$$

With the help of eq. (2·27) it follows that the contributions from the terms $l \geq 2$ in K_{ij} can be neglected, since the wave functions in the unperturbed system are regular at $r_{ij} = 0$. Hence the consideration of the term $l = 0$ is sufficient as in P_{ij}. The perturbed function singular at $r_{ij} = 0$ is at least of the order a and therefore it is permitted to put

$$j_0(k_r r) = 1 \quad \text{and} \quad n_0(k_r r) = -1/k_r r$$

in eq. (2·25). Again the equivalence of $A(i, j)$ to unity can be proved as previously done. Hence, with the help of eq. (2·22) we have

$$K_{ij} = -\frac{2}{r_{ij}^2} \int_0^a dr_{ij} r_{ij}^2 + \frac{2a}{r_{ij}^2} \int_0^a dr_{ij} r_{ij} \tag{4·4}$$

which should be interpreted as an integral operator. From eqs. (3·2), (4·2) and (4·4) we have

$$\sum_{i=1}^{N} (\nabla_i^2 + k^2)\,\Psi = H_b \Psi + H_t \Psi \tag{4·5}$$

where

$$H_b = 4\pi a \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \frac{\partial}{\partial r_{ij}} r_{ij} + \frac{4\pi a^3}{3} k^2 \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \frac{\partial}{\partial r_{ij}} r_{ij}$$

$$- \frac{\pi a^3}{3} \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \sum_K \boldsymbol{K}^2 \frac{\partial}{\partial r_{ij}} (r_{ij} \Omega_{K,0}(i, j)) \tag{4·6}$$

and

$$H_t = 2\pi \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \int_0^a dr_{ij} (r_{ij}^2 - a r_{ij}) \sum_{n \neq i, j} (\nabla_n^2 + k^2)$$

$$- 16\pi^2 a \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) \int_0^a dr_{ij} (r_{ij}^2 - a r_{ij}) \sum_{k<l}^{(i,j)\prime} \delta(\boldsymbol{x}_k - \boldsymbol{x}_l) \frac{\partial}{\partial r_{kl}} r_{kl}. \tag{4·7}$$

Here H_b corresponds to the pseudopotential for the binary collisions, H_t to that for the higher collisions. It is noted, however, that H_t does not correspond to the purely triple collisions, since it includes the term like $\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ or $\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)\,\delta(\boldsymbol{x}_3 - \boldsymbol{x}_4)$ corresponding to the binary collision of 1,2 or to the two independent binary one of 1,2 and of 3,4,

besides the term like $\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)$ which represents the triple collision of 1,2 and 3. Hereafter, however, we shall call H_t as corresponding to the triple collisions.

The lowest order term in eqs. (4·6) and (4·7) is the first in eq. (4·6) which is just the same as was treated by Huang and Yang.[1] For the ground state the perturbational treatment of this term under the periodic boundary condition yields

$$k^2=4\pi a\rho\frac{N-1}{N}+\cdots \tag{4·8}$$

$$\Psi=\psi_0+a\psi_1+\cdots \tag{4·9}$$

where

$$\psi_0=V^{-N/2}, \tag{4·10a}$$

$$\psi_1=\sum_{i<j}F(r_{ij})\psi_0, \tag{4·10b}$$

$$F(r_{ij})=-\frac{4\pi}{V}\sum_{k\neq 0}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_i-\boldsymbol{x}_j)}}{k^2} \quad\text{and}\quad F(r)\underset{r\to 0}{\to}-\frac{1}{r}+\frac{2.37}{L}. \tag{4·11}$$

Here the terms correct up to a which are necessary in the following calculations are given.

We wish now to calculate the contributions arising from H_t. Multiplying eq. (4·5) by ψ_0 and integrating over the coordinates of all particles, we have

$$Nk^2=\int\psi_0 H_b\Psi\,d\tau+\int\psi_0 H_t\Psi\,d\tau.$$

Let the second term on the right be Nk_t^2, i. e.,

$$Nk_t^2=(\psi_0 H_t\Psi)=\int\psi_0 H_t\Psi\,d\tau. \tag{4·12}$$

Substituting eq. (4·9) in this expression we have

$$Nk_t^2=-16\pi^2a^4N(N-1)\,(N-2)/3V^2-32\pi^2a^2N(N-1)\,(N-2)$$
$$\times\{\psi_0\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\int_0^a(r_{12}^2-ar_{12})\,dr_{12}\,A(1,2)\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_3)\,F(r_{23})\psi_0\}, \tag{4·13}$$

where the operator $A(1, 2)$ is written down explicitly. It is noted that the second term on the right represents the contribution from the cluster of particles 1, 2 and 3. From the defining equation for $A(1, 2)$, eq. (4·3), and eq. (4·11), we find

$$Nk_t^2=-64\pi^2a^4N(N-1)\,(N-2)/3V^2. \tag{4·14}$$

Or, if we define the corresponding energy as E_t, we have

$$E_t/N=-32\pi^2a^4\hbar^2(N-1)\,(N-2)/3mV^2\simeq-32\pi^2a^4\hbar^2\rho^2/3m \tag{4·15}$$

which is the energy due to the triple collisions.

On the other hand the second and the third terms in H_b also give the contributions

of the order a^4. The contribution from the third is easily proved to be zero and that from the second becomes

$$\{\psi_0 4\pi a^3 k^2/3 \sum_{i\neq j} \delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\,(\partial/\partial r_{ij})\, r_{ij}\psi_0\} = 16\pi^2 a^4 N(N-1)^2/3V^2. \qquad (4\cdot16)$$

Adding this to the value given by eq. (4·14), we have

$$-16\pi^2 a^4 N(N-1)\,(3N-7)/3V^2 \qquad (4\cdot17)$$

as the total Nk^2 except for the contributions from the first term in eq. (4·6). The corresponding energy is therefore

$$-8\pi^2 a^4 \hbar^2 \rho^2/m \qquad (4\cdot18)$$

per particle. It is noted that this energy is essentially identical with that given by the second term in eq. (4·13), i.e., the energy corresponding to the collision of three particles forming a closed cluster. In eq. (4·18) the appearance of a^4 is consistent with the suggestion given by Huang and Yang[1] that the triple collision energy is at least of the order a^4.

In closing this section we wish to mention some remarks about the contributions to the pseudopotentials from the terms of the higher order angular momentum. As is easily recognized, for the D-state P_{ij} and Q_{ij} are of the same order a^5. The larger l becomes, the lower the order of Q_{ij}. For instance for the G-state we have $P_{ij}\sim a^9$ and $Q_{ij}\sim a^7$. Hence it follows that for the large angular momentum the triple collision is more important than the binary one, which is not improbable from the physical point of view. For the particle pair with the larger angular momentum has a farther distance of closest approach, thus the chance for the binary collision will be diminished.

§ 5. The ground state energy and phonon spectrum in the ρ_k-representation

In § 4 we have calculated the ground state energy by a perturbational procedure, except for the contribution from the lowest order term in the pseudopotential. It is not appropriate, however, to treat this term by a perturbation method, for the energy thus calculated has not the correct volume dependence as was mentioned in § 1. We wish to show in this section that the ρ_k-representation developed by Bogolyubov and Zubarev[6] is useful to avoid this difficulty. The phonon spectrum will also be calculated in this representation.

One reason why the ρ_k-representation has not been applied hitherto to a system of hard-spheres lies in the divergence of the Fourier component of the corresponding potential function. On the other hand the pseudopotential has a definite Fourier component and enables one to apply the ρ_k-representation to the hard-sphere system.

If we retain only the first term in eq. (4·6) the Schrödinger wave equation becomes

$$-(\hbar^2/2m)\sum_i \nabla_i^2 \Psi + (2\pi a\hbar^2/m)\sum_{i\neq j}\delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\,\Psi + P'\Psi = E\Psi \qquad (5\cdot1)$$

where

$$P' = (2\pi a\hbar^2/m) \sum_{i \neq j} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j) r_{ij} (\partial/\partial r_{ij}) . \tag{5·2}$$

In order to carry out the computation it is convenient to neglect the term $P'\Psi$ for a time. Then eq. (5·1) represents the Hamiltonian of particles interacting with the potential of the δ-function type, and the ρ_k-representation can be applied. It turns out, however, that the energy thus calculated is divergent. Next we shall take account of the contribution from the term $P'\Psi$. It can be shown that this contribution becomes ∞, also. However, when two terms are added together, the total energy is no longer divergent, since the term $P'\Psi$ leads to a subtraction procedure which yields the cancellation of the divergence arising from the two terms.

From eq. (5·1) it follows that the potential function $V(\boldsymbol{x})$ is given by

$$V(\boldsymbol{x}) = 4\pi a\hbar^2 \delta(\boldsymbol{x})/m \tag{5·3}$$

of which Fourier component is

$$\nu(\boldsymbol{k}) = \int V(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} = 4\pi a\hbar^2/m . \tag{5·4}$$

Therefore with the help of eq. (2·15) of reference (6) the ground state energy E_0 is given by

$$E_0 = (N^2/2V)\nu(0) + (1/2) \sum_{k}' \{E(\boldsymbol{k}) - (\hbar^2 \boldsymbol{k}^2/2m) - (N/V)\nu(\boldsymbol{k})\} \tag{5·5}$$

where $E(\boldsymbol{k})$ is the phonon energy defined by

$$E(\boldsymbol{k}) = \sqrt{\hbar^2 \boldsymbol{k}^2 \nu(\boldsymbol{k}) \rho/m + (\hbar^2 \boldsymbol{k}^2/2m)^2} \tag{5·6}$$

or,

$$E(k) = (\hbar^2/2m) \sqrt{2k_0^2 k^2 + k^4} \tag{5·7}$$

where

$$k_0^2 = 8\pi a\rho . \tag{5·8}$$

From eqs. (5·4), (5·5) and (5·7) we have

$$E_0 = 2\pi a\hbar^2 \rho N/m + (\hbar^2/2m) \sum_{k}' \{(1/2) \sqrt{k^4 + 2k_0^2 k^2} - (k^2/2) - (k_0^2/2)\} . \tag{5·9}$$

If we replace the sum $\sum_k'$ by an integral above expression diverges, as expected. Although there may be several ways to compute the contribution from P', here we shall choose the following procedure. The wave function correct up to a was given by eq. (4·9). Using this result we compute the expectation value of P'. Up to the order a^2 this leads to

$$(\Psi P' \Psi) = (2\pi a^2 \hbar^2 N^2/m) \{\psi_0 \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) r_{12} F'(r_{12}) \psi_0\} .$$

From eq. (4·11) it follows

$$r_{12} F'(r_{12}) = -F(r_{12}) = (4\pi/V) \sum_{k}' e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}/\boldsymbol{k}^2$$

as $r_{12} \to 0$. Therefore we have

$$(\Psi P'\Psi) = (2\pi a^2\hbar^2 N^2/m)\,(4\pi/V^2)\sum_k 1/k^2 = (\hbar^2/2m)\,(k_0^4/4)\sum_k 1/k^2. \quad (5\cdot 10)$$

Adding this to eq. (5·9) we have

$$E_0 = 2\pi a\hbar^2\rho N/m + (\hbar^2/2m)\sum_k \{(1/2)\sqrt{k^4+2k_0^2k^2} - (k^2/2) + (k_0^2/2) + (k_0^4/4k^2)\}, \quad (5\cdot 11)$$

or, replacing the summation by an integration we have

$$E_0 = \frac{2\pi a\hbar^2\rho N}{m}\left\{1+\frac{128\,(\rho a^3)^{1/2}}{15\sqrt{\pi}}\right\}, \quad (5\cdot 12)$$

which is identical with a result given by Lee, Huang and Yang[3] with the explicit investigation of the eigenvalue problem of the pseudopotential. The coefficient of $(\rho a^3)^{1/2}$ is exact, since the correction term due to the triple collision is of the higher orders, as we have shown in § 4, and that due to the phonon-phonon interactions is also of the higher orders, as we shall show below.

The ground state energy expressed in eq. (5·5) corresponds to the unperturbed energy of a system of the independent phonons. If the interaction between phonons is taken into account the perturbed energy must be added. Its expression has already been given by Bogolyubov and Zubarev (eq. (4·20) of reference (6)). An exact calculation of this term based on the phonon spectrum given by eq. (5·7) is extremely tedious, yet its dependence on the physical quantities can be seen without any detailed calculations.

The perturbed energy is something of the form

$$E_2 \sim (N/\rho^2)\,(\hbar^2/m)\iint (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2/\sqrt{k^4+2k_0^2k^2}\,d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2.$$

If we measure the wave vector in units of k_0 we have $E_2 \sim (N/\rho^2)\,(\hbar^2/m)\,k_0^8 \sim (a\hbar^2\rho N/m)\,(\rho a^3)$. In the same manner the next order term is shown to be $(a\hbar^2\rho N/m)\,(\rho a^3)^{3/2}\times$ (constant). Hence, with the help of eq. (4·18), eq. (5·12) may be remedied as

$$E_0/N = \frac{2\pi a\hbar^2\rho}{m}\left\{1+\frac{128\,(\rho a^3)^{1/2}}{15\sqrt{\pi}}+(\alpha-4\pi)\,(\rho a^3)+\beta\,(\rho a^3)^{3/2}+\cdots\right\} \quad (5\cdot 13)$$

if the phonon-phonon interaction and the triple collision are taken into account. Here α and β are numerical constants. This expression clearly shows that E_0/N is no longer dependent on N explicitly. Furthermore we see that the expansion (5·13) is in powers of $(\rho a^3)^{1/2}$, which is in agreement with Lee and Yang's conclusion[7] drawn from the binary collision expansion method or Lee, Huang and Yang's conclusion[3] drawn from the explicit consideration of the eigenvalue problem.

It is interesting here to compare the excitation energy spectrum given by eq. (5·7) with one deduced by Landau[8] from the low-temperature properties of liquid He^4.

For small $k\,(k \ll k_0)$, k^4 can be neglected compared with $2k_0^2k^2$ and we have

$$E(k) = (\hbar^2/2m)\,(2k_0^2)^{1/2}k,$$

which is a linear function of k. This corresponds evidently to the long-wave excitations in liquid He^4. Then the sound velocity is given by

$$c_0 = (2\hbar/m)\sqrt{\pi a\rho}.$$

The substitution $a = 2.4\,A$ and the values of liquid He^4 leads to $c_0 = 129$ m/sec which is about a half of the experimentally observed value.

As k becomes larger, $E(k)$ is no longer dependent on k linearly and the sound velocity depends on the wave number explicitly ; in other words the sound wave becomes dispersive. Its expression is given by

$$c(k) = c_0(1 + k^2/4k_0^2 + \cdots)$$

up to the term of order k^2. On the other hand, for liquid He^4 the corresponding expression is given by Landau and Khalatonikov[9] and takes the form

$$c(k) = c_0(1 - \gamma^2 \cdot k^2 + \cdots)$$

$$\gamma = 6.2 \times 10^{-17}\ \mathrm{cm}^2.$$

Comparing these two expressions, we see that the theoretical $c(k)$ can not account for the positive value of γ in liquid He^4.

Furthermore we see that the theoretical $E(k)$ cannot account for the roton spectrum in liquid He^4, since $E(k)$ given by eq. (5·7) is a monotonic function of the wave number. At the same time, the form factor function $S(k)$, which can be calculated by the use of Feynman's[10] relation, is also monotonic, as is seen from the following equation

$$S(k) = k/\sqrt{k^2 + 2k_0^2}.$$

In actual liquid He^4, however, $S(k)$ obtained from the X-ray scattering experiments shows the characteristic, nonmonotonic behaviours.

From these considerations it follows that the excitation energy spectrum obtained by the pseudopotential method is no longer valid for the short-wave excitations, though it can predict the existence of phonon states in the long-wave regions. One may attempt to improve the spectrum by taking account of the higher order collision terms. This may be carried out by considering the term H_t, eq. (4·7), and by neglecting the differential operators as we have done from eq. (5·3) to eq. (5·7). Since H_t represents the many-body potential rather than the two-body potential, it may be transformed into the phonon-phonon interactions in the ω_k-representation, together with the alteration of the excitation spectrum. However it seems that the improved spectrum is also monotonic, since the potential is of the δ-function type of which Fourier component is constant and not oscillating.

In order to clarify these points, it may be instructive to reconsider the meaning of the pseudopotential. As Huang and Yang[1] have pointed out, the pseudopotential corresponds to the multipole expansions of surface charge induced on the $3N$-dimensional "tree". To make the understanding easier, it seems appropriate to use the analogy with the electrostatics. Consider a trivial problem to find the electrostatic potential when the sphere of perfect conductor is charged. If we confine ourselves to the spherical symmetric solution it is simply e/r, where e is the constant, r the distance from the origin. This solution has, of course, the meaning outside the sphere and the potential should be constant inside, owing to the surface charge distributed on the sphere. (Fig. 1 curve A). Now,

the above solution is clearly identical with the potential due to the point charge e at the origin, as far as the outside of the sphere is concerned. This charge is distributed on the surface of the sphere in the case of curve A.

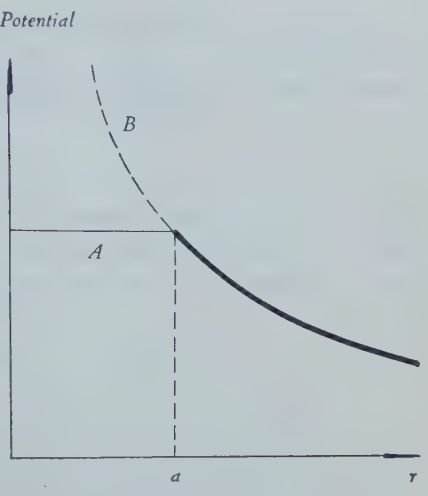

Fig. 1. The potential due to the spherical conductor of radius a. A corresponds to the surface charge, B to the point charge.

Returning now to the pseudopotential, we see easily that the pseudopotential corresponds to the curve B, since it includes the term $\delta(r)$ corresponding to the point charge at the origin. Then it may be inferred that the pseudopotential corresponding to the curve A can be constructed, which includes the term $\delta(r-a)$ instead of $\delta(r)$. In electrostatics the surface charge is given by the jump of $\partial(\textit{potential})/\partial r$ across the surface. The same may be applied to the hard-sphere system, with the potential replaced by the wave function Ψ, i. e., the surface charge is given by the discontinuous value of $\partial\Psi/\partial r$ at $r=a$. Further Ψ may be constant ($=0$) inside the hard-sphere corresponding to the fact that the potential is constant for $r\leq a$ in the electrostatics. Then we may infer that the surface charge becomes larger as the density increases, for the wave function is confined to the smaller regions and $\partial\Psi/\partial r$ at $r=a+0$ becomes larger in order that Ψ should be normalized.

Now, let the surface charge be $S\delta(r-a)$, which replaces the point charge $\delta(r)$, and substitute this in eq. (5·4) with $V(\boldsymbol{x})=S\delta(r-a)$. This leads to

$$\nu(k)=4\pi Sa^2\sin ka/ka,$$

and hence $E(k)$ becomes, with the help of eq. (5·6),

$$E(k)=(\hbar^2/2m)k\sqrt{S'(\sin ka/ka)+k^2} \qquad (5\cdot14)$$

where

$$S'=16\pi m\rho a^2S/\hbar^2.$$

From the reasoning we have mentioned above, it is expected that S' is the increasing function of the density.

Now, $E(k)$ expressed by eq. (5·14) has the remarkable properties that it is the oscillating function of k, contrary to $E(k)$ given by eq. (5·7). At low densities, S' is small and hence the term k^2 overcome the oscillating term, which yields rather a monotonic behaviour of $E(k)$. However, at high densities the oscillating term is expected to be comparable to the term k^2, and hence the nonmonotonic behaviour of $E(k)$ is expected. In this way the roton spectrum of liquid He^4 can be explained at least qualitatively.

From the above discussions, one may infer easily that the "surface charge pseudopotential" is suitable for the investigations of the spectrum in the short-wave regions, where the "point charge pseudopotential" seems to be no longer valid.

The reasoning discussed hitherto seems to be highly tentative; however it is not impossible to construct the "surface charge pseudopotential" in the more exact and systematic way. This will be discussed in a forthcoming paper.

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Two-Pion-Exchange Nuclear Potential

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The adiabatic nuclear potential involving the exchange of at most two pions is derived, the self-pimesic field of each nucleon being taken into account. Numerical values of this potential are calculated. The limit of the applicability of this potential is examined, and as a result it is found that this potential can be used in the region of the inter-nucleon distance $x \gtrsim 0.7\,\mu^{-1}$ (μ^{-1}: pion compton wave length).

From the analysis of the two-nucleon phenomena, we find that the quantitative agreement of the theory with experiments is much improved.

§ 1. Introduction

The pion-exchange nuclear potential has been extensively studied by many authors. However, previous derivations of pion potentials have been based more or less on the perturbation expansion. Hence any satisfactory treatment has not been made so far on the radiative correction.

The use of the dispersion relation has made it possible to correlate virtual pion-nucleon scattering with real scattering. Hence the effect of self-pimesic field of each nucleon can be treated exactly.[1] In Sec. 2 of this paper, numerical values of this potential will be presented. In Sec. 3, the limit of validity of our result will be investigated, by considering the approximation adopted, and by comparing with the previous two-pion-exchange potentials. In the last section, we shall compare our result with the experimental data on the two-nucleon phenomena.

§ 2. The evaluation of the potential

The nuclear potential is given as follows:[1]

$$V = V_2 + V_{4pp} + V_{4ps} + V_{4ss}\,, \tag{1}$$

$$V_2 = \frac{f^2}{4\pi} (\tau^1 \tau^2) (\sigma^1 \cdot \nabla) (\sigma^2 \cdot \nabla) \frac{e^{-\mu x}}{x}$$

$$= \frac{f^2}{4\pi} \frac{1}{3} (\tau^1 \tau^2) \left\{ (\sigma^1 \sigma^2) + S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \right\} \frac{e^{-\mu x}}{x}\,, \tag{2}$$

$$V_{4\eta\eta} = -[(\tau^1\tau^2)^2\sigma^1\cdot\nabla_y\sigma^1\cdot\nabla_z\sigma^2\cdot\nabla_y\sigma^2\cdot\nabla_z\Xi(y,z) + (\tau^1\tau^2)^2\sigma^1\cdot\nabla_y\sigma^1\cdot\nabla_z\sigma^2\cdot\nabla_z\sigma^2\cdot\nabla_y Y(y,z)$$
$$+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\sigma^1\cdot\nabla_y\sigma^1\cdot\nabla_z\sigma^2\cdot\nabla_y\sigma^2\cdot\nabla_z Y(y,z)$$
$$+\tau_i^1\tau_j^1\tau_j^2\tau_i^2\sigma^1\cdot\nabla_y\sigma^1\cdot\nabla_z\sigma^2\cdot\nabla_z\sigma^2\cdot\nabla_y Z(y,z)]_{y=z=x}, \tag{3}$$

$$\Xi(y,z) = -\frac{f^4}{\mu^4}F_{00}(y,z) - \frac{f^2}{9\pi\mu^2}\int\frac{dp}{\omega_p}\sigma_{33}(p)[5F_{\omega_p 0}(y,z) + G_{\omega_p 0}(y,z)]$$
$$+\frac{1}{162\pi^2}\iint\frac{dp}{\omega_p}\frac{dq}{\omega_q}\cdot\sigma_{33}(p)\,\sigma_{33}(q)[17F_{\omega_p\omega_q}(y,z) + 25G_{\omega_p\omega_q}(y,z)],$$
$$Y(y,z) = \frac{f^2}{3\pi\mu^2}\int\frac{dp}{\omega_p}\sigma_{33}(p)[F_{\omega_p 0}(y,z) + G_{\omega_p 0}(y,z)] \tag{3a}$$

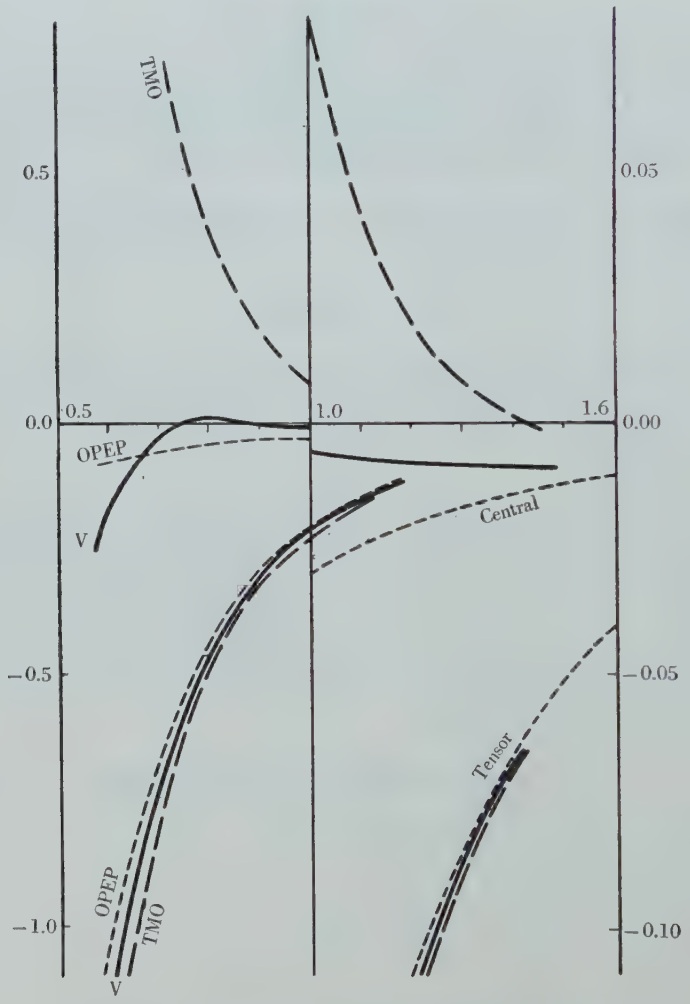

Fig. 1. Triplet even state
"OPEP" means one-pion exchange potential with $f^2/4\pi = 0.08$.

$$+\frac{5}{54\pi^2}\iint\frac{dp}{\omega_p}\frac{dq}{\omega_q}\sigma_{33}(p)\,\sigma_{33}(q)[F_{\omega_p\omega_q}(y,z)+G_{\omega_p\omega_q}(y,z)],$$

$$Z(y,z)=\frac{f^4}{\mu^4}F_{00}(y,z)+\frac{f^2}{9\pi\mu^2}\int\frac{dp}{\omega_p}\sigma_{33}(p)[5F_{\omega_p 0}(y,z)+9G_{\omega_p 0}(y,z)]$$

$$+\frac{1}{162\pi^2}\iint\frac{dp}{\omega_p}\frac{dq}{\omega_q}\sigma_{33}(p)\,\sigma_{33}(q)[17F_{\omega_p\omega_q}(y,z)+9G_{\omega_p\omega_q}(y,z)],$$

$$F_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3yz}\int\frac{k\sin k(y+z)}{\omega_k(\omega_k+\lambda)(\omega_k+\mu)}dk,$$

$$G_{\lambda\mu}(y,z)=\frac{1}{2(2\pi)^3yz(\lambda+\mu)}\int\frac{k\sin k(y+z)}{(\omega_k+\lambda)(\omega_k+\mu)}dk, \tag{3b}$$

$$V_{4ps}=-8\pi(a_1+2a_3)\left[\frac{f^2}{\mu^2}F_0(x)+\frac{4}{9\pi}\int\frac{dp}{\omega_p}\sigma_{33}(p)\,F_{\omega_p}(x)\right]$$

$$+(\tau^1\tau^2)\frac{16\pi}{3\mu}(a_1-a_3)\left[\frac{f^2}{\mu^2}G_0(x)-\frac{2}{9\pi}\int\frac{dp}{\omega_p}\sigma_{33}(p)\,G_{\omega_p}(x)\right], \tag{4}$$

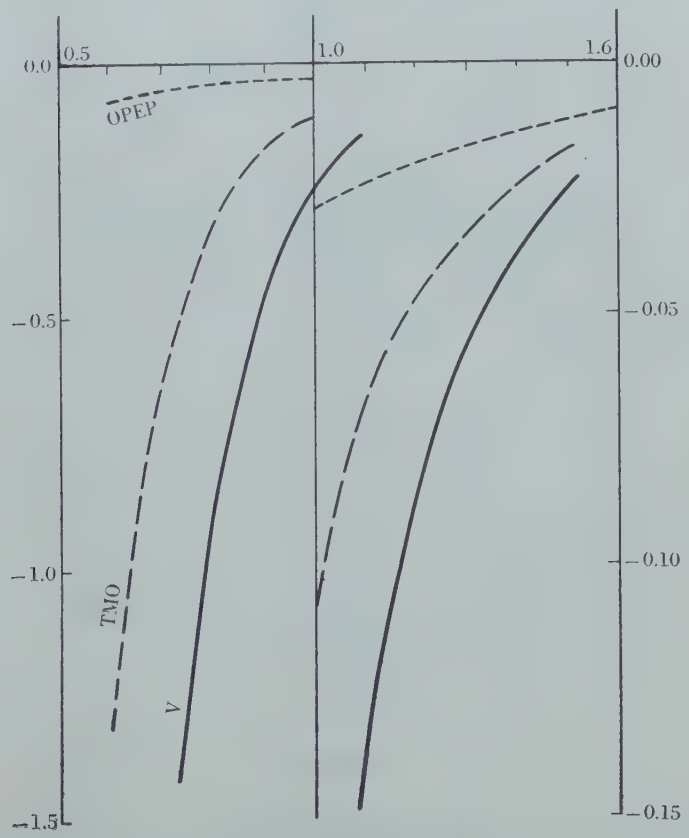

Fig. 2. Singlet even state

$$F_\lambda(x) = \Gamma_y \Gamma_z \frac{1}{2(2\pi)^3 yz} \int \frac{k \sin k(y+z)}{\omega_k(\omega_k+\lambda)} dk \Big]_{y=z=x},$$

$$G_\lambda(x) = \Gamma_y \Gamma_z \frac{1}{2(2\pi)^3 yz} \int \frac{k \sin k(y+z)}{\omega_k+\lambda} dk \Big]_{y=z=x}, \qquad (4a)$$

$$V_{4ss} = -\mu^3 \frac{(a_1+2a_3)^2}{6\pi} \frac{K_1(2\mu x)}{(\mu x)^2}$$

$$+ (\tau^1 \tau^2) \mu^3 \frac{(a_1-a_3)^2}{18\pi} \left[\frac{K_0(2\mu x)}{(\mu x)^3} + \frac{K_1(2\mu x)}{(\mu x)^4} \right], \qquad (5)$$

where f is the renormalized coupling constant, $\sigma_{33}(p)$ is the total cross section in the pure $J=3/2$, $I=3/2$ state, a_i is the scattering length* in the pure $I=i/2$ state, and

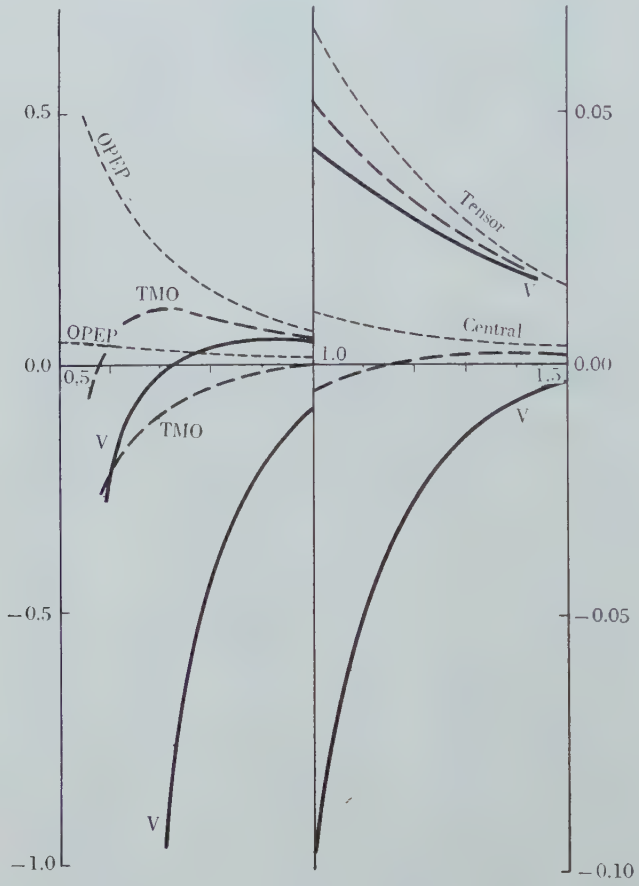

Fig. 3. Triplet odd state

* In our expressions, the sign of the scattering length is opposite to that defined by Fermi and Marshall, Phys. Rev. 75 (1949), 578.

$K_i(x)$ is the modified Hankel function of order i. In these expressions, the terms proportional to f^4 are fourth-order terms of perturbation theory (TMO potential).[2] As to the evaluation of (3), (4), the outline of the calculation will be found in Appendix 1.

Results are shown in Figs. 1, 2, 3 and 4 and Table 1. In these curves, the *S*-wave contributions are rather small as compared with the *P*-wave part for $x \gtrsim 0.7$ (see Fig. 5).

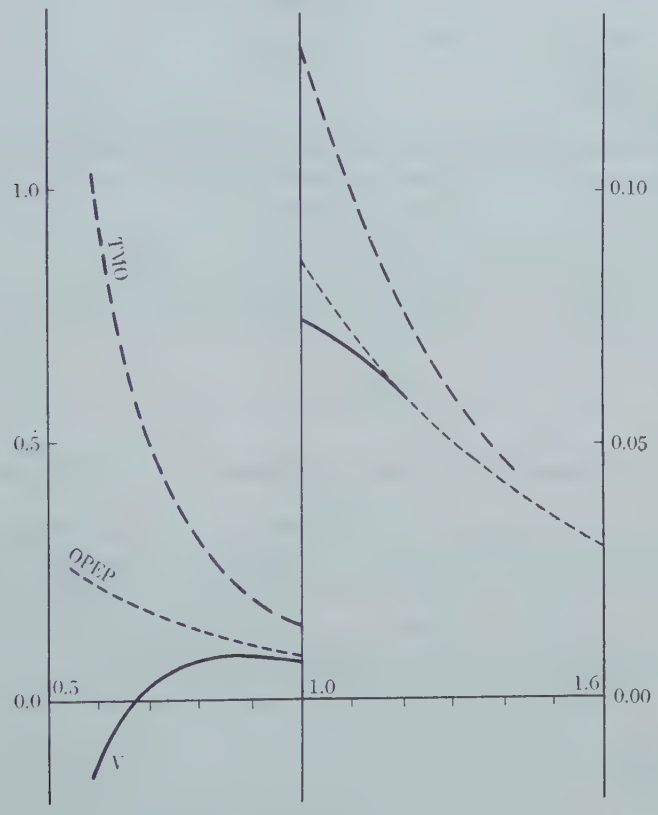

Fig. 4. Single odd state

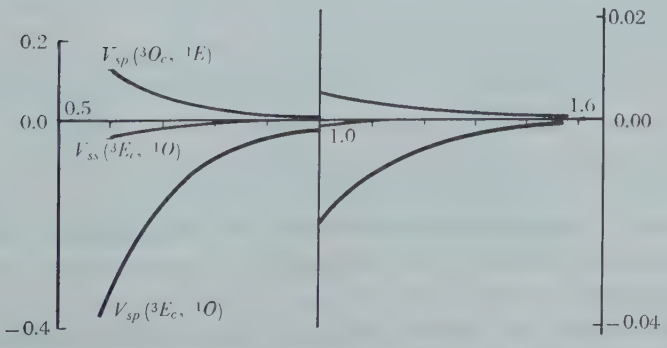

Fig. 5. *S*-wave contribution

In these figures $f^2/4\pi$ is taken as 0.08. The total cross section σ_{33} and the scattering lengths a_t are derived from the phase shifts given by Anderson :[2]

$$\tan\delta_{33}=\frac{0.2484\,p^3}{1+0.7700\,p^2}\,\frac{0.9427}{1.9427-\omega},$$

$$a_3=-0.0865,$$

$$a_1=+0.1950. \qquad (6)$$

Table 1. Character of the potential (for $x\gtrsim 0.7$)

Triplet even central	attractive (weak)
tensor	attractive (strong)
Singlet even	attractive (strong)
Triplet odd central	attractive (strong) [for outer region, repulsive (very weak)]
tensor	repulsive (weak)
Singlet odd	repulsive

§ 3. Merits of the present potential over other ones

We have calculated two-pion-exchange potential by using the total cross section of the pion-nucleon scattering for *P*- and *S*-waves. This potential corresponds to the expression of perturbation theory plus the correction of the self-pimesic field of each nucleon.

The *P*-wave part of this correction was also obtained by Klein and McCormick[4] and by Sato.[5] Henley and Ruderman,[6] previously, estimated the effect of the pion cloud. They also derived the potential relating to the pion-nucleon scattering. Unfortunately, they could not obtain good result, because they used an inadequate approximation similar to the Tamm-Dancoff method for the scattering matrix. Another derivation of the nuclear potential relating to the pion-nucleon scattering was performed by Hasegawa and Nogami.[7] They obtained the potential using the phase shifts of the pion-nucleon scattering calculated from the intermediate coupling theory. Their results are similar to ours qualitatively except for the central part of the triplet even state.

Another potential[8] following the F. S. T. method[9] has the form of a quotient of the "normal" part (expanded up to f^4) to the "probability" part (expanded up to f^2). If the whole expression is further expanded in powers of the coupling constant, it is seen that the potential contains the terms of iteration of the ladder type owing to the "probability" part. They used the perturbation expansion both in the denominator and in the numerator. As the Born approximation for the pion-nucleon scattering is not justified, it would be preferable to use the result of the pion-nucleon scattering for the two-pion-exchange part.

In our calculation, the number of virtual pions exchanged between the pair of nucleons is limited up to two. It is expected that the effects of the three or more pion

exchange are very small in the outer region, because in the energy region up to about 500 Mev inelastic process has very small cross section. It has also been estimated that the three-pion-exchange potential has rather small contribution for $x \gtrsim 0.6 \sim 0.7$.[10]

Before entering into the discussion of recoil effect, let us make a short remark on the pair terms. The previous conclusions[11] including pair suppression are that the contributions of the one- and two-pair terms of the fourth order potential are negligibly small for $x \gtrsim 0.7$. However, the previous calculations did not contain the large τ spin independent terms of our V_{4sp} and V_{4ss}, but contained only the small τ spin dependent terms of ours. Therefore, the effects of the pair terms are not necessarily negligible, particularly for the charge singlet state.

The other approximation we took is the neglect of nucleon recoil. Static approximation justified for the low energy pion phenomena is applicable for pion-nucleon scattering below several hundered Mev, while it may be in discord with experiment for two-nucleon problem at lower energy region. In order to be justified to use the adiabatic approximation, it is necessary that the kinetic energy of nucleon is smaller than the total energy of the exchanged pion:

$$-(\Delta/M)/\omega = (E-V)/\omega \ll 1 .$$

This leads to the condition

$$V/\mu \ll 1 . \tag{7}$$

As seen from Figs. 2 and 4, at $x \approx 0.7$ this condition breaks down for the central part of the charge triplet states. Therefore, in such cases, the potential based on the static approximation may not be applicable.

The recoil correction was estimated by Sato et al. on the assumption $p^2/M^2 \ll 1$.[12] Their result shows that, besides the appearance of spin-orbit coupling, the recoil effect increases rapidly at $x \lesssim 0.7$, although it is vanishingly small for $x \gtrsim 1.5$.

As they stress, recoil effects are not completely negligible in the region where the two-pion exchange potentials are of appreciable magnitude. We must consider this effects for states of weak potential such as the central part of the triplet even state. However, it should be remembered that their calculation does not contain any effect of pion cloud.

As a summary of this section we would conclude that our potential, which includes the effect of pion cloud, is a definite improvement over previous ones in the region $x \gtrsim 0.7$. However, for the central part of the triplet even state, which is very weak, other effects might become important.

§ 4. Check of the potential with two-nucleon phenomena

It has alreaey been shown clearly in S[13] that the pion theory of nuclear forces truly reflects the nature. The outer part of the potential due to the one-pion-exchange process has been *quantitatively* established by the comparison of its consequences with experimental data at low energies. At the same time, properties of the two-pion-exchange potential have been shown to be in *qualitative* agreement with experimental data at low and high

energies up to 100 Mev. Then the next question should be how *quantitatively* the two-pion-exchange potential reproduces the experimental data. Unfortunately, up to now the two-pion-exchange potential was not uniquely given when various methods are applied to the derivation.

It may be interesting to answer the above question assuming our two-pion-exchange potential, since it includes exactly the effects of the self-mesic field of each nucleon. The answer that will be given below is summarized as follows: The quantitative agreement with low and high energy data assuming our potential is much more satisfactory than assuming any other ones. In other words, our potential leaves all key points for the verification of the pion theory of nuclear forces as they were in the discussions in S, while our potential removes some weak points where the agreement between the theory and the experiment was only qualitative.

Discussions below will be developed on the basis of the results of S. The inter-nucleon distance is divided into three regions, I ($x \gtrsim 1.5$), II ($1.5 \gtrsim x \gtrsim 0.7$) and III ($x \lesssim 0.7$).

A. Low energy phenomena

1. *Triplet even state*

It was shown in S that if and only if the one-pion-exchange potential in the outer region I has the coupling constant $f^2/4\pi \approx 0.08$ and if the tensor potential in the region II is negative, a good fit to the deuteron data can be obtained.

In the triplet even state the two-pion-exchange part of our potential is weak and, in particular, its tensor part is vanishingly small. Therefore our potential with $f^2/4\pi \approx 0.08$ reproduces all deuteron data quantitatively.

Conversely, when our potential is assumed to be completely exact in the outer region $x \gtrsim 1$, the pion-nucleon coupling constant can be determined according to the argument developed in S. When only the one-pion-exchange potential was assumed in the outer region $x > 1$, the coupling constant was determined as $f^2/4\pi = 0.075 \pm 0.010$. Because the central part of our two-pion-exchange potential is not very strong, the value $f^2/4\pi$, when our potential is assumed, is little changed from the above value, and is estimated as

$$f^2/4\pi = 0.076 \pm 0.010 .$$

2. *Triplet odd state*

A recent work showed that when the effect due to the vacuum polarization established in the quantum electrodynamics is taken into account for the phase shift analysis of p-p scattering data, the resultant ^3P-wave phase shift is reduced appreciably.[14] Namely $^3\delta_p = -0.11 \pm 0.02^0$ at $E_{\text{lab}} = 3.9$ Mev when the vacuum polarization effect is neglected and -0.07 ± 0.02^0 when it is taken into account.

As discussed in S, the negative P-wave phase shift is a consequence of the fact that there appears a hump of the potential outside the pion range which is characteristic of the pion theory of nuclear forces. Namely, (i) the one-pion-exchange potential with $f^2/4\pi = 0.08$ gives a negative contribution of -0.16° to $^3\delta_p$ and (ii) a short range

attractive potential due to the two-pion-exchange processes gives a small positive contribution to ${}^3\delta_p$.

In the previous work,[15] a nice agreement between the theory and the apparent experimental result ${}^3\delta_p = -0.11°$ was obtained because the positive contribution (ii) was $+0.02 \sim 0.06°$, if the two-pion-exchange potential known at that time (the TMO potential[16] or Brueckner and Watson's one[17]) was assumed. However, when the vacuum polarization effect is taken into account, the contribution (ii) must be larger, that is, $+0.07 \sim 0.09°$.

The characteristic feature of our two-pion-exchange potential in the triplet odd state is that the central part is strongly attractive, while the tensor part is weak. As can be seen from Fig. 3, the central part is two or three times stronger than that of the TMO potential. Therefore our two-pion-exchange potential gives a contribution (ii) of $0.05 \sim 0.1°$ when it is cut off in the region III and can, in consequence, reproduce ${}^3\delta_p = -0.07 \pm 0.02°$.

3. *Singlet even state*

The discussion in S shows that in the presence of the one-pion-exchange potential with $f^2/4\pi \approx 0.08$ in the outer region $x \gtrsim 1$, the experimental data (summarized in Table 2) cannot be reproduced without assuming a strong attractive force in the region II. Although this strong attractive force is consistent with the prediction by the two-pion-exchange potential independently of the method of its derivation, the quantitative agreement was not so satisfactory as in the triplet even state. On the contrary, our two-pion-exchange potential gives a nice quantitative agreement as discussed in detail below.

The experimental data are summarized in Table 2.

Table 2. Experimental data in the singlet even state.[13] All figures are in unit of 10^{-13} cm.

scattering length ${}^1a$		effective range ${}^1r_e$	
$(n-p)$	-23.69 ± 0.06	$(n-p)$	2.1~2.8
$(p-p)$	$-15 \sim -20$	$(p-p)$	2.5~2.85

First, a fictitious assumption shall be made that the static pion-theoretical potential is valid even in the inner region $x_0 \lesssim x \lesssim 1$, where x_0 denotes the radius of a hard core. Table 3 shows that our potential with $f^2/4\pi \approx 0.08$ gives a precise agreement with the experimental data if it were to be valid even in such an inner region as $1 \gtrsim x \gtrsim 0.5$. When the same analysis is made assuming the TMO potential, however, the agreement is not so satisfactory. Namely, (i) when $f^2/4\pi = 0.08$ and $x_0 = 0.35$*, ${}^1a = -23.7 \times 10^{-13}$ cm and ${}^1r_e = 2.26 \times 10^{-13}$ cm, ${}^1r_e$ is too small.* (ii) When a reasonable assumption $x_0 \lesssim 0.6$ is made, a large effective range, ${}^1r_e > 2.5 \times 10^{-13}$ cm, requires a large coupling constant, $f^2/4\pi > 0.10$,[18] which is incompatible with the deuteron data as discussed before. It is to be noted that if any other form of the cut-off function is assumed inside x_0,

* ${}^1r_e$ and x_0 were obtained by an interpolation for Table 2b of reference 16.

such as the straight cut-off or the zero cut-off, the resultant ${}^1r_e$ becomes smaller and, consequently, the above discrepancies become larger. When other types of two-pion-exchange potential are assumed as above, the quantitative agreement becomes worse. Thus it may be concluded that our potential is preferable to the other ones.

Table 3. Theoretical results of our potential, when its validity is assumed for $x \geq x_0$.

$f^2/4\pi$	$x_0(1/\mu)$	${}^1a(10^{-13}\,\mathrm{cm})$	${}^1r_e(10^{-13}\,\mathrm{cm})$
0.08	0.49	−23.69	2.59
0.08	0.50	−14.15	2.73

The same conclusion is obtained when, instead of the fictitious assumption made above, the interaction in the inner region ($x \lesssim 1$) is replaced by a phenomenological square well potential as was done in S:

$$\begin{aligned} V &= \text{the pion-theoretical potential with } f^2/4\pi = 0.08 \quad \text{for } x \geq 1, \\ &= V_0 \ \text{(a constant)} \qquad\qquad\qquad\qquad \text{for } 1 > x \gtrsim x_0, \\ &= \infty \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for } x < x_0. \end{aligned} \tag{8}$$

Results when our potential is assumed in the outer region $x \geq 1$ are summarized in Table 4. It can immediately be concluded that a strong attractive force represented by V_0 must be present in the region between the pion range and a half of it. Just the similar conclusion was obtained in S when the one-pion-exchange potential with $f^2/4\pi = 0.08$ was assumed in the outer region as well as when the TMO and other pion-theoretical potentials were assumed. In order to make a quantitative comparison, it is convenient to define a ratio R of the pion-theoretical potential to the phenomenological one,

$$R = \frac{\text{value of the pion-theoretical potential at } x=1}{V_0 \text{ when the same potential as in the numerator is assumed for } x \geq 1}.$$

Table 4. Parameters of Eq. (8) for our potential that reproduce ${}^1a$ and ${}^1r_e$

${}^1a$ $(10^{-13}\,\mathrm{cm})$	${}^1r_e$ $(10^{-13}\,\mathrm{cm})$	x_0 $(1/\mu)$	V_0 (μ)
−14.15	2.50	0.09	−0.26
	2.65	0.26	−0.44
	2.85	0.46	−0.97
−23.65	2.50	0.25	−0.46
	2.65	0.40	−0.79
	2.85	0.59	−1.83

For our potential we can easily see that $R = 1 \sim 4$. For the TMO potential $R > 4$ and for the one-pion-exchange potential $R > 20$. At first sight, it might seem reasonable that $R \gg 1$, since the pion-theoretical potential rapidly decreases due to its singularity as $x \to 0$.

However, as was discussed in S, it is the attractive force around the pion range that serves to make ${}^1r_e$ as large as the experimental value. Hence the result $R>4$ for the TMO potential means a quantitative self-inconsistency that if the TMO potential were valid in the outer region $x\geq 1$, the same potential must be at least a few times larger than itself around $x\sim 1$. For any other two-pion-exchange potentials than the TMO potential, the quantitative agreement is poorer and therefore the quantitative self-inconsistency of the two-pion-exchange potential is removed only by our potential giving $R=1\sim 4$. This is due to the fact that our two-pion-exchange potential is much stronger around the pion range.

Summarizing the above discussions it may be concluded that our potential is preferable to the other two-pion-exchange potentials in that the former can precisely reproduce the experimental data without any self-inconsistency.

However, since such a strong attractive potential was not expected to be present in S, some conclusions obtained there have to be re-examined, which will be discussed in Appendix 2.

4. *Singlet odd state*

Discussions shall be omitted since no information is available on potential in the singlet odd state at low energies.

B. Intermediate and high energy phenomena

The preceding discussions about the low energy phenomena show that the outer part of our potential with $f^2/4\pi\approx 0.08$ reproduces all experimental data precisely. However, as the energy of the two-nucleon system goes higher, we are forced to limit ourselves only within qualitative discussions. The properties of the pion-theoretical potential were discussed up to $E_{\text{lab}}\sim 100$ Mev in S and was found that they fit the experimental data fairly well. As far as qualitative features are concerned, our potential is quite similar with the TMO as well as the other pion-theoretical potentials. Hence it can be expected that our potential also gives a fairly good fit to the experimental data up to 100 Mev. So in the present section only two comments supporting our potential will be made below.

1. *p-p scattering at 18.2 Mev*

Our potential with $f^2/4\pi\approx 0.08$ is in quantitative agreement with *p-p* scattering data at 18.2 Mev.

In S the data was analysed by assuming a potential of the form

$$\begin{aligned} V&=\text{the one-pion-exchange potential with } f^2/4\pi=0.08 \text{ for } x\geq 1,\\ &=V_{OC}+V_{OT}S_{12} \qquad\qquad \text{for } x\leq 1. \end{aligned} \tag{9}$$

(V_{OC} and V_{OT} are two constants).

The results are shown in Table 4.[19] As can be seen from Fig. 3, the tensor part of our potential is equivalent to V_{OT} between the straight and zero cut-off. Then it must be investigated whether the central part of our potential is equivalent to V_{OC} in Table 5. A numerical comparison tabulated in Table 6 shows that our potential can reproduce the *p-p* scattering data quantitatively if a suitable cut-off in the region III is assumed.

Table 5. Parameters of the potential of Eq. (9) fitting the p-p scattering data at 18.2 Mev.[19]

V_{0T}	V_{0C} (μ)
zero cut-off ($V_{0T}=0$)	$-0.29\sim-0.47$
straight cut-off ($V_{0T}=V_T(x=1)$)	$-0.20\sim-0.36$
twice the straight cut-off ($V_{0T}=2V_T(x=1)$)	$-0.10\sim-0.27$

Table 6. A ratio $R'=\dfrac{\int_{x_0}^{\infty}V^{(c)}x^2j_1^2dx-\int_0^{\infty}V_2^{(c)}x^2j_1^2dx}{\int_0^1|V_{0C}-V_2^{(c)}|x^2j_1^2dx}$ is calculated, where $V^{(c)}$ is the central part of our two-pion exchange potential and $V_2^{(c)}$ is that of the one-pion-exchange potential. $j_1=j_1(kx)$, where k is the wave number in unit of μ at 18.2 Mev. When $R'=1$, our two-pion-exchange potential is equivalent to V_{0C} in the Born approximation.

V_{0C} (μ)	R'		
	$x_0=0.6$	$x_0=0.8$	$x_0=1.0$
-0.20	1.2	0.6	0.4
-0.47	0.5	0.3	0.2

2. *n-p polarization at 98 Mev*

The TMO potential was found to be in disagreement with the n-p polarization experiment at 98 ± 3 Mev as was recently reported.[20] The polarization cross section can be expanded in terms of $\sin\theta P_L(\cos\theta)$, where P_L is the Legendre polynomial. The disagreement of the sign of the first term between the experiment (1.17 ± 0.11, the latest values*) and the theory (-0.2 by the TMO potential) may be significant since it is more sensitive than the n-p angular distribution to the detailed features of the potential in the triplet odd state. The first coefficient depends sharply on the ^3P-wave phase shifts which are under strong influence of the two-pion-exchange potential. A strong attractive central force in the triplet odd state has a trend to make the first coefficient positive and so our potential might be preferable to the other pion-theoretical potentials.[21] However, it would be premature to conclude something definite about the first coefficient assuming our potential without comprehensive and detailed investigations because (i) at such high energies the interaction may not be correctly described by the static approximation and (ii) the first coefficient also depends complicatedly on the inner interactions in the triplet even state.

The discussions of the present section may be summarized as follows: The quantitative agreement of the theory with the experiment is much improved if our two-pion-exchange potential is assumed. In particular, all low energy data are reproduced perfectly by the outer part of our potential. This may be due to the fact that our potential includes the effects of the pion cloud exactly.

* Private communication from P. Hillman and G. H. Stafford. [G. H. Stafford, C. Whitehead and P. Hillman, Nuov. Cim. 5 (1957), 1589].

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Appendix 1. The outline of the evaluation of Eqs. (3) and (4)

Differentiating with respect to y and z, and equating $y=z=x$, we obtain the following k-integrals :

$$I_F \equiv \int_{-\infty}^{\infty} \frac{k^n e^{2ikx} dk}{\omega_k(\omega_k+\lambda)(\omega_k+\mu)}, \quad I_G \equiv \frac{1}{(\lambda+\mu)} \int_{-\infty}^{\infty} \frac{k^n e^{2ikx} dk}{(\omega_k+\lambda)(\omega_k+\mu)} \qquad (A\cdot 1)$$

$$(n=1,\ 2,\ 3,\ 4,\ 5).$$

In the complex k plane, the integral can be replaced by an integral along the cut from the upper branching point i (meson mass$=1$) to the infinity $i\infty$.

Taking

$$k=i+s^2 e^{i\theta},$$

$$I_F = -4 i^n e^{-2x} \int_0^{\infty} \frac{(s^2+1)^n (s^4+2s^2-\lambda\mu)\, e^{-2xs^2} ds}{\sqrt{s^2+2}\,(s^4+2s^2+\lambda^2)(s^4+2s^2+\mu^2)} + i^n \frac{\pi}{\lambda} e^{-2x} \delta_{\mu 0},$$

$$I_G = 4 i^n e^{-2x} \int_0^{\infty} \frac{s^2 \sqrt{s^2+2}\,(s^2+1)^n e^{-2xs^2} ds}{(s^4+2s^2+\lambda^2)(s^4+2s^2+\mu^2)}.$$

After s-integral, the integrals in (3a) are evaluated numerically. No cut-off function is employed in the calculation.

Such a step leads us to the results, Figs. 1 to 5.

Appendix 2. Some consequences of a strong attractive force in the singlet even state

In the singlet even state our two-pion-exchange potential predicts a very strong-attractive force even outside the pion range. Since the discussions in S and reference 22 were made without anticipating such a strong attractive force, some conclusions obtained there have to be re-examined.

From the discussions made in 4, it can be easily understood that all qualitative conclusions obtained in the previous works remain valid. For example, it remains true that the properties of the pion-theoretical potential are as follows : the one-pion-exchange potential with $f^2/4\pi \approx 0.08$ in the region I and a strong attractive force in the region II consistent with the two-pion-exchange potential.

Then two quantitative conclusions should be re-examined. One was concerned with a lower limit of $f^2/4\pi$, i. e., $f^2/4\pi \gtrsim 0.07$. Its reasoning was that if $f^2/4\pi < 0.07$, $R > 10$. Since our potential is stronger than the other two-pion-exchange potentials, R is reduced

by a factor of about 4, and the above reasoning rejecting smaller values of $f^2/4\pi$ may not be justified. However, since $R \gg 1$ leads to a quantitative self-inconsistency as discussed in **4. A.** 3 it may be reasonable to conclude that $f^2/4\pi > 0.07$.

The other conclusion was that there must be a strong repulsive interaction in the region $x \lesssim 0.2$ that pushes the wave function outward, resulting in ${}^1r_e$ as large as the experimental value. For our potential the above conclusion should not be strictly applied, since the outer attractive force may make ${}^1r_e$ larger. However, as can be seen from Tables, 3, and 4, it may also be reasonable to predict hard core like repulsion in the inner region.

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Multipole Mixing in Odd Mass Spheroidal Nuclei

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The multipole mixing and half-lives of gamma transitions of odd mass nuclei are studied in the rare earth region. The relatively large E2/M1 ratios observed in rotational transitions are found to be consistent with the strong coupling model. Many long-life transitions have been found between different rotational families. Some of them are explained by using Nilsson's wave functions in spheroidal potential.

§ 1. Introduction

Multipole moments of γ-transitions in many nuclei have been determined by the measurements of angular correlation, internal conversion, Coulomb excitation, etc., and many interesting cases of multipole mixing have been observed, most of them being of M1+E2 type. According to Weisskopf's formula[1], M1 transition probabilities are much larger than E2 one in disagreement with experiments, and some of these discrepancies in medium weight nuclei have been qualitatively explained by the l-forbiddenness of M1 transitions.

In heavy nuclei, especially in the rare earth region, we have tried to account for this type of mixing and some other examples in terms of the strong coupling collective model.

The appearance of rotational spectra in heavy nuclei with large electric quadrupole moments is widely known. The available data on γ-transitions in these nuclei are mainly concerned with transitions between levels belonging to one and the same family, which are characterized by strong E2 components associated with the collective motion of nucleons in the nucleus. The collective motion is such as to deform a nucleus into a spheroid, which rotates adiabatically, giving rise to a rotational spectrum. These deformations are also associated with large electric quadrupole moments and fast E2 transitions as observed in these nuclei.

In the case of odd mass nuclei, M1 components are also present in transitions within a family, and they are important in analyzing the wave functions of the extra particles moving outside the even-even core. We present in § 3 the available data to show that the intensity of these M1 components is qualitatively consistent with the strong coupling collective model.

Only a little is known at present about transitions between levels belonging to different

families. This is due mainly to energy relations and partly to weak branching ratios for the γ-transitions to levels belonging to different families, or partly to the difficulties of Coulomb excitations. These γ-rays are expected to be valuable probes to know many properties of the excited states other than those informed by transitions within a rotational family. The available data on half-lives of these γ-transitions are compared with single proton values in § 4 where the qualitative features are also discussed.

Before the experimental indications are presented in the following sections, it seems convenient for us to recall briefly the outline of the strong coupling collective model as far as γ-transitions are concerned, in order to facilitate the analysis of empirical informations.

§ 2. Theoretical transition probability*

The transition probability for radiation of multipolarity λ and of frequency ω is given by

$$T(\omega, \lambda) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2} \frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2\lambda+1} B(\lambda), \tag{1}$$

where the reduced transition probability $B(\lambda)$ can be expressed in terms of the reduced matrix element of the multipole operator $\mathfrak{M}$

$$B(\lambda) = \frac{1}{2I_i+1} |\langle f \| \mathfrak{M} \| i \rangle|^2. \tag{2}$$

In calculating the transition probabilities, the special nuclear models are required to evaluate the expression (2), and for order-of-magnitude estimates, a single proton is often conveniently assumed to be responsible for the transition. The single proton value[1),2)] thus calculated has been found to be in fair agreement with experiments though considerable fluctuations are always to be expected. One has, however, noted systematically higher values of the E2 transition probability in the rare earth region, often 100 times as large as the single proton prediction. Bohr's strong coupling model[3)] can well account for this situation. In this model the multipole operator $\mathfrak{M}(\lambda, \mu)$ is written in the form

$$\mathfrak{M}(\lambda, \mu) = \sum_{\nu} \mathfrak{D}^{(\lambda)}_{\mu\nu}(\theta) M(\lambda, \nu), \tag{3}$$

where θ is the Euler angle giving the orientation of the nucleus, $\mathfrak{D}^{(\lambda)}_{\mu\nu}(\theta)$ is the matrix of the $2\lambda+1$ dimensional representation of the rotation group and $M(\lambda, \nu)$ is the multipole operator in the nuclear coordinate system.

The enhancement of E2 transition probability is accounted for in terms of the collective quadrupole oscillation of a nucleus, which gives rise to the quadrupole operator

$$M^{\text{coll}}(E2, \Delta K) = \frac{3}{4\pi} ZeR_0^2 a_{\Delta K}, \tag{4}$$

where $\Delta K = K_f - K_i$, and $a_{\Delta K}$ is the component of the deformation tensor a with respect to nuclear axis.

* In the following, we use the conventional notations.

For K-diagonal transitions, one obtains

$$M^{\text{coll}}(E2) = \frac{3}{4\pi} Ze R_0^2 \beta, \tag{5}$$

which is much larger than that of the single proton model

$$M^{\text{s.p.}}(E2) = er^2 Y_{20}(\theta_p).$$

The collective oscillation of a nucleus gives rise also to the magnetic dipole operator

$$M^{\text{coll}}(M1) = \frac{e\hbar}{2Mc} g_R \boldsymbol{R}, \tag{6}$$

where g_R and $\boldsymbol{R}$ are the g-factor and the angular momentum of the collective flow.

A restriction, besides the usual spin selection rule, can be deduced from eq. (3), namely the K-selection rule

$$-\lambda \leq \Delta K \leq \lambda. \tag{7}$$

The collective E2 transition is observed in K-diagonal transitions

$$\Delta K = 0, \tag{8}$$

in which case the reduced transition probability is given by

$$B(E2) = \frac{5}{16\pi} (eQ_0)^2 (I_i 2K0 | I_i 2 I_f K)^2. \tag{9}$$

The reduced transition probability of the M1 component accompanied by this E2 component is

$$B(M1) = \frac{3}{4\pi} (e\hbar/2Mc)^2 (g_\Omega - g_R)^2 (I_i 1K0 | I_i 1 I_f K)^2, \tag{10}*$$

where g_Ω is the g-factor of the extra proton.

For K-nondiagonal collective E2 transitions, one obtains

$$B(E2) = \left(\frac{3}{4\pi} Ze R_0^2\right)^2 (I_i 2K_i \Delta K | I_i 2 I_f K_f)^2 |(\chi_{K_f}, \chi_{K_i})|^2 |\langle a_{\Delta K}\rangle|^2, \tag{11}$$

where χ's are the wave functions of the extra proton and $\langle a_{\Delta K}\rangle$ is the matrix element of $a_{\Delta K}$. The right-hand side of eq. (11) is expected to be very small because (1) χ_{K_f} is almost orthogonal to χ_{K_i}, for the deformations corresponding to these proton orbits are not so much different, (2) $\langle a_{\Delta K}\rangle$ is nearly equal to zero for spheroidal deformation. In § 4, some examples are illustrated for which the collective E2 components are ineffective, in fact, the observed E2 transition probabilities are much smaller than single proton values in these cases. The collective E2 transition is expected to manifest itself only through the K-mixing in the case of K-nondiagonal transitions.

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* This equation is valid for $K \neq 1/2$. When $K = 1/2$ a further term must be included.

The reduced transition probability of an extra proton for K-nondiagonal transition is given by

$$B(\lambda) = (I_i \lambda K_i \Delta K | I_i \lambda I_f K_f)^2 | (\chi_{K_f}, M(\lambda, \Delta K) \chi_{K_i}) |^2, \qquad (12)^*$$

where $M(\lambda, \Delta K)$ is the multipole operator in the nuclear coordinate system. If the total or orbital angular momentum of the extra proton is a good quantum number, the second factor in the above expression would lead to the j or l selection rule as in the shell model. But, this is not to be expected because neither j nor l is a good quantum number for a proton in a deformed nucleus. In § 4, we use Nilsson's wave functions[4] for the single particle in a spheroidal potential to estimate this factor, and show that many long-life transitions can be explained by the smallness of this factor.

§ 3. M1+E2 transitions

The transitions within a rotational family and the transitions between different families are discussed in this section. For the transitions within a rotational family of an odd mass nucleus, M1 transitions are always accompanied with E2 transitions. The mixing ratios E2/M1 have been experimentally determined in numerous cases. The available data on M1+E2 transitions for odd mass nuclei in the rare earth region are summarized in Table I. Some of them are determined by K conversion coefficients and K/L ratios. They must be corrected for the finite nuclear size effect, but unfortunately there is no reliable method of this correction. However, this effect may be of the same order of magnitude in most cases with somewhat large experimental errors attached to these mixing ratios. From these ratios the values of $|Q_0/(g_\Omega - g_R)|$ are deduced by means of eqs. (9) and (10), and $|g_\Omega - g_R|$ in turn are determined from the knowledge of Q_0. The values of Q_0 are graphically estimated from the data for neighbouring nuclei. In fact the values of Q_0 have been measured by several methods, but they are fairly scattered above and below a smoothed curve and have large experimental errors, and moreover, the values for some nuclei are still missing. Thus we are compelled to adopt the values read on the smoothed curve.

Fig. 1 shows the values of $g_\Omega - g_R$ determined in this way, where the signs are so chosen as to give better agreement with the observed magnetic moments and to yield reasonable values for g_Ω and g_R. The curves in Fig. 1 are the "Schmidt-like" curves, representing $g_j - g_R$ with $g_R = 0.4$ and 0.2**, where g_j is the Schmidt g-factor and we have made the assignment $\Omega = j$. It is interesting to note that, though the points are pretty scattered, they are all enclosed in the "Schmidt-like" curves.

The mixing ratios E2/M1 of transitions between different families have been found to be of the same order as those within the same families, though transition probabilities themselves are much smaller. One reasoning to account for this rather unexpected fact

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* The case $K_i + K_f \leq \lambda$ is excluded here. We have met no such a case.

** Besides the hydrodynamical estimate $Z/A \sim 0.4$, a small value has also been suggested for g_R. (R. J. Blin-Stoyle, Rev. Mod. Phys. 28, (1956) 75.

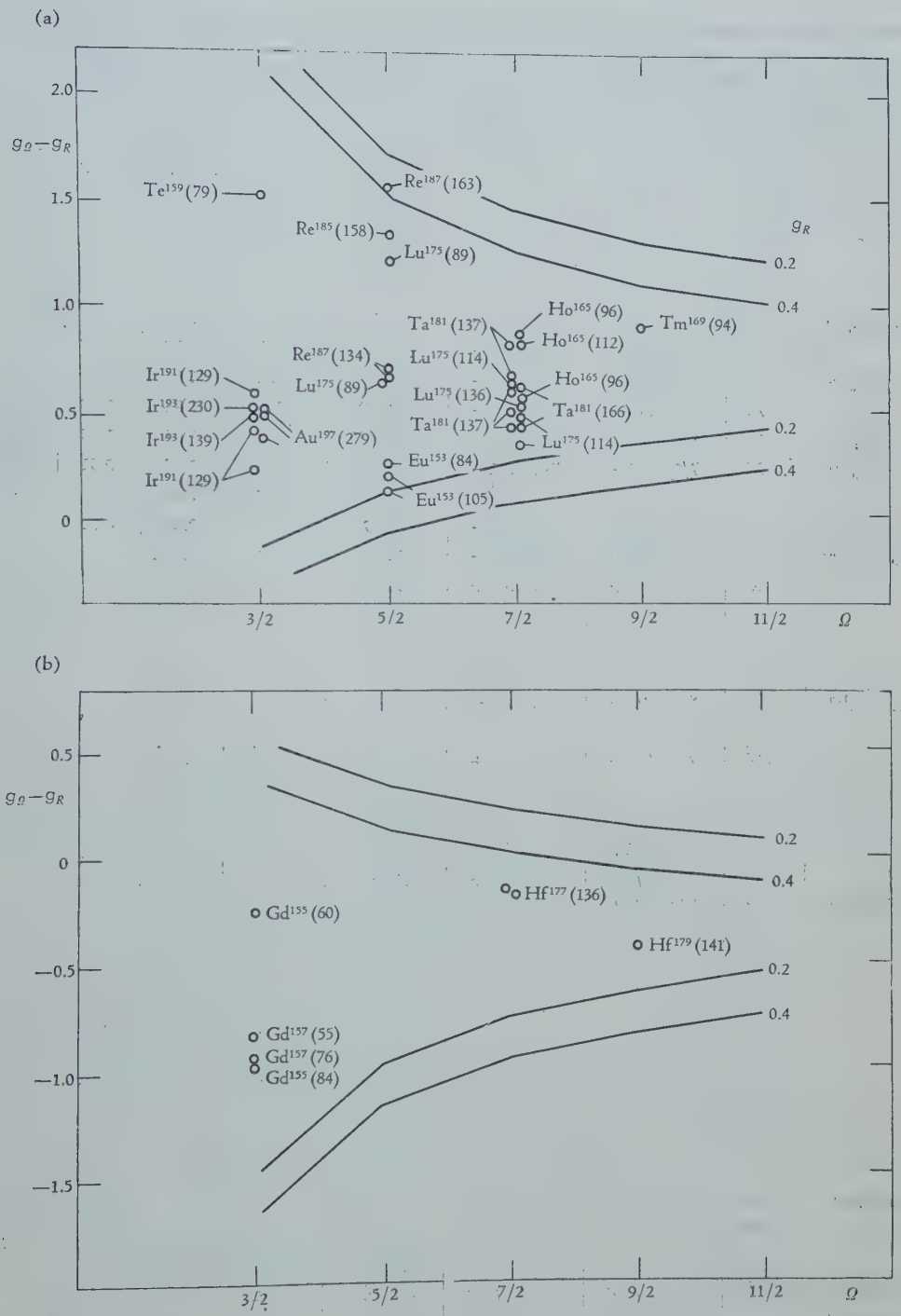

Fig. 1 Values of $g_\Omega - g_R$ determined from experiments and "Schmidt-like curves." (a) odd proton nuclei. (b) odd neutron nuclei.

Table I. M1+E2 transitions in odd mass nuclei. The values in the parentheses in the fifth column are calculated from the branching ratios of γ-rays from the second rotational excited state. The seventh column is obtained from the ratio $|Q_0/(g_\Omega - g_R)|$, using values of Q_0 on the smooth curve as described in § 3. The eighth column represents values reported in original papers.

Nucleus	γ-ray energy kev	Initial state K	Initial state I	Final state K	Final state I	E2/M1	$\left\|\frac{Q_0}{g_\Omega - g_R}\right\|$	$\|g_\Omega - g_R\|$	$\|g_\Omega - g_R\|$	Ref.
$_{63}Eu_{90}^{153}$	84	5/2	7/2+	5/2	5/2+	2.5	21.1	0.27	≤0.1	a)
	105	5/2	9/2+	5/2	7/2+	0.38	27.2	0.21	0.27	b)
						(1)	44.1	0.13		c)
$_{64}Gd_{91}^{155}$	60	3/2	5/2−	3/2	3/2−	0.05	28.5	0.25		c)
	84	3/2	7/2−	3/2	5/2−	(0.027)	6.9	0.97	0.97	b)
$_{64}Gd_{93}^{157}$	55	3/2	5/2−	3/2	3/2−	0.04	8.8	0.83		c)
	76	3/2	7/2−	3/2	5/2−	(0.028)	7.8	0.94	0.79	b)
$_{65}Tb_{94}^{159}$	79	3/2	7/2+	3/2	5/2+	(0.013)	3.5	1.53	1.65	c)
$_{67}Ho_{98}^{165}$	96	7/2	9/2−	7/2	7/2−	0.09	14.5	0.59	0.51	a)
						0.04	9.7	0.88		c)
	112	7/2	11/2−	7/2	9/2−	(0.039)	10.1	0.84	0.71	b)
$_{69}Tm_{100}^{169}$	94	9/2	11/2−	9/2	9/2−	0.024	9.4	0.092		d)
	110	1/2	5/2+	1/2	3/2+	0.025				d)
						(0.02)				c)
	178	7/2	7/2+	1/2	7/2+	0.11				d)
	199	7/2	7/2+	1/2	5/2+	0.15				d)
$_{71}Lu_{104}^{175}$	89	5/2	7/2+	5/2	5/2+	0.1	12.6	0.67		e)
						0.03	6.9	1.23		f)
	114	7/2	9/2+	7/2	7/2+	0.30	22.3	0.36		e)
						0.18	17.2	0.49		f)
						0.14	15.2	0.56		c)
	136	7/2	11/2+	7/2	9/2+	(0.14)	15.7	0.54	0.42	b), c)
	343	5/2	5/2+	7/2	7/2+	≤0.25				e)
$_{73}Ta_{108}^{181}$	137	7/2	9/2+	7/2	7/2+	0.25	16.9	0.46	0.45	g), h)
						0.075	9.3	0.84		i)
						0.11	11.2	0.70		j)
						0.19	14.7	0.53		c)
						0.14	12.7	0.61	0.56	a)
	166	7/2	11/2+	7/2	9/2+	0.26	17.5	0.45	0.53	h)
						0.13	12.4	0.63	0.58	b)
	480	5/2	5/2+	7/2	7/2+	1.6				g)
						34—55				k)
						8—23				k)
$_{72}Hf_{105}^{177}$	114	7/2	9/2−	7/2	7/2−	≥50			≤0.03	a)
	136	7/2	11/2−	7/2	9/2−	1.64	53.8	0.15	0.15	b)
						(1.5)	51.4	0.16		c)
$_{72}Hf_{107}^{179}$	141	9/2	13/2+	9/2	11/2+	(0.16)	19.2	0.42	0.36	b), c)
$_{75}Re_{110}^{185}$	158	5/2	9/2+	5/2	7/2+	(0.03)	5.08	1.36		c)
						(0.008)	2.62	2.63		l)

Nucleus	γ-ray energy kev	Initial state K	Initial state I	Final state K	Final state I	$E2/M2$	$\lvert Q_0/(g_\Omega - g_R)\rvert$	$\lvert g_\Omega - g_R\rvert$	$\lvert g_\Omega - g_R\rvert$	Ref.
$_{75}Re_{112}^{187}$	134	5/2	7/2+	5/2	5/2+	0.11	8.8	0.72		m)
						0.12	9.2	0.68	0.63	a)
	163	5/2	9/2+	5/2	7/2+	(0.02)	4.02	1.57		l)
$_{77}Ir_{114}^{191}$	82	1/2	1/2+	3/2	3/2+	0.67				n)
	129	3/2	5/2+	3/2	3/2+	0.2	8.38	0.60		n)
						0.43	12.30	0.41		c)
						1.2	20.5	0.24		a)
	216	3/2	7/2+	3/2	5/2+	(0.03)	2.84	1.76		l)
$_{77}Ir_{116}^{193}$	139	3/2	5/2+	3/2	3/2+	0.28	9.21	0.49	0.53	a)
	230	3/2	7/2+	3/2	5/2+	(0.30)	8.43	0.53		l)
$_{78}Pt_{117}^{195}$	210			1/2	1/2−	~1				a)
$_{79}Au_{116}^{195}$	61	1/2	1/2+	3/2	3/2+	0.20				n)
$_{79}Au_{118}^{197}$	77	1/2	1/2+	3/2	3/2+	0.14				n)
	279	3/2	5/2	3/2	3/2+	0.33	4.98	0.50		i)
						0.30	4.75	0.53		m)
						0.56	6.49	0.39	0.39	h)
$_{81}Tl_{122}^{203}$	279	1/2	3/2+	1/2	1/2+	4.26				h)
						3				o)
						2.25				m)
						1.5				p)
						2.7				c)
						1.9				q)
	410	1/2	5/2+	1/2	3/2+	0.78				h)
$_{81}Tl_{124}^{205}$	205	1/2	3/2+	1/2	1/2+	0.27				h)

a) Huus, Bjerregaard and Elbek, Dan. Mat. Fys. Medd. **30** (1955), No. 17.
b) N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104** (1956), 981.
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n) J. W. Mihelich and A. de-Shalit, Phys. Rev. **93** (1954), 135.
o) Z. O'Friel and A.H. Weber, Phys. Rev. **101** (1956), 1076.
p) A. H. Wapstra and G. J. Nijgh, Nuclear Phys. **1** (1956), 245.
q) Nordling, Siegbahn, Sokolowski and Wapstra, Nuclear Phys. **1** (1956), 326.

is that M1 transition probabilities in deformed nuclei are sometimes reduced from single

proton values. We have also to consider the effect of K-mixing, to the effect that even a small mixing may be effective to enhance the transition $K \to K'$, which is normally very small.

These arguments of general character, however, will not contribute much to the explanation of the observed facts. In next section, examples will be discussed individually with regard to transition probabilities as well as multipole mixing.

§ 4. Discussions of the observed half-lives

For the experimental test of the selection rule $\Delta K \leq \lambda$, we have collected the available data on half-lives of γ-transitions between levels belonging to different families. In Table II, the ratios of experimental half-lives to the single proton values are tabulated.

As is shown in the first column the E1 as well as the M1 transitions are much weaker than single proton values, this could easily be understood in terms of the selection rule $\Delta K \leq \lambda$ in operation. The half-life of the 425 kev level in Tm^{171}, the decay scheme of which is shown in Fig. 2, has been measured to be 2.5 μ sec, and the two E1 transitions from this level, 290 kev and 310 kev, which are not listed in Table II, are reduced at least by order of 10^8 from single proton values. The E2 transitions in Table II, with $\Delta K = 3$, are also reduced, but the reduction factor is not so large as in the cases of E1 or M1 transitions mentioned above. It seems, however, to be more adequate to compare the E2 transition probabilities with theoretical values based on collective description rather than with single proton ones, in which case, a factor as large as those in the cases of E1 or M1 transitions would be obtained.

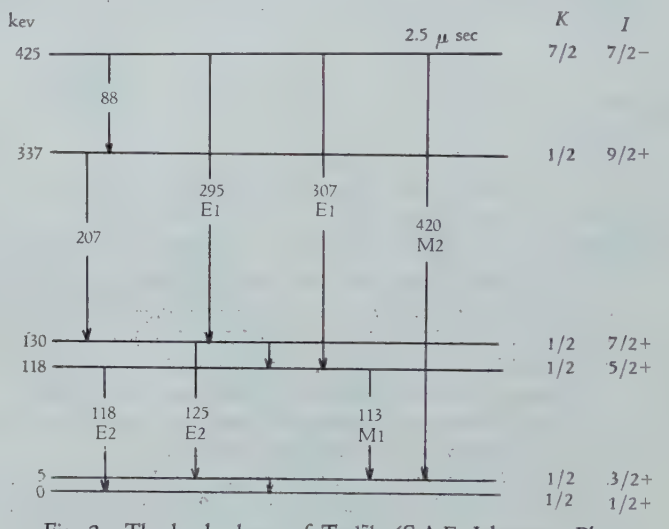

Fig. 2. The level scheme of Tm^{171}. (S.A.E. Johansson, Phys. Rev. **105** (1957), 189.)

As we have already seen, the selection rule (7) seems to be in operation to forbid those transitions with changes in K larger than their multipolarity. It remains now to be investigated whether those transitions with the K-changes smaller than their multipolarities are equally noticeable. For E1 and M1 transitions tabulated in the last column of Table II, the transition probabilities are much smaller than single proton values, and this contrasts with the fact that M1 transition probabilities within same families ($\Delta K = 0$) are of the same order with single proton values. E2 transitions with $\Delta K = 1$ listed in Table II also are all reduced from single proton values by factors about 10^2, while $E2$

Table II. The ratio of single proton transition probabilities to experimental ones. The figure in parentheses indicates the energy in kev of the γ-ray. The 63 kev γ-ray in Tm^{169} is assigned $\Delta K=0$ rather than $\Delta K=1$, but this is included here because parity is changed.

multipolarity	$\Delta K=3$	$\Delta K=2$	$\Delta K=1$
E1	4×10^{8}, Tm^{169} (240) 9×10^{7}, Tm^{169} (261)		2×10^{5}, Tm^{169} (63)
M1	6×10^{5}, Tm^{169} (177) 6×10^{5}, Tm^{169} (198)		5×10^{4}, Tm^{169} (94) 6×10^{5}, Ta^{181} (482)
E2	1×10^{3}, Tm^{169} (177) 1×10^{3}, Tm^{169} (198) 4×10^{3}, Tm^{169} (308)	5×10, Ta^{181} (133)	1×10^{2}, Tm^{169} (94) 9×10, Ta^{181} (346) 8×10, Ta^{181} (482)

Tm^{169}: Hatch, Boehm, Marmier and DuMond, Phys. Rev. **104** (1956), 745.
Mihelich, Ward and Jacob, Phys. Rev. **103** (1956), 1285.

Ta^{181}: F. Boehm and P. Marmier, Phys. Rev. **103** (1956), 342.
F. K. McGowan, Phys. Rev. **93** (1954), 471.
S. H. Vegors and P. Axel, Phys. Rev. **101** (1956), 1067.
Gimmi, Herr and Scherrer, Helv. Phys. Acta **28** (1956), 470A.

transitions within same families are known to be enhanced by factors about 10^2 from single proton values.

It is quite interesting that some transitions are, even for the case $\Delta K \leq \lambda$, much reduced from single proton values. We have calculated the transition probabilities using Nilsson's wave fuctions[4] for the nucleons in a spheroidal potential, in order to observe whether these long-life transitions can find proper accounts in the strong coupling model. The results are given in Table III, and the calculated transition probabilities are obviously, in their general trend, much smaller than single proton values.

Table III. Long-life transitions.

Nucleus	γ-energy kev	Multipolarity	Transition probability in sec^{-1}		
			observed	calculated	single proton
Ta^{181}	482	E2	5×10^{7}	5×10^{5}	4×10^{9}
	346	E2	1×10^{7}	7×10^{4}	8×10^{8}
	133	E2	2×10^{4}	3×10^{4}	6×10^{6}
	482	M1	6×10^{6}	8×10^{9}	3×10^{12}
Tm^{169}	94	M1	5×10^{5}	2×10^{10}	2×10^{10}
	63	E1	7×10^{6}	3×10^{7}	3×10^{12}

Now we enter into individual analysis of the cases included in Table III. For the 482 kev M1 transition the calculated value is reduced by a factor about 10^3 from the single proton estimate, whereas the observed value is reduced by a further factor about 10^3. The reduction of the calculated value is easily understood if one considers two extreme cases: no deformation and large deformation. In these cases the transitions are

essentially $d_{5/2}\rightarrow g_{7/2}$ and 4, 0, 2, 5/2→4, 0, 4, 7/2*, [5], respectively, and are forbidden in each case because of $\Delta l=2$ or $\Delta\Lambda=2$.

The three E2 transitions in Ta^{181} listed in Table III are also much smaller than the single proton values. This is accounted for as the effect of deformation, i.e., these transitions are allowed in the spherical limit ($d_{5/2}\rightarrow g_{7/2}$ or $d_{3/2}\rightarrow d_{5/2}$) while they are forbidden (4, 0, 2, 5/2→4, 0, 4, 7/2 or 4, 1, 1, 1/2→4, 0, 2, 5/2) in the limit of large deformation.

The calculated value for the 133 kev E2 transition is almost equal to the observed one, whereas values for 346 kev and 482 kev E2 transitions are smaller than those observed. It must, however, be noted that the calculated probabilities correspond to the single proton transitions and the effect of possible *K*-mixing has been disregarded, for even a small *K*-mixing may greatly increase the transition probability. Since it is rather awkward to estimate the degree of *K*-mixing, we have contented ourselves with the confirmation that the observed values are greater than the smallest possible theoretical values without *K*-mixing. Although these calculations can not be believed literally, we could say that the long-life E2 transitions in Ta^{181} is quite consistent with the deformed nuclear model.

The case of Tm^{169} is somewhat perplexing. The level scheme of this nucleus is given in Fig. 3. The retardation of the 63 kev E1 transition ($7/2^-\rightarrow 7/2^+$) can properly be accounted for by Nilsson's wave functions. The retardation is complete in two extreme cases, i.e., for spherical potential the transition is essentially $h_{11/2}\rightarrow g_{7/2}$ which is *j*-forbidden, and for large deformation 5, 2, 3, 7/2→4, 0, 4, 7/2 which is also forbidden, because $\Delta\mu=2$ and spin flip is required. The transition probability may be small between these two extreme cases. The main parts of the wave functions are not operative in the transition and the remaining small contributions always tend to cancel out. This situation is quite similar to that pointed out by

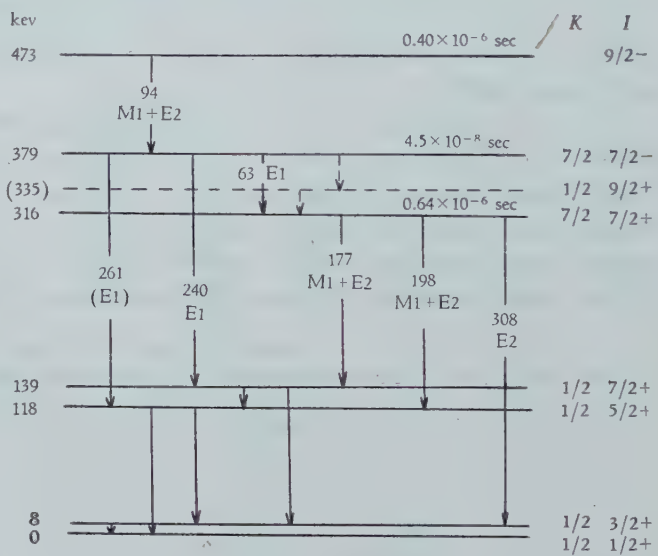

Fig. 3. The level scheme of Tm^{169}.

* These sets of four figures represent the asymptotic quantum numbers N, μ, Λ, Ω of a single particle in a strongly deformed harmonic potential in which the third axis is a symmetry axis. N and μ are the principal and the third component of the oscillation quantum number, Λ and Ω are the third components of the orbital and total angular momenta, respectively.

Strominger and Rasmussen[6] for slow E1 transitions in the actinide region.

The perplexing feature is met in the 94 kev M1 transition. Hatch et al. regard the $9/2^-$ level as the rotational excitation of the $7/2^-$ level, in fact, the spacing between the two levels is consistent with the moment of inertia inferred from the ground state band. The observed long half-life of the 473 kev level seems, however, to be inconsistent with their assignment; we therefore assumed this level to be a particle excitation level so that the 94 kev M1 transition may well be retarded. But the calculation did not yield any retardation due to the fact that initial and final levels are like levels, i.e., $h_{11/2}$ $(K=)$ $9/2$ and $h_{11/2}$ $7/2$, and we could find no other proton orbits with odd parity and such large values to produce small transition rates.

The ratios, single proton to experimental, of transition probabilities, as listed in Table II, cannot be obtained for transitions

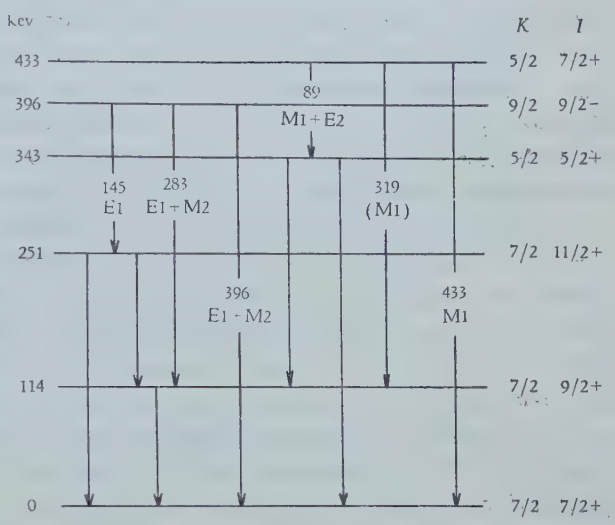

Fig. 4. The level scheme of Lu^{175}. (Hatch, Boehm, Marmier and DuMond, Phys. Rev. **104** (1956), 745.)

Table IV. The ratios of single proton transition probabilities to experimental ones for γ-rays in Lu^{175}*. These ratios are determined up to a common factor. (a) refers to the γ-rays from the 396 kev level, and (b) refers to those from the 433 kev level.

(a) γ-rays from the 396 kev level.

Energy kev	Multipolarity	Single proton transition probability in sec^{-1}	Experimental relative intensity	Ratio	ΔK
145	E1	1.5×10^{13}	5.9	2.5×10^{5}	1
283	E1	1.1×10^{14}	61	1.8×10^{5}	1
	M2	7.0×10^{6}	1.2	5.8×10^{-1}	1
396	E1	3.0×10^{14}	80	3.7×10^{5}	1
	M2	3.5×10^{7}	20	1.8×10^{-1}	1

(b) γ-rays from the 433 kev level.

Energy kev	Multipolarity	Single proton transition probability in sec^{-1}	Experimental relative intensity	Ratio	ΔK
89	E2	9.0×10^{5}	1.2	7.5×10^{-3}	0
	M1	2.0×10^{10}	39	5.1	0
319	M1	8.0×10^{11}			1
433	M1	2.3×10^{12}	16	1.4×10^{3}	1

* Hatch, Boehm, Marmier and DuMond Phys. Rev. **104** (1956), 745.

with unknown half-lives. But these ratios can be determined up to a common factor, provided that the branching ratios for these transitions are available. The nuclide Lu^{175}, illustrated in Fig. 4, provides examples of these cases, the available data are shown in Table IV. In Table IV (a), which refers to transitions with $\Delta K=1$, the entries differ very much for different multipolarities, but if one assumes M2 components to be of the order of single proton values, E1 components turn out to be reduced by factors of order 10^5, the factor 10^5 being compatible with those tabulated in Table II. This reduction and the consequent increase of M2/E1 ratios from single proton values have already been discussed by Chase and Wilets[7]. The assumption that M2 components are of the order of single proton values seems to be compatible with the M2 transition from the 425 kev level of Tm^{171} (see Fig. 2).

In Table IV (b), the 433 kev M1 transition are compared with the 89 kev rotational transition, and it is observed, if one assumes the 89 kev M1 component to be of the order of single proton value, the 89 kev E2 component is enhanced by a factor about 10^2, whereas the 433 kev M1 transition with $\Delta K=1$ is reduced by a factor about 10^3.

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Dispersion Relation for the Photoproduction of *K*-Mesons*

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Dispersion relation for the photoproduction of *K*-mesons by nucleon are derived both covariantly and non-covariantly. It contains the renormalized Born term with inclusion of the magnetic moments of baryons.

§ 1. Introduction

One interesting problem involving the *K*-meson is the determination of its parity. Since there seems to be no direct experiment to determine it at present unlike the pion case, one must be satisfied with a less incisive type of experiment.

Fujii and Marshak[1] found the magnitude and angular dependence of the cross-section for the photoproduction of *K*-mesons in hydrogen depends rather sensitively on the parity of the *K*-mesons. Their calculation is based on the lowest order perturbation expression with inclusion of the anomalous magnetic moments of the proton and the hyperon, which turned out to give considerable effects.

The purpose of this paper is to extend their calculation by using the dispersion relation. Although our derivation of the dispersion relation is not very satisfactory from the standpoint of rigor, the result seems to justify the inclusion of the anomalous magnetic moments of baryons in the Born term.[2] The complete amplitude of the process has a continuous integral, in addition to the Born term which is renormalized. If no resonance phenomena exist in this process, then we might expect small contributions from the continuous integral, and thus the Born term alone will be a good approximation.

We will discuss this relativistic dispersion relation in section (III). In section (II), we first study the application to this problem of Chew and Low's[3] static theory.

§ 2. Static theory of the photo-production of *K*-mesons

In this section, we consider the static theory. Its relativistic extension will be given in the next section. Here, we will show that extension of the Chew and Low[3] theory of photo-pion production is possible and only renormalized quantities appear in the resulting expressions.

The interaction Hamiltonian with the electromagnetic field is given by

* Work supported in part by the Atomic Energy Commission.

$$H_{\text{elec.}} = \int d^3\boldsymbol{r}\, \boldsymbol{j}\cdot\boldsymbol{A}$$

where

$$\boldsymbol{j}_\mu = ie\left\{\bar{\psi}_N \frac{1+\tau_3}{2}\gamma_\mu\psi_N + (\partial_\mu\phi_K^*)\frac{1+\tau_3}{2}\phi_K - \phi_K^*\frac{1+\tau_3}{2}(\partial_\mu\phi_K)\right\}$$
$$+e(\varphi_1\partial_\mu\varphi_2-\varphi_2\partial_\mu\varphi_1)+e(\bar{\psi}_1(\Sigma)\gamma_\mu\psi_2(\Sigma)-\bar{\psi}_2(\Sigma)\gamma_\mu\psi_1(\Sigma)) \qquad (2\cdot1)$$

where ψ_N and ϕ_K are the nucleon and K-meson operators, respectively and φ_α and $\psi_\alpha(\Sigma)$ are the charge-symmetrical pion and Σ-hyperon operators, respectively.

First, consider the process $\gamma+N\to\Lambda+K$. The matrix element for this process is given by[3]

$$H_k(q) = (\Psi_q^{(-)}(\Lambda+K),\ \int \boldsymbol{j}\boldsymbol{A}_k d^3\boldsymbol{r}\,\Psi_N) \qquad (2\cdot2)$$

where Ψ_N and $\Psi(\Lambda+K)$ are the Heisenberg state vectors of the initial nucleon and the final K-meson and Λ system, respectively, and $\boldsymbol{k}$ and $\boldsymbol{q}$ are the incident photon and the final K-meson momenta, respectively. $\boldsymbol{A}_k$ is defined by

$$\boldsymbol{A}_k = (1/2k)^{1/2}\,\mathcal{E}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad (2\cdot3)$$

where $\mathcal{E}$ is the polarization vector of the incident photon.

Now, as in the photo-pion production case,[3] it is convenient to divide the current $\boldsymbol{j}$ into vector, scalar and meson parts.

$$\boldsymbol{j}=\boldsymbol{j}_v+\boldsymbol{j}_s+\boldsymbol{j}_m. \qquad (2\cdot4)$$

In the similar way, $\boldsymbol{j}_v+\boldsymbol{j}_s$ can be defined as independent of any meson variables (so, it commutes with those) and by the condition

$$(\Psi_N\boldsymbol{j}\,\Psi_N) = (\Psi_N(\boldsymbol{j}_v+\boldsymbol{j}_s)\Psi_N). \qquad (2\cdot5)$$

Unlike the pion case, $\boldsymbol{j}_v+\boldsymbol{j}_s$ is not uniquely defined here, because we can choose the following arbitrary combination for it.

$$\boldsymbol{j}_v+\boldsymbol{j}_s=\bar{\psi}_N(\alpha+\beta\tau_3)\gamma_\mu\psi_N+\delta\bar{\psi}_\Lambda\gamma_\mu\psi_\Lambda$$
$$+\epsilon(\bar{\psi}_1(\Sigma)\gamma_\mu\psi_2(\Sigma)-\bar{\psi}_2(\Sigma)\gamma_\mu\psi_1(\Sigma))$$
$$+\theta(\bar{\psi}_3(\Sigma)\gamma_\mu\psi_\Lambda-\bar{\psi}_\Lambda\gamma_\mu\psi_3(\Sigma)) \qquad (2\cdot6)$$

where α, β, δ, ϵ and θ are arbitrary numbers. To fix $\boldsymbol{j}_v$ and $\boldsymbol{j}_s$ uniquely, we can impose additional conditions besides (2·5). How to impose these additional conditions is by no means trivial. Actually, if we take the combination arbitrarily, then we find that we cannot express the matrix element (2·2) only by means of the renormalized coupling constants.

The scalar part $\boldsymbol{j}_s$ which is defined as scalar in the isotopic spin space can be fixed uniquely by imposing the additional condition

$$(\Psi_\Lambda,\ \boldsymbol{j}\Psi_\Lambda)\equiv(\Psi_\Lambda(\boldsymbol{j}_v+\boldsymbol{j}_s)\Psi_\Lambda)\equiv(\Psi_\Lambda\boldsymbol{j}_s\Psi_\Lambda) \qquad (2\cdot7)$$

where Ψ_Λ is the true one Λ-hyperon state. The last identity comes as result of isotopic spin selection rule. The reason why (2·5) and (2·7) fix the scalar part j_s uniquely can be seen from the fact that there are two arbitrary numbers α and δ in (2·6) for the scalar part.

To determine the vector part uniquely, noticing that in the static limit,

$$\int \boldsymbol{j}_v \cdot \boldsymbol{A}_k \, d^3\boldsymbol{r}$$

must have the form proportional to $(\sigma, \nabla \times \boldsymbol{A}_k)$ from the transverse character of the photon field,[3] then we impose the condition

$$\int \boldsymbol{j}_\alpha(\boldsymbol{r})\, \varepsilon\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r} = C F_1(p^2)\, (2\omega_p)^{1/2} V_p^{(0)}(\pi_0) \tag{2·8}$$

where

$$\boldsymbol{p} = \boldsymbol{k} \times \boldsymbol{\varepsilon}. \tag{2·9}$$

F_1 is an unknown function of p^2, normalized to

$$F_1(0) = 1 \tag{2·10}$$

C is a constant, and

$$V_p^{(0)}(\pi_0) = [H_\pi,\ a_p^*(\pi_0)] \tag{2·11}$$

where H_π is the interaction Hamiltonian involving pion and $a_p^*(\pi_0)$ is the creation operator of π_0-meson. In the static limit, H_π is given by

$$\begin{aligned} H_\pi = (\sigma \cdot \nabla)\, \varphi_\alpha [& f_1^{(0)} \cdot \tau_\alpha \psi_N^* \psi_N + i f_2^{(0)} (\psi^*(\Sigma') \times \psi(\Sigma'))_\alpha \\ & + f_3^{(0)} \psi_\Lambda^* \psi_\alpha(\Sigma') + f_3^{(0)} \psi_\alpha^*(\Sigma') \psi_\Lambda] \end{aligned} \tag{2·12}$$

where we have assumed that the Λ and Σ' have the same parity and the K-meson is scalar, and where all the $f_i^{(0)}$ and $g_i^{(0)}$ are unrenormalized coupling constants. Furthermore, ψ_N, ψ_Λ and $\psi_\alpha(\Sigma')$ are the destruction operators for baryons, evaluated at the origin and so $\psi_N^* \psi_N$ or $\psi_N^* \psi_\Lambda$, etc., here are simply the projection operator for the nucleon or operator changing Λ into N, etc. The right-hand side of (2·12) must be interpreted in this way, although it is rather cumbersome convention.

In the same way, the K-meson interaction is given by

$$\begin{aligned} H_K \equiv & g_1^{(0)} \tau_\alpha \phi_K \psi_N^* \psi_\alpha^*(\Sigma') + g_2^{(0)} \phi_K \psi_N^* \psi_\Lambda \\ & + \text{complex conjugate} \end{aligned} \tag{2·13}$$

where the same convention is understood.

So, (2·11) means

$$\begin{aligned} V_p^{(0)}(\pi_0) = (2\pi)^{-3/2} \frac{(\sigma \cdot \boldsymbol{p})}{(2\omega_p)^{1/2}} \{ & i f_1^{(0)} \tau_3 \psi_N^* \psi_N + f_2^{(0)} (\psi_1^*(\Sigma') \psi_2(\Sigma') - \psi_2^*(\Sigma') \psi_1(\Sigma')) \\ & + i f_3^{(0)} \psi_\Lambda^* \psi_{\Sigma_3} + i f_3^{(0)} \bar{\psi}_{\Sigma_3} \psi_\Lambda \}. \end{aligned} \tag{2·14}$$

The condition (2·5) and (2·8) determine $\boldsymbol{j}_v$ uniquely. Then, as in the photo-pion case,[3] the constant C in (2·8) can be determined by

$$C=\frac{(2\pi)^{3/2}}{f_1}\frac{\mu_p-\mu_n}{2}(e/2M) \tag{2·14}$$

where μ_p and μ_n are the total magnetic moments of proton and neutron, respectively, and f_1 is the renormalized pion-nucleon conpling constant which is defined by

$$(\Psi_N{}^* V_\mu^{(0)}(\pi_0)\Psi_N)\equiv u_N{}^* V_\mu(\pi_0)u_N=-\frac{if_1}{(2\pi)^{3/2}}\frac{1}{(2\omega_p)^{1/2}}u_N{}^*\tau_3(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_N. \tag{2·15}$$

In the above expression, u_N means the Pauli spinor of the nucleon. Then, the contribution from the vector part for (2·2) is given by

$$H_k^{(V)}(q)=(2\pi)^{3/2}\frac{e}{f_1}\frac{\mu_p-\mu_n}{2M}(\omega_p/k)^{1/2}F_1(p^2)\,T(\pi_0(\boldsymbol{p})+N\to K(\boldsymbol{q})+\Lambda) \tag{2·16}$$

where

$$T(\pi_0(\boldsymbol{p})+N\to K(\boldsymbol{q})+\Lambda)\equiv(\Psi_q^{(-)}(\Lambda+K)V_q^{(0)}(\pi_0)\Psi_N)$$

is the transition matrix[3] of the process indicated in the bracket of the left-hand side, namely $\pi_0+N\to K+\Lambda$. The formula (2·16) corresponds to the similar one in the photo-pion process[3] and this simple result is due to our special choice of the condition (2·10). Although $F_1(p^2)$ is unknown, we may put it equal to unity as in the photo-pion case.

Next, we calculate the contribution from the scalar part.

$$H_k^{(S)}(q)\equiv(\Psi_q^{(-)}(K+\Lambda),\ \int d^3\boldsymbol{r}\cdot\boldsymbol{j}_s\boldsymbol{A}_k\Psi_N). \tag{2·17}$$

In the same way as in the photo-pion case,[3] (2·17) can be estimated and it gives

$$H_k^{(S)}(q)=-\sum_n\left\{\frac{(\Psi_\Lambda\int\boldsymbol{j}_s\cdot\boldsymbol{A}_k d^3\boldsymbol{r}\Psi_n)(\Psi_n V_q^{(0)*}(K)\Psi_N)}{E_n+E_q}\right.$$

$$\left.+\frac{(\Psi_\Lambda, V_q^{(0)*}(K)\Psi_n)(\Psi_n\int\boldsymbol{j}_s\boldsymbol{A}_k d^3\boldsymbol{r}\Psi_N)}{E_n-E_q-i\epsilon}\right\} \tag{2·18}$$

where

$$V_q^{(0)}(K)=[H_K,\ A_q{}^*(K)] \tag{2·19}$$

and $A_q{}^*(K)$ is the creation operator for a K-meson with momentum $\boldsymbol{q}$.

The lowest discrete levels, which appear in the summation of the right-hand side are $n=N$, Λ and Σ' states.

Contributions from these can be calculated by using the next relations,

$$(\Psi_N,\int\boldsymbol{j}_s\boldsymbol{A}_k d^3\boldsymbol{r}\Psi_N)=(1/2k)^{1/2}\frac{\mu_p+\mu_n}{2}(e/2M)F_2(p^2)u_N{}^*(i\boldsymbol{\sigma}\boldsymbol{p})u_N \tag{2·20}$$

$$(\Psi_\Lambda, \int \boldsymbol{j}_s \cdot \boldsymbol{A}_k d^3\boldsymbol{r}\, \Psi_\Lambda) = (1/2k)^{1/2} \mu_\Lambda (e/2M) F_\Lambda(p^2) u_\Lambda^* (i\sigma \boldsymbol{p}) u_\Lambda \qquad (2\cdot21)$$

with $$F_2(0) \equiv F_\Lambda(0) \equiv 1$$

and

$$(\Psi_\Lambda, V_q^{(0)*}(K)\Psi_N) = (u_\Lambda^* V_q^*(\Lambda, N, K) u_N) = -\frac{g_2^*}{(2\pi)^{3/2}} \frac{1}{(2E_q)^{1/2}} (u_\Lambda^* u_N) \qquad (2\cdot22)$$

where F_2 and F_Λ are some unknown functions and μ_Λ is the magnetic moment of Λ-hyperons. The derivation of $(2\cdot20)$ and $(2\cdot21)$ is similar to the vector case and we used the conditions $(2\cdot5)$ and $(2\cdot7)$. In $(2\cdot22)$, g_2 defines the renormalized coupling constant for $g_2^{(0)}$ and E_q is the K-meson energy.

Actually, $n=\Sigma$ states in $(2\cdot18)$ does not contribute at all, because of the conservation of the strangeness quantum number and the isotopic spin selection rule.

Then, the sum of contributions from $n=N$, Λ and Σ can be written as

$$B^{(S)} = \frac{1}{E_q} (1/2k)^{1/2} (e/2M) \left\{ V_q^*(\Lambda, N, K) \frac{\mu_p + \mu_n}{2} i(\sigma \boldsymbol{p}) F_2(p^2) \right.$$
$$\left. - \mu_\Lambda (i\sigma \boldsymbol{p}) F_\Lambda(p^2) V_q^*(\Lambda, N, K) \right\} \qquad (2\cdot23)$$

where we omitted mass difference between Λ and N for simplicity, although this is not essential.

Similarly to the pion production case,[3] we can investigate the property of $H_k^{(S)}(q)$ as a function of the complex variable z replacing the energy E_q, by putting

$$\widetilde{V}_q = (2E_q)^{1/2} V_q^{(0)}(K), \qquad (2\cdot24)$$

$$B^{(S)} = \cdot \frac{1}{E_q (2E_q)^{1/2}} R_{qk},$$

and

$$H_k^{(S)}(q) = \frac{1}{(2E_q)^{1/2}} \widetilde{H}_{qk}(E_q).$$

Then, $\widetilde{V}_q$ and R_{qk} are independent of E_q and $(2\cdot18)$ gives

$$\widetilde{H}_{qk}(Z) = \frac{R_{qk}}{z} + \frac{1}{\pi} \int_{m_\pi}^{\infty} d\omega \left[\frac{G_1(\omega)}{\omega - z} + \frac{G_2(\omega)}{\omega + z} \right] \qquad (2\cdot25)$$

where m_K and m_π are pion and K-meson mass, respectively, and

$$G_1(\omega) = (1/2i) \{ \widetilde{H}_{qk}(\omega + i\epsilon) - \widetilde{H}_{qk}(\omega - i\epsilon) \},$$
$$G_2(\omega) = (1/2i) \{ \widetilde{H}_{qk}(-\omega + i\epsilon) - \widetilde{H}_{qk}(-\omega - i\epsilon) \}. \qquad (2\cdot26)$$

In the right-hand side of $(2\cdot25)$, the region $m_K > |\omega| > m_\pi$ is the continuous unphysical region. From $(2\cdot25)$, we can see that $\widetilde{H}_{qk}(z)$ is an analytic function of z and has a pole at $z=0$, and branch lines along the real axis at $\infty > z > m_\pi$ and $-\infty < z < m_\pi$.

Finally, consider the meson current contribution. The method is analogous to the scalar case and so we do not bother ourselves in its derivation. It gives the form similar to (2·18) with an addition of meson current term. In the same way, by separating the contributions from $n=N$, Λ and Σ states in the intermediate states, we have

$$H_k^{(m)}(q)=B^{(m)}-\sum_{n>0}^{\infty}\left\{\frac{(\Psi_\Lambda,\int \boldsymbol{j}_m \boldsymbol{A}_k d^3\boldsymbol{r}\,\Psi_n)(\Psi_n, V_q^{(1)*}(K)\Psi_N)}{E_n+E_q}\right.$$

$$\left.+\frac{(\Psi_\Lambda, V_q^{(0)*}(K)\Psi_n)(\Psi_n,\int \boldsymbol{j}_m \boldsymbol{A}_k d^3\boldsymbol{r}\,\Psi_N)}{E_n-E_q-i\epsilon}\right\} \tag{2·27}$$

where

$$B^{(m)}\equiv -\frac{eg_2^*}{(4kE_q)^{1/2}}\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{q})}{(2\pi)^{3/2}}(1+\tau_3)-\left[\left(\frac{1}{2k}\right)^{1/2}\mu_{\Lambda-\Sigma_3}\left(\frac{e}{2M}\right)F_{\Lambda-\Sigma_3}(p^2)(i\sigma\boldsymbol{p})\right.$$

$$\left.-\frac{(2\pi)^{3/2}}{f_1}\frac{\mu_p-\mu_n}{2}\left(\frac{e}{2M}\right)F_1(p^2)\left(\frac{\omega_p}{k}\right)^{1/2}V_p(\Lambda,\Sigma,\pi_0)\right]\frac{V_q^*(N,\Sigma_3,K)}{E_q} \tag{2·28}$$

where $\mu_{\Lambda-\Sigma_3}$ is an analogue of the magnetic moment for the $\Lambda-\Sigma_3$ and photon interaction, and $V(N,\Sigma_3,K)$ and $V(\Lambda,\Sigma_3,\pi_0)$ are the renormalized vertex operators.

If we sum $H_k^{(V)}(q)$, $B^{(S)}$ and $B^{(m)}$, we have

$$H_k^{(V)}+B^{(S)}+B^{(m)}$$

$$=\frac{(2\pi)^{3/2}}{f_1}\frac{\mu_p-\mu_n}{2}\left(\frac{e}{2M}\right)F_1(p^2)\left(\frac{\omega_p}{k}\right)^{1/2}\left[T(\pi_0(\boldsymbol{p})+N\to K(\boldsymbol{q})+\Lambda)\right.$$

$$\left.+\frac{V_p(\Lambda,\Sigma,\pi_0)V_q^*(N,\Sigma_3,K)}{E_q}\right]$$

$$+\left(\frac{1}{2k}\right)^{1/2}\left(\frac{e}{2M}\right)\frac{1}{E_q}\left\{V_q^*(\Lambda,N,K)i\sigma\boldsymbol{p}\cdot\frac{\mu_p+\mu_n}{2}F_2(p^2)\right.$$

$$\left.-i(\sigma\boldsymbol{p})\cdot\mu_\Lambda F_\Lambda(p^2)V_q^*(\Lambda,N,K)-i(\sigma\boldsymbol{p})\mu_{\Lambda-\Sigma_3}F_{\Lambda-\Sigma_3}(p^2)V_q^*(N,\Sigma_3,K)\right\}$$

$$-\frac{1}{(4kE_q)^{1/2}}\frac{eg_2^*}{(2\pi)^{3/2}}\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{q})}{m_K^2+(\boldsymbol{k}-\boldsymbol{q})^2}(1+\tau_3). \tag{2·29}$$

Furthermore, if we calculate $T(\pi_0+N\to K+\Lambda)$ by the lowest order perturbation theory and put all $F_i(p^2)$ equal to be unity, we find that (2·29) gives the renormalized Born approximation with inclusion of the magnetic moments of the baryons and $\Lambda-\Sigma_3-\gamma$ interaction.

Finally, we comment on the process $\gamma+N\to\Sigma+K$. In this case, the condition (2·7) is not convenient. Instead of (2·7), if we impose the condition

$$(\Psi_\Sigma,\ \boldsymbol{J}_S\Psi_\Sigma)\equiv(\Psi_\Sigma,\ \boldsymbol{j}_S\Psi_\Sigma) \tag{2·30}$$

where $\boldsymbol{J}_S$ is the total scalar part of $\boldsymbol{j}$ (so, it contains the meson variables), then we can proceed similarly and only the renormalized coupling constants appear in the theory. The final result is a little more complicated than (2·29), but in the same way, we can show that it contains the renormalized Born approximation with inclusion of the magnetic moments of baryons, as before.

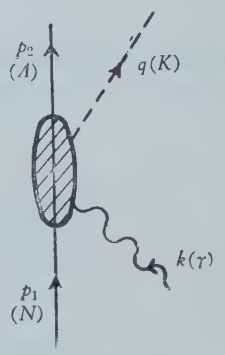

Fig. 1

§ 3. Relativistic dispersion relation

Now, let us consider the covariant calculation of the process, in which an incoming proton with momentum k collides with a nucleon of the momentum p_1 creating a K-meson and a Λ-hyperon, whose momenta are q and p_2, respectively.

Apart from a kinematical factor, the S-matrix for this process is given by

$$S=\iint d^4x\,d^4y\,e^{ikx-iqy}(\Psi_\Lambda^*(p_2)\,\{P(j_\mu(x)\,\theta(y))\,\epsilon_\mu - \frac{(\mathcal{E}q)}{(kq)}\,\delta(x_0-y_0)[j_4(x),\ \theta(y)]\}\,\Psi_N(p_1)) \tag{3·1}$$

where

$$j_\mu(x)=\Box_x A_\mu(x)\qquad \theta(y)=(\Box_y-m^2)\,\phi(y).$$

The second term of the right-hand side of (3·1) assures the gauge-invariance of S. To see covariance of S, we have only to notice that

$$\delta(x_0-y_0)[j_4(x),\ \theta(y)]=-ie\frac{1+\tau_3}{2}\,\delta^{(4)}(x-y)\,\theta(y). \tag{3·2}$$

Now, as usual,[4] we can replace the causal propagator by the retarded one and, using the translation invariance of the theory, we can rewrite (3·1) as follows:

$$S=-(2\pi)^4 i\,\delta^{(4)}(p_1+k-p_2-q)\,T \tag{3·3}$$

$$T=i\int d^4x\,e^{-iQx}(\Psi_\Lambda^*(p_2)\,\{-\theta(x_0)[j_\mu(-ax),\ \theta(bx)]\,\epsilon_\mu - \frac{(\mathcal{E}q)}{(kq)}\,\delta(x_0)[j_4(-ax),\ \theta(bx)]\}\,\Psi_N(p_1)) \tag{3·4}$$

where

$$Q=a\cdot k+b\cdot q \tag{3·5}$$

and a and b are arbitrary numbers satisfying

$$a+b=1. \tag{3·6}$$

From the covariance of the theory, we may put

$$T=\frac{1}{(2\pi)^3}\left(\frac{M_\Lambda\cdot M_N}{E_\Lambda(p_2)E_N(p_1)}\right)^{1/2}(\bar{u}_\Lambda(p_2)Uu_N(p_1)) \tag{3·7}$$

where the u are normalized to

$$\bar{u}_N(p_1)\,u_N(p_1)=\bar{u}_\Lambda(p_2)\,u_\Lambda(p_2)=1$$

so that U is relativistically invariant. Furthermore, from the gauge-invariance of the theory, U can be expanded as[5]

$$U=\sum_{\mu=1}^{4}U^{(\mu)}M_\mu \tag{3·8}$$

$$\begin{aligned}
M_1&=i\gamma_5(\gamma\cdot\varepsilon)\,(\gamma\cdot k)\\
M_2&=i\gamma_5[(p_1\cdot\varepsilon)\,(q\cdot k)-(p_1\cdot k)\,(q\cdot\varepsilon)]\\
M_3&=\gamma_5[(\gamma\cdot\varepsilon)\,(p_1\cdot k)-(\gamma\cdot k)\,(p_1\cdot\varepsilon)-iM_N(\gamma\cdot\varepsilon)\,(\gamma\cdot k)]\\
M_4&=[(\gamma\cdot\varepsilon)\,(p_2\cdot k)-(\gamma\cdot k)\,(p_2\cdot\varepsilon)+iM_\Lambda(\gamma\cdot\varepsilon)\,(\gamma\cdot k)]\gamma_5
\end{aligned} \tag{3·9}$$

where we have assumed that K-meson is pseudoscalar, and $U^{(\mu)}$ in (3·8) is independent of γ-matrices. In the case of the scalar K-meson, if we replace all $i\gamma_5$ in (3·9) by unity, the same formulas given later hold true exactly. The position of γ_5 in (3·9) must be retained, in order to treat the pseudoscalar K-meson case as well as the scalar case at the same time. Although we have assumed space-reflection invariance, this is not essential for the argument which follows.

Now, we are in a position to construct the dispersion relation. However, difficulty arises from the treatment of the 2nd term of the right-hand side of (3·1). As can be seen from (3·3), this term is actually the constant, but we cannot treat it separately from the first term in (3·1), because both terms are not separately gauge-invariant. From the mathematical view-point, we can take only the first term and construct the dispersion relation in the usual way, by separating dispersive and absorptive parts, according to the decomposition

$$\theta(x_0)=(1/2)[\epsilon(x_0)+1].$$

Then, we have the dispersion relation, which express the dispersive part in the terms of the absorptive part. However, this is contradiction, because the absorptive part is gauge-invariant, whereas the dispersive part is not, as can be seen easily. This difficulty may be connected with ambiguity of $\epsilon(x_0)$ at $x_0=0$. As was pointed out by Bogolieubov,[6] due to this ambiguity of $\epsilon(x_0)$ at $x_0=0$ and the singular character of the commutator in the dispersive part, the dispersive part is ambiguous and may contain an additional unknown polynomials with respect to the energy. Thus, we must include some unknown gauge-dependent additional term in the dispersion relation and if we add the second term of (3·5) to the dispersive part, these gauge-dependent terms may cancell each other.

Because of this difficulty, we do not aim at a rigorous derivation of the dispersion relation here. In addition, the fact that the K-meson mass is larger than the pion mass, makes a rigorous treatment more difficult,[7] because of the appearance of virtual pions.

For the sake of gauge-invariance, it seems better to treat the 2nd term of (3·2) together with the first, as was marked in the above. To do so, in the second term, notice that

$$\delta(x_0) = \frac{\partial}{\partial x_0}\theta(x_0)$$

and integrate partially over x_0. Then, we have

$$T = \int d^4x\, e^{-iQx}\theta(x_0)\, g_{\alpha\omega}(x) \tag{3·10}$$

where α and ω represent the initial nucleon and the final Λ states, respectively, and

$$g_{\alpha\omega}(x) = i\Big(\Psi_\Lambda^*(p_2)\Big\{\Big[-j_\mu(-ax),\ \theta(bx)\Big]\epsilon_\mu + \Big[j_4(-ax),\ i\frac{(\varepsilon q)}{(qk)}\frac{\partial}{\partial x_4}\theta(bx)\Big]$$

$$+\frac{(\varepsilon q)}{(qk)}\Big[Q_4 j_4(-ax) + i\frac{\partial}{\partial x_4}j_4(-ax),\ \theta(bx)\Big]\Big\}\Psi_N(p_1)\Big). \tag{3·11}$$

Define the absorptive and dispersive parts by

$$A_{\alpha\omega}(Q) = \frac{1}{2i}\int d^4x\, e^{-iQx} g_{\alpha\omega}(x)$$

$$D_{\alpha\omega}(Q) = \frac{1}{2}\int d^4x\, e^{-iQx}\epsilon(x_0)\, g_{\alpha\omega}(x) \tag{3·12}$$

and so

$$T = D_{\alpha\omega}(Q) + iA_{\alpha\omega}(Q). \tag{3·13}$$

Only the first term of the right-hand side of (3·11) contributes to the absorptive part as we can see by performing the inverse operation, in which we got (3·10) and (3·11).

Corresponding to the decomposition (3·12) and (3·13), we can define absorptive and dispersive parts for $U^{(\mu)}$ defined by (3·7) and (3·8):

$$U^{(\mu)} \equiv U_D^{(\mu)} + iU_A^{(\mu)}. \tag{3·14}$$

Now, we assume the invariance of the theory under the weak reflection,[8] i. e. space-time reflection. Then, from (3·11), we get the next identity.

$$D_{\alpha\omega}^*(Q) = D_{\tilde{\alpha}\tilde{\omega}}(Q), \quad A_{\alpha\omega}^*(Q) = A_{\tilde{\alpha}\tilde{\omega}}(Q) \tag{3·15}$$

where $\tilde{\alpha}$ and $\tilde{\omega}$ are the reversed states of α and ω under the weak reflection, respectively. From (3·15), (3·8) and (3·9), we can show that $U_D^{(\mu)}$ and $U_A^{(\mu)}$ are real quantities. Thus, from (3·14) we have

$$\mathrm{Re}\, U^{(\mu)} \equiv U_D^{(\mu)}$$

$$\mathrm{Im}\, U^{(\mu)} \equiv U_A^{(\mu)}. \tag{3·16}$$

With these preparation, we can undertake the proof of the dispersion relation.

We define K by

$$K=p_1-p_2=q-k \tag{3·17}$$

and take the arbitrary numbers a and b introduced in (3·5) and (3·6) as follows:

$$a=\frac{1}{2}\left(1-\frac{m^2}{K^2}\right) \quad b=\frac{1}{2}\left(1+\frac{m^2}{K^2}\right). \tag{3·18}$$

Furthermore, we define

$$p=\frac{1}{2}(p_1+p_3)-\frac{1}{2K^2}(M_\Lambda^2-M_N^2)\cdot K. \tag{3·19}$$

Then we have

$$Q\cdot K=P\cdot K=0 \tag{3·20}$$

$$Q^2=-\frac{1}{4K^2}(K^2+m^2)^2$$

$$p^2=-\frac{1}{2}(M_\Lambda^2+M_N^2)-\frac{1}{4}K^2-\frac{1}{4K^2}(M_\Lambda^2-M_N^2)^2 \tag{3·21}$$

and

$$P\cdot Q=\frac{1}{4}(p_1+p_2)(q+k)+\frac{m^2}{4K^2}(M_\Lambda^2-M_N^2). \tag{3·22}$$

In proving the dispersion relation, it is convenient to take the following reference system,

$$P_\mu=0 \quad (\mu=1, 2, 3). \tag{3·23}$$

In this reference system, we can put

$$K=(\boldsymbol{V}, 0) \tag{3·24}$$

then

$$\boldsymbol{p}_1=\frac{1}{2}\left[1+\frac{M_\Lambda^2-M_N^2}{\boldsymbol{V}^2}\right]\cdot\boldsymbol{V}$$

$$\boldsymbol{p}_2=-\frac{1}{2}\left[1-\frac{M_\Lambda^2-M_N^2}{\boldsymbol{V}^2}\right]\cdot\boldsymbol{V} \tag{3·25}$$

and

$$\boldsymbol{Q}\cdot\boldsymbol{V}=0 \tag{3·26}$$

$$Q^2=-\frac{1}{4\boldsymbol{V}^2}(\boldsymbol{V}^2+m^2)^2.$$

From the first equation of (3·26), $\boldsymbol{Q}$ is orthogonal to $\boldsymbol{V}$ and so we can fix $\boldsymbol{V}$ and the direction of $\boldsymbol{Q}$. If we put

$$Q_0=E, \quad \boldsymbol{Q}=\lambda\boldsymbol{e} \tag{3·27}$$

where $\boldsymbol{e}$ is a fixed unit vector, which is orthogonal to $\boldsymbol{V}$, then from the 2nd equation of (3·26)

$$\lambda=\left[E^2-\frac{1}{4\boldsymbol{V}^2}(\boldsymbol{V}^2+m^2)^2\right]^{1/2}. \tag{3·28}$$

Rewriting (3·10) as follows:

$$T_{\alpha\omega}(E,\ \boldsymbol{e})=\int d^4x\ e^{iEx_0-i\lambda(\boldsymbol{e}\cdot\boldsymbol{r})}\ g_{\alpha\omega}(x), \tag{3.29}$$

defining the symmetrization and anti-symmetrization operators, S_e and A_e by

$$\begin{aligned} S_eT(E,\ \boldsymbol{e})&=(1/2)\left[T(E,\ \boldsymbol{e})+T(E,\ -\boldsymbol{e})\right]\\ A_eT(E,\ \boldsymbol{e})&=(1/2\lambda)\left[T(E,\ \boldsymbol{e})-T(E,\ -\boldsymbol{e})\right] \end{aligned} \tag{3·30}$$

we can avoid the double-valuedness of T as a function of E.

Then, from the causality condition

$$g_{\alpha\omega}(x)=0 \qquad \text{for} \quad x^2>0 \tag{3·31}$$

we may use the usual Goldberger method to derive the dispersion relation. Here, instead of doing this, we use the analytic property of $S_e\cdot T$ and $A_e\cdot T$ as function of E on the upper half plane, due to the presence of $\theta(x_0)$ in (3·29) and the causality condition (3·31). This means that $U^{(\mu)}$ has the same property. If we return to the general reference system and notice that $U^{(\mu)}$ is relativistically invariant, this implies that $U^{(\mu)}$ is an analytic function of $-P\cdot Q$ on the upper half-plane for fixed value of K^2. More conveniently, if we define

$$\nu=-\tfrac{1}{2}k(p_1+p_2), \quad \nu_1=\tfrac{1}{2}(kq) \tag{3·32}$$

$U^{(\mu)}$ is an analytic function of ν on the upper half-plane for fixed value of ν_1. Therefore, we have the following dispersion relation.

$$U^{(\mu)}=\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\frac{\operatorname{Im}U^{(\mu)}(\nu',\ \nu_1)}{\nu'-\nu-i\epsilon}. \tag{3·33}$$

The next problem is the separation of the discrete unphysical region in the integral (3·33). According to (3·16), $\operatorname{Im}U^{(\mu)}$ can be obtained by studying U_A or more conveniently $A_{\alpha\omega}$ in (3·12). As we mentioned before, only the first term in (3·11) contributes to $A_{\alpha\omega}$ and so, as in the usual way,

$$\begin{aligned} A_{\alpha\omega}\equiv\tfrac{1}{2}(2\pi)^4\Big[&\sum_n(\Psi_\Lambda^*(p_2)\theta(0)\Psi_n)\,(\Psi_n^*\epsilon_\mu j_\mu(0)\Psi_N(p_1))\,\delta(k+p_1-p_n)\\ &-\sum_n(\Psi_\Lambda^*(p_2)j_\mu(0)\epsilon_\mu\Psi_n)\,(\Psi_n^*\theta(0)\Psi_N(p_1))\,\delta(k-p_2+p_n)\Big]. \end{aligned} \tag{3·34}$$

From this, we see that the integral in (3·33) is confined to the following region by performing the usual analysis.

$$\begin{aligned} \nu'>\nu_0&=\nu_1+\tfrac{1}{2}(m_\pi{}^2+2m_\pi M_N)\\ \nu'<-\nu_0'&=-\nu_1-\tfrac{1}{2}(m_\pi{}^2+2m_\pi M_\Lambda) \end{aligned} \tag{3·35}$$

and the discrete points

$$\nu=\pm\nu_1$$

and

$$\nu=-\nu_2=-\nu_1-\tfrac{1}{2}(M_\Sigma{}^2-M_\Lambda{}^2) \tag{3·36}$$

where the discrete points $\nu=\pm\nu_1$ and $\nu=-\nu_2$ come from $n=N$, Λ and Σ in (3·34) and the continuous region (3·35) from the other states like $n=N+\pi$ or $\Lambda+\pi$, etc. We can calculate the contributions from these discrete points. For example, consider the contribution from $n=N$ in the first term of (3·34). Then, we obtain

$$I=\tfrac{1}{2}(2\pi)^4(\Psi_\Lambda{}^*(p_2)\theta(0)\Psi_N(p_1+k))\,(\Psi_N{}^*(p_1+k)j_\mu(0)\epsilon_\mu\Psi_\mu(p_1))$$
$$\times\delta((k)_0+(p_1)_0-(p_1+k)_0). \tag{3·37}$$

Because of 4-momentum conservation and covariance, we may write

$$(\Psi_\Lambda{}^*(p_2)\theta(0)\Psi_N(p_1+k))=(\Psi_\Lambda{}^*(p_2)\theta(0)\Psi_N(p_2+q))$$
$$=\frac{ig_2{}^*}{(2\pi)^3}\left(\frac{M_\Lambda\cdot M_N}{E_\Lambda(p_2)E_N(p_1+k)}\right)^{1/2}\bar{u}_\Lambda(p_2)\gamma_5 u_N(p_2+q) \tag{3·38}$$

where g_2 defines the renormalized coupling constant.[6] In the same way, from the covariance, we may write

$$(\Psi_N{}^*(p_1+k),\ j_\mu(0)\epsilon_\mu\Psi_N(p_1))$$
$$=\left(\frac{M_N{}^2}{E_N(p_1+k)E_N(p_1)}\right)^{1/2}\bar{u}_N(p_1+k)\{A\gamma_\mu+B\tfrac{1}{2}[\gamma_\mu,\ i\gamma k]\}u_N(p_1)\epsilon_\mu \tag{3·39}$$

where A and B are pure constants and, therefore, give the renormalized electric charge and anomalous magnetic moments of the nucleon by

$$A=\frac{ie}{(2\pi)^3}\frac{1+\tau_3}{2}\qquad B=\frac{i}{(2\pi)^3}\left[\frac{1+\tau_3}{2}\mu_p+\frac{1-\tau_3}{2}\mu_n\right]. \tag{3·40}$$

From (3·38), (3·39) and (3·40), after summing over the two spin states in the intermediate nucleon state, (3·37) can be written finally as

$$I=-\frac{g_2{}^*}{2(2\pi)^2}\left(\frac{M_\Lambda\cdot M_N}{E_\Lambda(p_2)E_N(p_1)}\right)^{1/2}\bar{u}_\Lambda(p_2)\left\{e\frac{1+\tau_3}{2}M_1-\frac{2e}{(qk)}\frac{1+\tau_3}{2}M_2\right.$$
$$\left.-2\left(\frac{1+\tau_3}{2}\mu_p+\frac{1-\tau_3}{2}\mu_n\right)M_3\right\}u_N(p_1)\,\delta(2kp_1) \tag{3·41}$$

where M_μ are given by (3·9). Comparing this with (3·7), we can get the following contribution for U_A from this discrete level $n=N$

$$\frac{\pi}{2}g_2{}^*\left\{e\frac{1+\tau_3}{2}M_1-\frac{e}{\nu_1}\frac{1+\tau_3}{2}M_2-2\left(\frac{1+\tau_3}{2}\mu_p+\frac{1-\tau_3}{2}\mu_n\right)M_3\right\}\delta(\nu-\nu_1).$$

In the same way, we can evaluate the contributions from $n=\Lambda$ and $n=\Sigma$ states.

Summing up these three contributions, we have the following contributions from the discrete levels.

$$U_A^{(1)}=\frac{\pi}{2}\left\{eg_2^*\frac{1+\tau_3}{2}\delta(\nu-\nu_1)-g_1^*\mu_{\Lambda-\Sigma_3}(M_\Sigma-M_\Lambda)\tau_3\delta(\nu+\nu_3)\right\},$$

$$U_A^{(2)}\equiv-\pi\frac{eg_2^*}{\nu_1}\delta(\nu-\nu_1),$$

$$U_A^{(3)}=-\pi g_2^*\left(\frac{1+\tau_3}{2}\mu_p+\frac{1-\tau_3}{2}\mu_n\right)\delta(\nu-\nu_1),$$

$$U_A^{(4)}=-\pi\{g_2^*\mu_\Lambda\delta(\nu+\nu_1)+g_1^*\mu_{\Lambda-\Sigma_3}\tau_3\delta(\nu+\nu_2)\} \qquad (3\cdot42)$$

where g_1 is the renormalized coupling constant of Σ, N, K interaction and $\mu_{\Lambda-\Sigma_3}$ is the analog of the magnetic moment for the interaction $\Lambda-\Sigma_3-\gamma$.

Thus, the dispersion relation (3·35) becomes

$$U^{(\mu)}(\nu,\nu_1)=\frac{A^{(\mu)}}{\nu_1-\nu-i\epsilon}+\frac{B^{(\mu)}}{\nu_1+\nu+i\epsilon}+\frac{C^{(\mu)}}{\nu_2+\nu+i\epsilon}$$
$$+\frac{1}{\pi}\left[\int_{\nu_0}^{\infty}d\nu'\frac{\mathrm{Im}\,U^{(\mu)}(\nu',\nu_1)}{\nu'-\nu-i\epsilon}-\int_{\nu_0'}^{\infty}d\nu'\frac{\mathrm{Im}\,U^{(\mu)}(-\nu',\nu_1)}{\nu'+\nu+i\epsilon}\right], \qquad (3\cdot43)$$

$$A^{(1)}=\frac{1}{2}eg_2^*\frac{1+\tau_3}{2},\quad B^{(1)}=0,\quad C^{(1)}=\frac{1}{2}g_1^*\mu_{\Lambda-\Sigma_3}(M_\Sigma-M_\Lambda)\cdot\tau_3,$$

$$A^{(2)}=-\frac{1}{2}\frac{eg_2^*}{\nu_1}\frac{1+\tau_3}{2},\quad B^{(2)}=0,\quad C^{(2)}=0,$$

$$A^{(3)}=-g_2^*\left(\frac{1+\tau_3}{2}\mu_p+\frac{1-\tau_3}{2}\mu_n\right),\quad B^{(3)}=0,\quad C^{(3)}=0,$$

$$A^{(4)}=0,\quad B^{(4)}=g_2^*\mu_\Lambda,\quad C^{(4)}=g_1^*\mu_{\Lambda-\Sigma_3}\cdot\tau_3. \qquad (3\cdot44)$$

The same formulas hold for the scalar K-meson case, if we define $M^{(\mu)}$ by replacing all $i\gamma_5$ in (3·9) by unity.

We can check that the inhomogeneous terms in the right-hand side of (3·43) give exactly the renormalized Born approximation with inclusion of the anomalous magnetic moments of N and Λ as well as $\mu_{\Lambda-\Sigma_3}$. Unlike the dispersion relation in the static limit (2·29), the unknown function $F_4(p^2)$ is not present.

The region of integration in (3·44) consists of physical and unphysical regions. The latter is given by

$$\tfrac{1}{2}[(M_\Lambda+m)^2-M_N^2]\geq\nu'\geq\nu_0 \quad \text{in the 1st integral} \qquad (3\cdot45)$$
$$\tfrac{1}{2}[(M_N+m)^2-M_\Lambda^2]\geq\nu'\leq\nu_0' \quad \text{in the 2nd integral}$$

and the former is

$$\nu'\geq\tfrac{1}{2}[(M_\Lambda+m)^2-M_N^2] \quad \text{in the 1st integral}$$
$$\nu'\geq\tfrac{1}{2}[(M_N+m)^2-M_\Lambda^2] \quad \text{in the 2nd integral.} \qquad (3\cdot46)$$

Actually, (3·46) still contains a kind of unphysical region as in the non-forward scattering pion-nucleon dispersion relation, but similarly, analytic continuation may be possible if we expand the amplitude into spherical harmonics. The appearance of the non-physical region (3·45) is essentially due to the fact that K-meson is heavier than the pion. This appearance makes the dispersion relation (3·43) less useful in this case. In the physical region (3·46), $\mathrm{Im}\, U^{(\mu)}(\nu', \nu_1)$ in the first integrals and $\mathrm{Im}\, U^{(\mu)}(-\nu', \nu_1)$ in the second can be expressed in terms of the amplitudes of $\gamma + N \to K + \Lambda$ and $\bar{K} + N \to \gamma + \Lambda$ processes, respectively.

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On the Form Factor of Liquid He^4 at Absolute Zero, I

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An integro-differential equation is evolved for the radial distribution function in liquid He^4 at absolute zero assuming a particular form of the ground state wave function, the validity of the superposition approximation and a simple relation between the radial distribution function and the wave function. For a system of hard-spheres this equation is transformed into an integral equation for the Fourier transform of the radial distribution function which is closely related to the form factor, and an approximate solution is obtained under the linear approximation. The calculated form factor is compared with the experimental data.

§ 1. Introduction

As the wellknown methods for dealing with the liquid states in the classical theory there are, on the one hand, the cell method and, on the other hand, the method of an integral equation for the distribution functions developed by Kirkwood[1),2)] and Born and Green.[3)] Both have provided a useful means, for instance, to determine the radial distribution function, to calculate the equation of state, to study the condensation phenomena, etc. Theory of liquid helium along the former line has also been developed by various workers.[4)] In particular Matsubara and Matsuda have succeeded in establishing the lattice model keeping the essential features of the real phase transition of liquid helium which could not be explained without a "liquid-like" treatment of many Boson system.

The purpose of this paper is also to attempt a liquid-like description for the many Boson system along the latter line, as a generalization of the method of an integral equation in the classical theory. For the present, we shall confine ourselves to liquid He^4 at absolute zero, that is, the ground state of the Bose system.

It may be appropriate here to describe the motivation underlying such an attempt. To facilitate understanding, let us consider a system of hard-spheres as a typical model of liquid helium. At low densities, quantum mechanical or quantum statistical treatment of the system can be carried out in a systematic way with the help of the pseudopotential method[5)] or of the binary collision expansion method.[6)] However such a systematic treatment is confined only to a dilute system and a formal application to the actual liquid helium leads to the disagreements with the experimental facts or to the lack of the self-consistency of the theory in some respects. For instance, we may cite the following points: (a) the gas-like behaviour of the λ-line as was pointed out by Matsubara and Matsuda,[4)] (b) the monotonic functional form of the form factor which is contrary to the observed

oscillatory one, (c) the serious discrepancy of the sound velocity when calculated in the first and in the second approximations ($\sim$130 m/sec in the first, $\sim$320 m/sec in the second), (d) the occurrence of the ferromagnetism of liquid He^3 when applied to the Fermi system. In order to clarify these points, it would be necessary either to refine above methods so as to be applied to the condensed system or to take another way. One of the corresponding ways in the classical theory is a method of an integral equation, and it may not be unnatural to expect that an analogous method in the quantum theory should yield a satisfactory description for the liquid states.

Of course we shall be encountered with much more difficulties in the quantum theory in the course of the practical applications. One of them is that we have not enough knowledge about the behaviours of the ground state wave function and hence about the characteristic nature of the probability distribution of particles in the configurational space, in contrary to the classical theory where it is characterized by the Boltzmann distribution. However, from a view-point of the generalization of the classical theory, it may be useful to assume that the similar dependence of the probability distribution holds also in the quantum theory. This implies that the ground state wave function is assumed to be a product of functions of atomic pairs, which we shall hereafter call a "pair approximation". It is noted that the same wave function has already been worked out by various authors.[7),9)]

It is easy to see that the pair approximation is in a certain sense a generalization of the Hartree approximation, for if the function of atomic pairs is a product of one-body functions the former is reduced to the latter. An optimum wave function consistent with the pair approximation will be determined by a variational procedure and one may expect that the two-body distribution function will be determined by the self-consistent two-body potential function which is in turn characterized by the two-body distribution itself, just as the one-body distribution is determined by the self-consistent potential field in the Hartree approximation. In contrast to the Hartree one, pair approximation can take account of the atomic correlation in the system which is neglected in the former.

In § 2 we shall show that the procedure mentioned above is in fact possible and gives an integro-differential equation for the radial distribution function under the superposition approximation and under the further simplifications. In § 3 this equation will be transformed into a more convenient form for a system of hard-spheres and the general nature of the solution will be discussed. In § 4 an approximate solution for the form factor will be obtained and compared with the experimental data.

Throughout this paper we shall engage mainly in the form factor which is directly obtained from the X-ray scattering experiments and is defined by

$$S(k) = 1 + \rho \int (g(R) - 1)\, e^{ik \cdot R} dv \qquad (1 \cdot 1a)$$

$$= 1 + 4\pi\rho \int_0^\infty (g(R) - 1)\, R \frac{\sin kR}{k} dR \qquad (1 \cdot 1b)$$

where $g(R)$ is the radial distribution function and ρ is the average density. The investigation of the $S(k)$ is interesting from the theoretical point of view, for it plays an

important role in the determination of the excitation energy spectrum in liquid He^4 as has been pointed out by Bijl[7] and by Feynman.[9]

§ 2. An integro-differential equation for the radial distribution function

Let us consider an assembly of N interacting Bose particles in a box of volume V. The Hamiltonian for this system is written

$$H=-\sum_i(\hbar^2/2m)\nabla_i^2+\sum_{i<j}V(i,\ j) \tag{2·1}$$

where $V(i,\ j)$ is the potential of the force between i-th and j-th particles.

In accordance with the statements mentioned in § 1, we assume that the ground state wave function Ψ is a product of functions including coordinates of two particles. Since Ψ is everywhere real and positive and is symmetric for interchange of particles, Ψ will be written as

$$\Psi=\exp\{(1/2)\sum_{i<j}u(i,\ j)\}. \tag{2·2}$$

We assume that $u(i,\ j)$ is real and depends only on the distance between two particles and vanishes at large separations. The best choices for the function $u(i,\ j)$ are obtained by an optimum variation calculation with the trial function (2·2). We shall first compute the expectation value of the kinetic energy

$$\langle K.\,E.\rangle=(\hbar^2/2m)\int\sum_i\nabla_i\Psi\cdot\nabla_i\Psi\,d\tau\Big/\int\Psi\Psi\,d\tau$$

where the integral $d\tau$ is over all configurations of all particles. Substitution of eq. (2·2) in this expression gives

$$\langle K.\,E.\rangle=(\hbar^2/8m)\int\nabla_1u(1,\ 2)\cdot\nabla_1\rho_2(1,\ 2)\,dv_1dv_2, \tag{2·3a}$$

with the help of the equation

$$\nabla_1\rho_2(1,\ 2)=\rho_2(1,\ 2)\nabla_1u(1,\ 2)+\int\rho_3(1,\ 2,\ 3)\nabla_1u(1,\ 3)\,dv_3, \tag{2·4}$$

where ρ_2 and ρ_3 are, respectively, distribution functions for sets of two and three particles defined by

$$\rho_2(1,\ 2)=N(N-1)\int\Psi\Psi\,dv_3\cdots dv_N\Big/\int\Psi\Psi\,d\tau, \tag{2·5a}$$

$$\rho_3(1,\ 2,\ 3)=N(N-1)\,(N-2)\int\Psi\Psi\,dv_4\cdots dv_N\Big/\int\Psi\Psi\,d\tau. \tag{2·5b}$$

We write

$$\rho_2(1,\ 2)=\rho^2g(1,\ 2)=\rho^2g(R_{12})$$

where R_{12} is the distance between the particles 1 and 2, and $g(R)$ is the radial distribution

function which is defined as the ratio of the local density at a distance R from an arbitrary particle to the average density. If we introduce the superposition approximation in which the ρ_3 is related to ρ_2 itself in the following manner

$$\rho^3 \rho_3(1, 2, 3) = \rho_2(1, 2)\, \rho_2(2, 3)\, \rho_2(3, 1), \tag{2·6}$$

eq. (2·4) enables us to express the $\Gamma_1 u(1, 2)$ in terms of $\rho_2(1, 2)$ or $g(1, 2)$ by an iteration procedure and the substitution of the resulting equation in eq. (2·3a) leads to

$$\langle K.E. \rangle = (\hbar^2 \rho^2/8m) \int (\Gamma_1 g(1, 2))^2 / g(1, 2)\, dv_1 dv_2 + (\hbar^2 \rho^2/8m) \sum_{n=1}^{\infty} (-1)^n \rho^n$$
$$\times \int \{\Gamma_1 g(1, 2) \cdot \Gamma_1 g(1, n+2)\}\, g(n+1, n+2) \prod_{i=3}^{n+1} \{g(1, i)\, g(i-1, i)\}\, dv_1 dv_2 \cdots dv_{n+2}. \tag{2·3b}$$

On the other hand the expectation value of the potential energy is given always in terms of $g(R)$ as follows

$$\langle \sum_{i<j} V(i, j) \rangle = (\rho^2/2) \iint V(1, 2)\, g(R_{12})\, dv_1 dv_2. \tag{2·7}$$

Hence the expectation value of the Hamiltonian, $\langle H \rangle$, can be expressed as a functional of $g(R)$ and the solution of Euler's equation which makes the $\langle H \rangle$ stationary should yield an approximation to $g(R)$ consistent with the pair approximation and the superposition approximation. Hereafter, however, we shall introduce an approximation in which the higher order terms except $n=1$ are neglected in eq. (2·3b). In other words, we assume

$$\Gamma_1 u(1, 2) = \Gamma_1 g(1, 2)/g(1, 2) - \rho \int g(2, 3) \Gamma_1 g(1, 3)\, dv_3 \tag{2·8}$$

which corresponds to the first term of the power series expansion of $\Gamma_1 u(1, 2)$ in terms of the density. This simplification involves, however, some difficulties associated with the normalization of the radial distribution function which will be referred to in § 4.

Under the approximation just mentioned we have an equation for the $\langle H \rangle$ from eqs. (2·3b) and (2·7)

$$\langle H \rangle = (\pi \hbar^2 V \rho^2/2m) \int_0^{\infty} \{(dg/dR)^2/g\}\, R^2 dR + 2\pi V \rho^2 \int_0^{\infty} V(R)\, g(R)\, R^2 dR$$
$$- (\hbar^2 \rho^3/8m) \int \{\Gamma_1 g(1, 2) \cdot \Gamma_1 g(1, 3)\}\, g(2, 3)\, dv_1 dv_2 dv_3. \tag{2·9}$$

Employing the bipolar coordinates in the third term, we obtain Euler's equation for $g(R)$ which makes eq. (2·9) stationary,

$$-\frac{2\hbar^2}{m} \frac{1}{\sqrt{g}} \frac{d^2}{dR^2} (R\sqrt{g}) - 2V(R) R + \frac{3\hbar^2 \rho \pi}{2m} \int_0^{\infty} ds \int_{|R-s|}^{R+s} dt\, g'(s)\, g'(t)\, (s^2 + t^2 - R^2) = 0. \tag{2·10a}$$

The third term of this equation corresponds to the self-consistent field in the Hartree approximation and represents the effects of the presence of the particles other than two under consideration on the radial distribution for the pair. This " self-consistent " potential is transformed as follows,

$$\int_0^\infty ds \int_{|R-s|}^{R+s} dt\, g'(s)\, g'(t)\, (s^2+t^2-R^2) = \int_{-\infty}^{\infty} g'(s)\, g(R+s)\, (2s^2-R^2)\, ds .$$

Here the integration with respect to s or t is performed on the left side noting that the region of the integration and the integrand are symmetric in s and t. Further we extended the definition of $g(R)$ to the entire real axis by the requirement $g(-R)=g(R)$. Eq. (2·10a) therefore becomes

$$\frac{2\hbar^2}{m}\,\frac{1}{\sqrt{g}}\,\frac{d^2}{dR^2}\,(R\sqrt{g})-2V(R)\,R+\frac{3\hbar^2\rho\pi}{2m}\int_{-\infty}^{\infty} g'(s)\, g(R+s)\, (2s^2-R^2)\, ds=0 . \tag{2·10b}$$

This is an integro-differential equation for the radial distribution function which is to be solved under the boundary conditions that $g(R)$ approaches 1 for large R and is finite at $R=0$. The solution may be obtained by using a method of successive approximation for an appropriate potential function, for example, one of the Lennard-Jones type. Here, however, instead of solving this equation directly, we shall transform it into a more convenient form to deal with under the further approximation for a system of hard-spheres, anticipating that the attractive forces play a secondary role in determining the radial distribution.

§ 3. An integral equation for the Fourier transform for a system of hard-spheres

In a system composed of hard-spheres of diameter a, the radial distribution function has the property

$$g(R)=0, \quad |R|<a$$

and the potential of force $V(R)$ vanishes for $|R|>a$. Denoting the reduced distance R/a by x, we obtain the equation for $g(x)$

$$g(x)=0, \quad |x|<1 \tag{3·1a}$$

$$\frac{2}{\sqrt{g}}\,\frac{d^2}{dx^2}\,(x\sqrt{g})+\frac{3\rho\pi a^3}{2}\,U(x)=0, \quad |x|>1 \tag{3·1b}$$

where $U(x)$ is defined in the following manner,

$$U(x)=\int_{-\infty}^{\infty} g'(\sigma)\, g(x+\sigma)\, (2\sigma^2-x^2)\, d\sigma . \tag{3·2}$$

The form of the integrand on the right side can be simplified, if we introduce a new function $\varphi(x)$ defined by

$$g(x)=1+\varphi(x)/x. \tag{3·3}$$

The substitution of this equation in eq. (3·2) leads to

$$U(x)=2\int_{-\infty}^{\infty}\varphi'(\sigma)\varphi(x+\sigma)\,d\sigma \tag{3·4}$$

where we have used the partial integration noting that $\varphi(x)$ is an antisymmetric function of x.

It is convenient to split $\varphi(x)$ into two parts $\varphi_0(x)$ and $\varphi_1(x)$,

$$\varphi(x)=\varphi_0(x)+\varphi_1(x) \tag{3·5}$$

where

$$\left.\begin{aligned}\varphi_0(x)&=-x, \quad |x|<1\\ &=0, \quad |x|>1\end{aligned}\right\}, \qquad \varphi_1(x)=0, \quad |x|<1.$$

From eqs. (3·1a), (3·1b), (3·3) and (3·5) the equation for $\varphi_1(x)$ becomes

$$\varphi_1(x)=0, \quad |x|<1 \tag{3·6a}$$

$$(d^2/dx^2)\,\varphi_1(x)+(3\rho\pi a^3/2)\,U(x)=0, \quad |x|>1. \tag{3·6b}$$

In deriving eq. (3·6b), we have expanded $g(x)$ in the first term in eq. (3·1b) in powers of φ_1/x and retained only the lowest term, assuming that φ_1/x is small compared with 1.

The form of $U(x)$ expressed in terms of $\varphi(x)$, which is the convolution of $\varphi'(x)$ with $\varphi(x)$, suggests the use of the method of Fourier transforms. Hence we introduce $f(k)$ defined by

$$f(k)=(i/2\pi)\int_{-\infty}^{\infty}\varphi(x)\,e^{-ikx}\,dx=(1/\pi)\int_{0}^{\infty}\varphi(x)\sin kx\,dx,$$

$$\varphi(x)=-i\int_{-\infty}^{\infty}f(k)\,e^{ikx}\,dk=2\int_{0}^{\infty}f(k)\sin kx\,dk, \tag{3·7}$$

with the corresponding equations for $\varphi_0(x)$ and $\varphi_1(x)$, respectively. From eq. (3·5) it follows

$$f(k)=f_0(k)+f_1(k) \tag{3·8}$$

where

$$f_0(k)=(1/\pi)\,(k\cos k-\sin k)/k^2. \tag{3·9}$$

This corresponds to the step function if we return to the $g(x)$. Expressed in terms of $f(k)$, the form factor function becomes, with the help of eq. (1·1b),

$$S(k) = 1 + 4\pi^2 \rho a^3 f(k)/k \tag{3·10}$$

if the scale of k is measured in units of $1/a$.

Substitution of eq. (3·7) in eq. (3·4) leads to an equation for $U(x)$

$$U(x) = -4\pi i \int_{-\infty}^{\infty} k f^2(k)\, e^{ikx}\, dk. \tag{3·11}$$

Now the integro-differential equation for $\varphi_1(x)$, eq. (3·6b), together with the auxiliary condition, eq. (3·6a), is shown to be united in a single integral equation for $f(k)$:

$$-f_1(k) + 6\pi^2 \rho a^3 \frac{f^2(k)}{k} = \frac{6\pi^2 \rho a^3}{\pi} \int_{-\infty}^{\infty} \frac{f^2(k')}{k'} \frac{\sin(k-k')}{k-k'} dk'. \tag{3·12}$$

If we multiply this equation by $\exp(ikx)$ and integrate over k, with the help of the following equation,

$$\int_{-\infty}^{\infty} \frac{\cos kx \sin k}{k} dk = \pi, \quad |x| < 1$$
$$= 0, \quad |x| > 1 \tag{3·13}$$

and of eq. (3·11), the right side vanishes for $|x| > 1$ and differentiation of the left side twice with respect to x immediately leads to eq. (3·6b), while for $|x| < 1$, the second term on the left is cancelled by the right and so the first term on the left should vanish, which is nothing but eq. (3·6a). Thus in our approximations, the form factor of a system of hard-spheres is obtained as soon as the solution of the integral equation (3·12) is found. We mention some general nature of the solution of eq. (3·12) before attempting to solve this equation.

We introduce a parameter λ and a function $h(k)$, defined in the following manner

$$\lambda = 12\rho\pi a^3, \tag{3·14}$$

$$\lambda\pi \int_{-\infty}^{\infty} \frac{f^2(k')}{k'} \frac{\sin(k-k')}{k-k'} dk' = h(k). \tag{3·15}$$

Using these equations and solving the quadratic eq. (3·12) by formula, we have

$$f(k) = \frac{k - \sqrt{k^2 - 2\lambda\pi k f_0 + \lambda k h(k)}}{\lambda\pi}. \tag{3·16}$$

Of the two signs of the square root we choose the one for which $f(k)$ approaches zero as k becomes very large.

Now consider the Fourier transform of $h(k)$:

$$v(x) = (1/\pi) \int_{-\infty}^{\infty} h(k) \sin kx\, dk. \tag{3·17}$$

It is easily proved that $v(x)$ vanishes for $|x|>1$ with the help of eq. (3·13), and therefore we have inversely

$$h(k)=\int_0^1 v(t)\sin kt\,dt=(1/2)\int_{-1}^1 v(t)\sin kt\,dt, \tag{3·18}$$

noting that $v(t)$ is an antisymmetric function of t. From eqs. (3·16) and (3·18) it follows that $f(k)$ is determined by the function $r(x)$, that is, the solution of the integral equation (3·12) is characterized by an appropriate function $r(x)$ which is different from zero only on the interval $|x|<1$. The function $v(x)$ is also related to $f(k)$ directly as

$$v(x)=\lambda\pi\int_{-\infty}^{\infty}\frac{f^2(k)}{k}\sin kx\,dk \tag{3·19}$$

which is easily proved with the use of eqs. (3·13), (3·15) and (3·17).

Finally we have an equation for the form factor with the help of eqs. (3·9), (3·10), (3·16) and (3·18),

$$S(k)=1-(1/3)\{\sqrt{1-2\lambda G(k)+\lambda H(k)}\ -1\}, \tag{3·20}$$

where

$$G(k)=(k\cos k-\sin k)/k^3 \tag{3·21}$$

and

$$H(k)=\int_0^1 v(t)\,(\sin kt/k)\,dt. \tag{3·22}$$

§ 4. Linear integral equation for $v(x)$ and its approximate solution

We have shown in § 3 that the determination of the $S(k)$ is carried out by finding the function $v(x)$ which is zero outside the interval $|x|<1$. It is easier to deal with the $v(x)$ than with the integral equation (3·12) itself, since the $r(x)$ is defined only on the finite interval. Elimination of $h(k)$ and $f(k)$ from eqs. (3·16), (3·18) and (3·19) leads to an integral equation for $v(x)$ which is of non-linear type and is not easily solved. Therefore we introduce the linear approximation in which f_1^2 is neglected compared with f_0^2 and f_0f_1, assuming that the $g(x)$ does not considerably differ from the step function. Under this approximation we have

$$f^2(k)/k=\{f_0(k)h(k)-\pi f_0^2(k)\}/\{\lambda\pi^2 f_0(k)-\pi k\} \tag{4·1}$$

from eqs. (3·12) and (3·15). Substitution of this equation in eq. (3·19) leads to a linear integral equation for $v(x)$, using eqs. (3·7), (3·9) and (3·21):

$$v(x)=x+\frac{2}{\lambda}K'(x)+\int_{-1}^1 K(x-t)\,v(t)\,dt \tag{4·2}$$

where

$$K(x)=-\frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{G(k)\cos kx}{1-\lambda G(k)}\,dk. \tag{4·3}$$

Eq. (4·2) is quite similar to the integral equation derived by Kirkwood and Boggs[1] for the radial distribution function of a fluid of hard-spheres in the classical statistical mechanics. The kernel of the integral equation, $K(x)$, is identical with theirs, the only differences being the appearance of $2K'(x)/\lambda$ in the inhomogeneous part and the parity of the unknown function, i. e., symmetric in their case, while antisymmetric in our case with respect to x. Such differences, however, do not prevent us from solving the equation along the line conjectured by Kirkwood and Boggs.

The function $K(x)$ defined by eq. (4·3) is evaluated by means of the theory of residues to give for $|x|<2$

$$K(x)=\frac{3\lambda}{4(\lambda+3)}\left[\frac{15+6\lambda}{15+5\lambda}-x^2\right]-\lambda\sum_{n=1}^{\infty}\frac{(1+z_n)\cosh z_n x}{z_n{}^3F'(z_n)}e^{-z_n} \tag{4·4}$$

where

$$F(z)=1+\frac{\lambda}{z^3}(z\cosh z-\sinh z). \tag{4·5}$$

The sums in the Cauchy series of eq. (4·4) extend over all zeroes with the real part positive, of the function $F(z)$. The ordering of the zeroes is so chosen that the conjugate pairs are ordered according to increasing real parts. Introducing eq. (4·4) into eq. (4·2), we have rigorously

$$\begin{aligned}v(x)=-\frac{\lambda}{\lambda+3}x-2\sum_{n=1}^{\infty}\frac{(1+z_n)\sinh z_n x}{z_n{}^2F'(z_n)}e^{-z_n}+\frac{3\lambda x}{2(\lambda+3)}M_0\\+\sum_{n=1}^{\infty}\lambda\frac{(1+z_n)\sinh z_n x}{z_n{}^3F'(z_n)}e^{-z_n}M_n\end{aligned} \tag{4·6}$$

where

$$M_0=\int_{-1}^{1}v(t)\,t\,dt \quad\text{and}\quad M_n=\int_{-1}^{1}v(t)\sinh z_n t\,dt. \tag{4·7}$$

This is an implicit form of the solution of the integral equation (4·2), and the coefficients M_n should be determined self-consistently, solving an infinite set of simultaneous equations in which the unknowns are the M_n, $n=0, 1, 2, \cdots$, thus

$$\sum_{n=0}^{\infty}\gamma_{mn}M_n=A_m \tag{4·8}$$

where the γ's and A's are known functions of λ and z_n.

Now, if we consider the approximate kernel with higher order terms except $n=1, 2$ neglected, the sums in eq. (4·6) become finite and the set (4·8) is reduced to one of three equations in three unknowns, M_0, M_1 and M_2. We have solved this finite set of linear equations for $\lambda=10$, 15 and 20. In order to carry out the computations it is

Table I. Zeroes of $F(z)$, $z_1=\alpha_1+i\beta_1$, and $D(\lambda)$

λ	α	β	$D(\lambda)$
10	1.93	5.40	11.8
15	1.48	5.53	20.2
20	1.15	5.62	29.9

necessary to find explicitly the first zeroes of $F(z)$. The real and imaginary parts of the first pairs are shown in Table I, where we have slightly corrected the values given by Kirkwood and Boggs.

A value which is to be assigned to the single parameter λ can be obtained through the normalization condition for the radial distribution function. It has been shown that[10]

$$S(0)=1+\rho\int(g(R)-1)dv=kT\kappa\rho \qquad (4\cdot9)$$

holds where κ is the compressibility, k Boltzmann's constant and T the temperature. From this relation it follows that $S(0)$ should vanish at absolute zero and the condition $S(0)=0$ leads to

$$D(\lambda)=(2\lambda/3)+(\lambda M_0(\lambda)/2)=15 \qquad (4\cdot10)$$

with the help of eqs. (3·20), (3·21), (3·22) and (4·7). In Table I, we have tabulated the values of $D(\lambda)$. From this table we see that the value of λ which satisfies eq. (4·10) lies intermediate between 10 and 15. The determination of λ is carried out by making use of the standard interpolation formula for $M_0(\lambda)$ in eq. (4·10) and this leads to $\lambda=12.0$. This value of λ corresponds to $a=2.44$ Å, if in eq. (3·14) the molar volume of the liquid state 27.6 cm^3/mole and Avogadro's number 6.02×10^{23}/mole are introduced. Experimentally the value of a is obtained by analyzing the X-ray scattering diagrams, and by determining the distance at which the radial distribution function vanishes. According to Goldstein and Reekie[11] this procedure yields about 2.4 Å for the diameter which is very close to the theoretically calculated one.

All numerical values which are necessary to compute $v(x)$ explicitly are obtained by means of the interpolation from the values for $\lambda=10$, 15 and 20. Thus we have for $\lambda=12.0$

$$v(x)=2.19x+0.200\sinh 1.74x\cos 5.46x+0.266\cosh 1.74x\sin 5.46x. \qquad (4\cdot11)$$

The form factor computed from eqs. (3·20), (3·21), (3·22) and (4·11) with $\lambda=12.0$ and $a=2.44$ Å, is presented in Fig. 1, where for comparison the experimental data obtained at $T=2.06°K$ are shown.[11] The finite value of the experimentally observed form factor at $k=0$ is due to the finite temperature at which the experiment is performed, as is seen from eq. (4·9), and ought to approach zero as the temperature is lowered. The form of the form factor for k values larger than 0.9 or 1.0 $Å^{-1}$ does not show any easily detectable variation over a relatively wide temperature range.

We see from Fig. 1 that the curve A shows the deviations from the B towards the rightward direction, which are probably due to the neglection of the attractive forces in our theory, yet it seems that A has the characteristic form of the form factor in actual liquid helium.

In closing this section, we mention some remarks concerning the normalization condition for the radial distribution function. The $g(R)$ which is subject to eq. (2·8) does not necessarily satisfy the relation $S(0)=0$ for an arbitrary value of λ, and therefore our assumption, eq. (2·8), relating the $u(R)$ to the $g(R)$, turns out to be inconsistent with the normalization condition. The determination of λ we have performed may partly remove this inconsistency and a fairly good coincidence between the theoretical and experimental values of the diameters of the hard-spheres seems to indicate that this inconsistency does not produce a serious effect on the determination of the form factor.

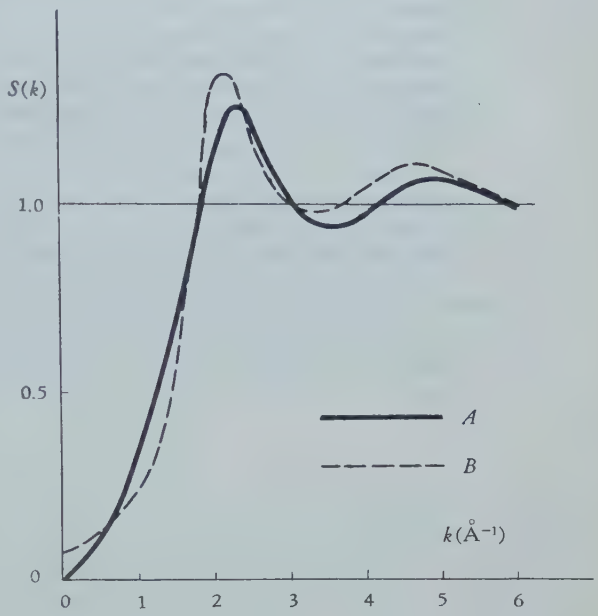

Fig. 1. Theoretical and experimental form factor. Curve A theoretical. Curve B from Goldstein and Reekie[11] at $T=2.06°K$.

§ 5. Conclusions

According to Feynman,[9] the behaviour of the $S(k)$ for small k values is related to the phonon excitations in liquid helium and is given by $S(k)\simeq\hbar k/2mc$ where c is the sound velocity. On the other hand, in our theory, the $S(k)$ is proportional to k^2 at small k, as is seen from eqs. (3·20), (3·21) and (3·22). The origin of the discrepancy lies also in our crudest approximation, (2·8), and it is necessary to employ a more refined approximation in relating the $u(R)$ to the $g(R)$. In the next paper we shall undertake such a refinement and present a detailed calculation of the form factor, including the attractive forces besides the hard core repulsion, and of the cohesive energy based on our approximations.

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On the Propagation of the Hydromagnetic Waves in Compressible Ionized Fluid

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A mathematical treatment of the wave propagation due to the coupled motion of hydrodynamic flow and magnetic field is given on the basis of the method of characteristics.

Two simplifying assumptions are introduced: first the conductivity of the medium is infinite, and second, the effect of the electric displacement current may be neglected.

Special consideration is given to the occurrence of shock waves in unidimensional motion.

§ 1. Introduction

Some aspects of the hydromagnetic behavior of compressible ionized fluids have been mathematically studied by several authors.[1),2),3),4)] The basic equations of motion, however, are non-linear and highly intractable from the view-point of obtaining analytical solutions, and the characteristic features of solutions of these equations have been little known. On the other hand, in problems involving the aerodynamic flow of compressible fluids, we have encountered many peculiar phenomena essentially connected with the non-linearity of the underlying equations, among which the occurrence of shock fronts is the most conspicuous. Hence, in the magneto-hydrodynamics of compressible fluid, we may analogously expect that there exist similar phenomena which cannot be explained by linear approximation. In fact, hydromagnetic shock waves can occur as was analyzed by de Hoffman and Teller.[1)] In their calculations, however, it was not clarified how shock discontinuities may arise in the course of propagation, and the mechanism of occurrence of shock fronts is not obvious. Discussions of this paper are mainly concerned with this problem, in which the non-linearity of the basic equations plays a decisive role.

In the following discussions, we assume that the electrical conductivity is infinite and the displacement current can be neglected. At the same time, some assumption must be made concerning the thermal conditions in the gas, and it is convenient to adopt the usual postulate of isentropic conditions in the gas. Under these assumptions, the equations of motion of an ionized fluid yield a set of simultaneous first order partial differential equations for the magnetic field strength $\boldsymbol{H}$, the velocity of fluid $\boldsymbol{v}$, and the density ρ. In section 2 below, the general propagation characters of this field are investigated. In section 3, the solutions for unidimensional flow problems are discussed on the basis of the method of characteristics, it is then described how shock waves of matter and magnetic field occur under continuous initial conditions.

§ 2. The characteristics of the basic equations

Under the assumptions made in section 1, the basic equations of motion can be given by the following set of simultaneous partial differential equations[2] for $\boldsymbol{v}$, $\boldsymbol{H}$ and ρ:

$$\operatorname{curl}[\boldsymbol{v}\boldsymbol{H}]-\partial\boldsymbol{H}/\partial t=0\,, \tag{2·1a}$$

$$\partial\boldsymbol{v}/\partial t+(\boldsymbol{v}\operatorname{grad})\boldsymbol{v}+(\mu_0/\rho)[\boldsymbol{H}\operatorname{curl}\boldsymbol{H}]=-(\operatorname{grad}p)/\rho\,, \tag{2·1b}$$

$$\partial\rho/\partial t+\operatorname{div}(\rho\boldsymbol{v})=0\,, \tag{2·1c}$$

in which μ_0 is the permeability divided by 4π.

The equation for the characteristic surface of (2·1), $\varphi(x, y, z, t)=0$, reduces to

$$(\operatorname{grad}\varphi)^{-2}\Lambda^4-\bar{c}^2\Lambda^2+(\mu_0/\rho)c_s^2(\boldsymbol{H}\operatorname{grad}\varphi)^2=0\,, \tag{2·2}$$

$$\Lambda=\pm\sqrt{\frac{\mu_0}{\rho}}\,(\boldsymbol{H}\cdot\operatorname{grad}\varphi)\,,^{*} \tag{2·2}'$$

in which c_s is the sound velocity defined by $\sqrt{p'(\rho)}$, Λ is given by

$$\Lambda=\varphi_t+\boldsymbol{v}\operatorname{grad}\varphi,$$

and $\bar{c}^2$ is the sum of c_s^2 and the square of the Alfvén velocity $c_a=\sqrt{(\mu_0/\rho)}\,|\boldsymbol{H}|$, i. e. $\bar{c}^2=c_s^2+c_a^2$. The latter equation (2·2)′ indicates that there always exists a wave whose propagation velocity relative to fluid is equal to the Alfvén velocity. The equation (2·2) can be solved with respect to φ_t, then it yields the following four equations:

$$\varphi_t+\boldsymbol{v}\operatorname{grad}\varphi \pm(1/2)\sqrt{(\operatorname{grad}\varphi)^2}\{\bar{c}^2\pm[\bar{c}^4-4(\mu_0/\rho)c_s^2(\boldsymbol{H}\operatorname{grad}\varphi)^2/(\operatorname{grad}\varphi)^2]^{1/2}\}^{1/2}=0\,. \tag{2·3}$$

The left-hand sides of (2·3) are always real, hence the original equations can represent the propagation of waves for any value of $\boldsymbol{H}$, $\boldsymbol{v}$ and ρ. As the equations (2·3) are of Hamilton-Jocobi type, the trajectories of the characteristic rays are equivalent to the paths of particles whose Hamiltonians are given by

$$\mathcal{H}=\boldsymbol{v}\cdot\boldsymbol{p}\pm(1/2)\sqrt{\boldsymbol{p}^2}\{\bar{c}^2\pm[\bar{c}^4-4(\mu_0/\rho)c_s^2(\boldsymbol{H}\cdot\boldsymbol{p})^2/\boldsymbol{p}^2]^{1/2}\}^{1/2}. \tag{2·4}$$

In the above Hamiltonians of the particles, $\boldsymbol{p}$ is the momentum and $\boldsymbol{v}$, $\boldsymbol{H}$ and ρ should be considered as the external fields acting on them. The paths of the particles or the trajectories of the characteristic rays can, of course, be given as the solutions of the canonical equations of motion:

$$dx_i/dt=\partial\mathcal{H}/\partial p_i,\quad dp_i/dt=-\partial\mathcal{H}/\partial x_i. \tag{2·5}$$

We cannot perform the integration of (2·5) unless the solutions of the original equations (2·1) are known. Even if the solutions of (2·1) could be obtained, it seems to be

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* Besides these characteristic relations (2·2) and (2·2)′, we have the relation $\varphi_t=0$.

quite difficult to obtain the analytical solutions of (2·5).

It may, however, be assumed that there are many cases in which the Alfvén velocity is exceedingly small compared with the sound velocity; then the Hamiltonians (2·4) can be simplified and reduce to

$$\mathcal{H}^{+} \cong \boldsymbol{v}\cdot\boldsymbol{p} \pm c_s\{1+(1/2)\,(c_a/c_s)^2\chi^2\}\sqrt{\boldsymbol{p}^2}, \tag{2·6}$$

$$\mathcal{H}^{-} \cong \boldsymbol{v}\cdot\boldsymbol{p} \pm \sqrt{\mu_0/\rho}\,\boldsymbol{H}\cdot\boldsymbol{p}, \tag{2·7}$$

in which only the first order terms with respect to $(c_a/c_s)^2$ are retained; χ^2 is defined by

$$\chi^2 = 1-\{(\boldsymbol{p}\cdot\boldsymbol{H})/|\boldsymbol{p}|\cdot|\boldsymbol{H}|\}^2. \tag{2·8}$$

The Hamiltonians $\mathcal{H}^{+}$ yield the canonical equations:

$$d\boldsymbol{x}/dt \cong \boldsymbol{v} \pm (c_s/\sqrt{\boldsymbol{p}^2})[\boldsymbol{p}+(c_a/c_s)^2\{(1-\chi^2/2)\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{H})\boldsymbol{H}/\boldsymbol{H}^2\}]. \tag{2·9}$$

For the magnitude of the velocity relative to $\boldsymbol{v}$, we have

$$(d\boldsymbol{x}/dt-\boldsymbol{v})^2 \cong c_s^2(1+(c_a/c_s)^2\chi^2). \tag{2·10}$$

From (2·7) we have also

$$d\boldsymbol{x}/dt \cong \boldsymbol{v} \pm \sqrt{\mu_0/\rho}\,\boldsymbol{H}, \tag{2·11}$$

or

$$(d\boldsymbol{x}/dt-\boldsymbol{v})^2 \cong c_a^2. \tag{2·12}$$

Let θ be the angle between the directions of $\boldsymbol{p}$ and $\boldsymbol{H}$, then, the definition (2·8) reduces to

$$\chi^2 = \sin^2\theta;$$

hence the relations (2·9) and (2·10) indicate that the propagation velocity $\boldsymbol{v}'$ relative to the fluid velocity $\boldsymbol{v}$ is maxmum in its magnitude equal to $\sqrt{c_s^2+c_a^2}$ if $\boldsymbol{p}$ and $\boldsymbol{H}$ are perpendicular to each other or $\boldsymbol{v}'$ is normal to $\boldsymbol{H}$, and minimum, c_s, if $\boldsymbol{p}$ and $\boldsymbol{H}$, or $\boldsymbol{v}'$ and $\boldsymbol{H}$ are parallel (or antiparallel). The relations (2·11) and (2·12) show that $\boldsymbol{v}'$ is parallel (or antiparallel) to $\boldsymbol{H}$ and its magnitude is equal to the Alfvén velocity, c_a. If the Alfvén velocity is, contrary to the above assumption, much greater than the sound velocity, we have another limitting case, in which the characteristic equations yield the following results: $|\boldsymbol{v}'|$ is minimum, equal to c_a, if $\boldsymbol{v}'$ is parallel or (antiparallel) to $\boldsymbol{H}$, and maximum, equal to $\bar{c}$ in the direction normal to $\boldsymbol{H}$; besides these propagation velocities, we have also the velocity whose magnitude is equal to the sound velocity c_s and whose direction is parallel (or antiparallel) to $\boldsymbol{H}$.

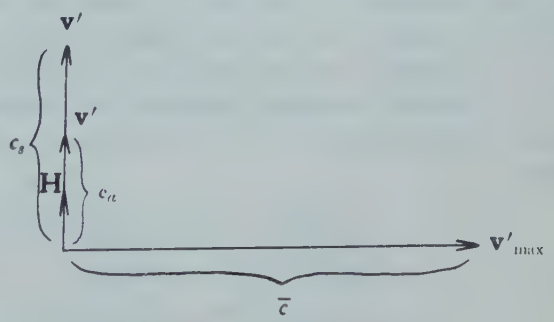

Fig. 2·1 The propagation velocity $\boldsymbol{v}'$ relative to $\boldsymbol{v}$ $(c_s \gg c_a)$.

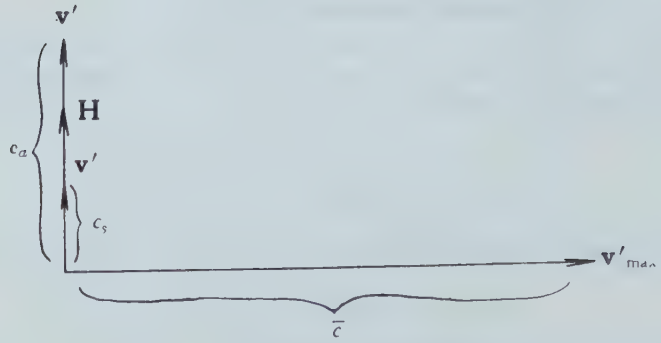

Fig. 2·2 The propagation velocity v' relative to v ($c_s \ll c_a$).

In both cases, the magnetic field acts to increase the propagation velocity in the direction normal to itself.

§ 3. One-dimensional propagation

In problems of purely one-dimensional flow in the current hydrodynamical sense, one cannot find any coupling between electromagnetic field and hydrodynamical flow ; therefore, it seems to be rather trivial to discuss those problems on the magneto-hydrodynamical view-point. The simplest, but nontrivial case to be solved may be the flow problems in which all quantities depend only on z and t and there does not exist any degree of freedom of motion in x-direction. (In y direction, therefore, field and flow are constant.) Under these restrictions, it can easily be seen that the z component of magnetic field, H_z, does not change in the course of time ; hence, as for the magnetic field in the z-direction it is sufficient to take account of only the external one.

In the following discussions, we assume, for simplicity, that in z-direction there does not exist external magnetic field. We have then, instead of (2·1 a–c), the following equations :

$$\rho \cdot \partial v/\partial t + \rho v \cdot \partial v/\partial z + p' \cdot \partial \rho/\partial z + \mu_0 h \cdot \partial h/\partial z = 0 , \tag{3·1}$$

$$h \cdot \partial v/\partial z + v \cdot \partial h/\partial z + \partial h/\partial t = 0 , \tag{3·2}$$

$$\rho \cdot \partial v/\partial z + \partial \rho/\partial t + v \cdot \partial \rho/\partial z = 0 , \tag{3·3}$$

$$\partial v_y/\partial t + v \cdot \partial v_y/\partial z = 0 , \tag{3·4}$$

in which we denote the z component of the fluid velocity and the y component of the magnetic field as v and h, instead of v_z and H_y, respectively. As is obviously seen from (3·4), the y component of the velocity v_y can be determined by v ; hence we try to solve the equations (3·1), (3·2) and (3·3) first.

Multiplying (3·1), (3·2) and (3·3) by λ_1, λ_2 and λ_3 respectively and denoting the line element of characteristics as $d\sigma$, we can transform (3·1), (3·2) and (3·3) into the following characteristic form :[6]

$$\rho(z_\sigma - v t_\sigma)\lambda_1 - h t_\sigma \lambda_2 - \rho t_\sigma \lambda_3 = 0, \tag{3·7·1}$$

$$-c_s^2 t_\sigma \lambda_1 + (z_\sigma - v t_\sigma)\lambda_3 = 0, \tag{3·7·2}$$

$$-\mu_0 h t_\sigma \lambda_1 + (z_\sigma - v t_\sigma)\lambda_2 = 0, \tag{3·7·3}$$

$$\rho v_\sigma \lambda_1 + h_\sigma \lambda_2 + \rho_\sigma \lambda_3 = 0, \tag{3·7·4}$$

$$(\rho v v_\sigma + c_s^2 \rho_\sigma + \mu_0 h h_\sigma)\lambda_1 + (h v_\sigma + v h_\sigma)\lambda_2 + (\rho v_\sigma + v \rho_\sigma)\lambda_3 = 0. \tag{3·7·5}$$

From (3·7·1), (3·7·2) and (3·7·3),* we have

$$dz/dt = v \pm \bar{c}, \tag{3·8a}$$

and

$$dz/dt = v. \tag{3·8b}$$

If $z_\sigma - v t_\sigma \neq 0$, from (3·7·2), (3·7·3) and (3·7·4) we obtain

$$c_s^2 \rho_\sigma + \mu_0 h h_\sigma \pm \rho \bar{c} v_\sigma = 0. \tag{3·9}$$

If $z_\sigma - v t_\sigma = 0$, from (3·7·1), (3·7·2) and (3·7·4) we have

$$h\rho_\sigma - \rho h_\sigma = 0,$$

or

$$\rho/h = \text{const.} \tag{3·10}$$

Denoting the characteristics and their line elements given by (3·8a) and (3·8b) as $C^\pm$ and C^0 and $d\alpha$, $d\beta$ and $d\tau$ respectively, we can compendiously write the above relations as follows:

$$\Gamma^+ : c_s^2 \rho_\alpha + \mu_0 h h_\alpha + \rho \bar{c} v_\alpha = 0$$
$$\text{along } C^+ : dz/dt = v + \bar{c}, \tag{3·9a}$$

$$\Gamma^- : c_s^2 \rho_\beta + \mu_0 h h_\beta - \rho \bar{c} v_\beta = 0$$
$$\text{along } C^- : dz/dt = v - \bar{c}, \tag{3·9b}$$

$$\Gamma^0 : h\rho_\tau - \rho h_\tau = 0$$
$$\text{along } C^0 : dz/dt = v. \tag{3·10}$$

The integral (3·10) indicates that the ratio between the density ρ and the magnetic field h is constant along each particle path or the magnetic field is frozen into the material.[7]

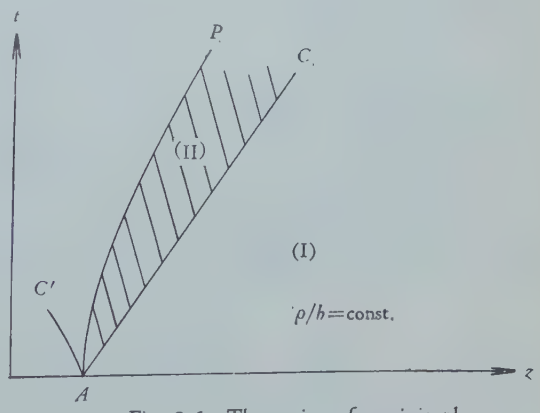

Fig. 3·1 The region of semi-simple wave.

Very often we encounter the problems involving the region through-

* These equations can, of course, be derived directly from the general expression (2·6).

out which ρ/h is constant. For instance, if a region (II) is adjacent to a region of constant ρ and h, (I), each particle path issuing out of (I) and invading (II) carries a constant value of ρ/h determined in (I). Hence, in the region (II) bounded by the C^+-characteristics AC and the particle path AP (c. f. Fig. 3·1), ρ/h is constant everywhere. Solutions or flows of this type will be called "Semi-Simple Waves". (The existence of solutions of this type can be seen directly from the equations (3·2) and (3·3)). In regions of semi-simple waves, the equations (3·9) can easily be integrated. Putting $h=\kappa\rho$ (κ is a constant) we have

$$v+l(\rho)=\gamma(\beta), \tag{3·11}$$

$$v-l(\rho)=s(\alpha), \tag{3·12}$$

in which $l(\rho)$ is defined by

$$l(\rho)=\int(\bar{c}/\rho)\,d\rho, \tag{3·13}$$

and the Alfvén velocity, c_a, reduces to $\sqrt{\mu_0\rho}\,\kappa$. Therefore, in regions of semi-simple waves, problems of the hydromagnetic flow reduce to ordinary hydrodynamic flow problems where sound velocity is replaced by $(p'+\mu_0\kappa^2\rho)^{1/2}$. Accordingly, in the former as well as in the latter, one can find regions of "simple waves" in which $r(\beta)$ or $s(\alpha)$ is constant. It should, however, be noted that in the ordinary hydrodynamics, the simple wave region, e. g. in Fig. 3·1, is to be bounded by the $C^{\pm}$ characteristics AC and AC', if the region (I) is a constant state; but in the magnetohydrodynamics the region of simple wave in Fig. 3·1 is equal to that of the semi-simple wave circumscribed by C^+ characteristics and "the particle path". In regions of simple waves we can prove that the propagation velocity dz/dt increases as the particle velocity v increases. The proof can be performed in the same way as in the hydrodynamics, provided that $p_{\rho\rho}>0$.[8] It results consequently that in the forward facing simple waves the density ρ increases as the particle velocity v increases, and in the backward facing simple waves the density ρ decreases as v increases. Hence we can conclude that dynamical behaviours of ionized fluid are the same as those of ordinary fluid whose sound velocity is given by $\bar{c}$, so far as the compression and the rarefaction in semi-simple wave regions are concerned.

This conclusion leads obviously to the occurrence of discontinuity surfaces of magnetic field strength, which are frozen in hydrodynamical shock waves.

We illustrate the above situation in the following simple example. Let us consider the following initial conditions at $t=0$: ρ and h are everywhere constant, equal to ρ_0 and h_0 respectively, the initial velocity of fluid $v_0(z)$ is given by

$$v_0(z)=\begin{cases}=0 & \text{for } z>a \quad (>0)\\ =\omega(z) & \text{for } z\leqq a,\end{cases}$$

where $\omega(z)$ is a monotonously decreasing function and non-negative for $z\leqq a$, and $v_0(z)$ is smooth as a whole. We hereafter restrict the solution within the region of positive z. The above conditions say that the fluid of a constant density ρ_0 has been at rest

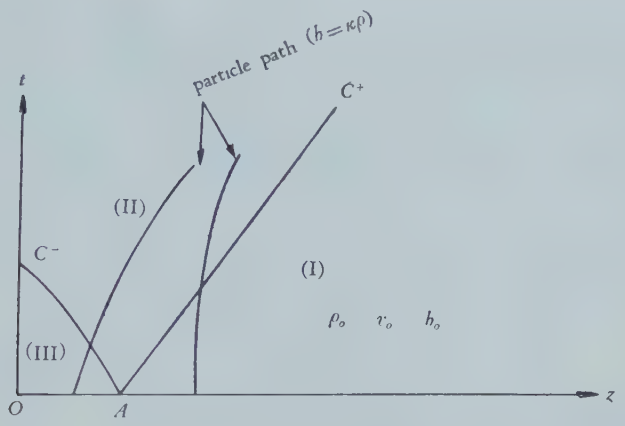

Fig. 3·2 The region of simple wave.

under the existence of a constant magnetic field h_0 in y direction, and at $t=0$ it is suddenly compressed to the positive z direction. The region D, defined by $0 \leqq z < \infty$, $0 \leqq t < t_c$ with an appropriate time t_c, is obviously of semi-simple waves, and throughout this region it holds $h/\rho = h_0/\rho_0 = \kappa$ (=constant).

By the $C^{\pm}$ characteristics AC, AC' issuing out of $A(0, a)$ the region D may be divided into three regions, (I), (II) and (III) (c. f. Fig. 3·2).

The region (I) is of constant state and there we have $v=0$, $l=l_0$. ($l(\rho_0)$ is simply denoted as l_0). The region (II) is of simple wave and throughout this region holds

$$v - l(\rho) = -l_0;$$

the C^+ characteristics become straight lines given by $z=(v+\bar{c})t$. Considering the initial conditions, from (3·9a) we finally have

$$v = (1/2)\omega(\alpha),$$
$$l(\rho) = (1/2)\omega(\alpha) + l.$$

As $\omega(z)$ is a decreasing function, the above results directly yield the occurrence of a shock wave in this region.

Fig. 3·3 shows how the waves of greater velocities overtake the waves in front propagating with smaller velocities, and lead to the occurrence of the shock front.[8]

As the magnetic field is always frozen in the material, this shock front always accompanies the discontinuity surface of the magnetic field. This also indicates that the y component of the magnetic field H_y, induced by the fluid motion in z direction, propagates in the same mode of variation as that of the density ρ.

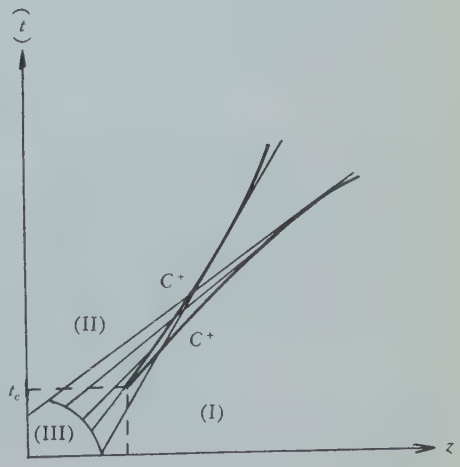

Fig. 3·3

The motion of fluid in the y-direction can easily be obtained from the equation (3·4) (It should be noted that in this case v_y is not proportional to H_y, contrary to the incompressible case). In the neighborhood near the shock front, the higher order terms neglected, such as the displacement current, the finite conductivity and the viscosity, have to be taken into account. Especially in gaseous conductors, the effect of the displacement current seems to be important. The discussion with respect to this point will be done elsewhere.

The writer would like to express his sincere thanks to Professor K. Husimi for his usefull advices and to Mr. M. Nishimoto for valuable information.

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Note on the Collision Matrix for the Nuclear Reactions, I

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The relation between the general theory of nuclear reactions due to Kapur and Peierls and that due to Wigner and Eisenbud is discussed. It is shown that the exact transformation of the R-matrix into the collision matrix yields the expression for the latter which is identical term by term with that obtained by Kapur and Peierls. The intuitive interpretation of the transformation procedure is given in terms of the "multiple reflexion interpretation".

§ 1. Introduction

Several kinds of the method of derivation have been presented of the collision matrix describing the nuclear reaction. The most typical and mathematically rigorous derivations of it have been given by Kapur and Peierls[1] and later by Wigrer and his collaborators.[2)~5)]

As is well known, Kapur and Peierls have successfully obtained the general expression for the collision matrix on the basis of the idea of the compound nucleus formation in the intermediate stage of the reaction process. The derivation due to these authors is to a large extent free from special assumptions and ambiguities in the mathematical treatment of the problem and may be regarded as providing a rigorous foundation of the earlier works[6),7),8)] on the problem of the compound nucleus reactions.

In the mentioned theory, however, the state of the compound nucleus, Ψ_n, is defined as a "decaying state" formed by the all relevant nucleons. The decaying state is nothing but a state in which the wave function satisfies the boundary condition of outgoing waves in all of the open channels. The state, therefore, cannot be a real state and, then, the corresponding eigenvalue of energy becomes a complex quantity, the imaginary part of which being connected with the life of decay of the state. The energy-eigenvalue is also dependent on the energy of the total system, since the boundary condition imposed on the wave function is energy-dependent.

The situation mentioned above gives rise to the fact that the quantities characterizing the compound state, *i. e.* not only the energy-eigenvalue, but also the normalization constant

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of the wave function and the reduced widths for the channels, are all complex and energy-dependent.

In view of the above situations, Wigner and his collaborators formulated their theory based on the definition of the compound state, X_λ, that satisfies the real and energy-independent boundary condition in the channels, so that the parameters characterizing the compound states are all real and independent of the energy of the total system, in contradistinction to those for the decaying states. However, the boundary condition mentioned above does not fit to the scattering problem, since the wave function which satisfies this boundary condition becomes the standing wave, instead of the progressive wave. These channel functions of standing wave are connected with each other in terms of the R-matrix. In order to obtain the cross sections for the reactions, it is necessary to carry out a transformation of the R-matrix into the collision matrix, the latter being the matrix which connects the channel wave functions satisfying boundary condition appropriate for the scattering problem. The transformation mentioned above is difficult to be actually carried through, since it amounts to making an inversion of a matrix with infinite number of rows and columns.

In fact, no exact calculation of the collision matrix from the R-matrix has been carried out so far, except for very simple special cases, and it has been usual to make some simplifying assumptions to facilitate the calculation. Several useful approximations have been obtained in this way.[3),5),9),11)] The properties of the collision matrix derived from the R-matrix formalism along these lines have been extensively studied by Wigner and his collaborators[3),5),11),12)] and recently by Thomas.[10)]

The two formalisms of the general theory of the nuclear reactions described above are, of course, expected to be equivalent to each other. However, there has been no detailed investigations on the explicit relation between them in general case.

In this paper we attempt to work an exact calculation of the transformation of the R-matrix into collision matrix, by means of a technique given by Wigner[3),4)] and reformulated recently by Thomas.[10)] Eventually, it reduces to an eigenvalue problem. The latter is shown to be equivalent to the one which appears in the transformation of the complete set of K-P wave functions of the compound states, $\{\Psi_n\}$, into the W-E one, $\{X_\lambda\}$. The solution to the eigenvalue problem gives the energy levels of the decaying states, Ψ_n, of the K-P formalism in terms of the characteristic parameters of the R-matrix formalism and the energy of the total system. Furthermore, the explicit calculation shows that the resultant expression, obtained by the above mentioned procedure, of the collision matrix is identical, term by term, with that given by K-P. It is, of course, expected that the collision matrices obtained by these two formalisms are the same as a whole because of the equivalence of the both formalisms. It seems to the authors, however, that the term-by-term identity of the collision matrices mentioned above is not *a priori* evident.

We also investigate the implications of the above stated transformation. It is found that the scattered wave obtained by K-P can be interpreted as the superposition of scattered waves each of which undergoes, before it leaves the internal region, the reflexions certain number of times at the boundary between the internal region and the channels,

and excites a compound states, X_λ, in between each of the successive reflexions. This interpretation might be called "multiple reflexion interpretation".

In § 2, for the sake of convenience for later discussions, the theory of *K-P* and that of *W-E* are briefly described and the explanation of the notations is given.

In § 3, the proof of identity is given of the collision matrix derived from the *R*-matrix formalism with that from *K-P* formalism in three steps, § 3*A*, § 3*B* and § 3*C*.

In § 4, the implications of the transformation described in § 3 is investigated and the multiple reflexion interpretation is introduced.

In § 5, the concluding remarks are given.

§ 2. The formalisms of Kapur-Peierls and Wigner-Eisenbud

We consider the nuclear reactions of the type $a+X\rightarrow b+Y$ in which particle a collides with nucleus X, and then particle b and nucleus Y emerge after the collision. A channel* of the reaction is specified by the type and the states of the colliding or emerging two particles together with the channel spin j_s which we denote by the suffix s, and the relative orbital angular momentum l_s.

The wave function of the total system $\Psi_{sl_s\nu_s m_s}$, in which the incoming wave $I_{sl_s\nu_s m_s}$, with the z-components ν_s and m_s of the angular momenta j_s and l_s, respectively, is present only in the channel s, is defined to have the asymptotic form in the external region,

$$\Psi_{sl_s\nu_s m_s}=I_{sl_s\nu_s m_s}-\sum_{tl_t\nu_t m_t} U_{sl_s\nu_s m_s;\,tl_t\nu_t m_t}E_{tl_t\nu_t m_t}, \tag{1}$$

where $E_{tl_t\nu_t m_t}$ is the outgoing wave and $U_{sl_s\nu_s m_s;\,tl_t\nu_t m_t}$ is the element of the collision matrix. $I_{tl_t\nu_t m_t}$ and $E_{tl_t\nu_t m_t}$ in (1) are defined by

$$I_{tl_t\nu_t m_t}=(\hbar k_t/M_t)^{-1/2}r_t^{-1}u_{l_t}^{(-)}(k_t r_t)\,\psi_{t\nu_t}(i_t)\,Y_{l_t m_t}(\Omega_t), \tag{2a}$$

and

$$E_{tl_t\nu_t m_t}=(\hbar k_t/M_t)^{-1/2}r_t^{-1}u_{l_t}^{(+)}(k_t r_t)\,\psi_{t\nu_t}(i_t)\,Y_{l_t m_t}(\Omega_t), \tag{2b}$$

respectively, where k_t and M_t are the wave number and the reduced mass of the relative motion, and r_t and Ω_t are the length and the direction of the vector connecting the centers of mass, of the colliding or emerging pair t, respectively. $u_{l_t}^{(\pm)}(k_t r_t)$ are the radial outgoing and incoming wave functions with the asymptotic form $\exp[\pm i(k_t r_t-l_t\pi/2)]$ as $r_t\rightarrow\infty$.[13] $\psi_{t\nu_t}(i_t)$ is the internal wave function, i_t being the internal coordinates of the particles of the pair t, and $Y_{l_t m_t}$ is the normalized spherical harmonics.

According to *K-P*, the collision matrix is given by the dispersion formula,

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* The definitions of the terms, the internal and external regions and the channels of the configuration space, the channel spins and channel radii are omitted here, since they are of common use and presumably well known.

$$U_{sl_s\nu_s m_s;\, tl_t\nu_t m_t} = \{u_{l_s}^{(-)}(k_s a_s)/u_{l_s}^{(+)}(k_s a_s)\}\,\delta_{st}\delta_{l_s l_t}\delta_{\nu_s\nu_t}\delta_{m_s m_t}$$
$$+ i\sum_{n,J}\frac{\tilde{u}_{sl_s\nu_s m_s}^{n,JM*}\, u_{tl_t\nu_t m_t}^{n,JM}}{N_n^J(E_n^J - E - i\Gamma_n^J/2)}\,\delta_{\nu_s+m_s,\,M}\,\delta_{M,\,\nu_t+m_t}, \tag{3}$$

in terms of the compound nuclear state Φ_n^{JM} and its associated state $\tilde{\Phi}_n^{JM}$. Φ_n^{JM} satisfies the Schrödinger equation with the total Hamiltonian H,

$$(H - W_n^J)\,\Phi_n^{JM} = 0, \tag{4}$$

in the internal region and the boundary condition of the decaying state,

$$\int \psi_{s\nu_s}^* Y_{l_s m_s}^* \,\mathrm{grad}_n\, \Phi_n^{JM}\, dS = (f_{sl_s} - a_s^{-1}) \int \psi_{s\nu_s}^* Y_{l_s m_s}^*\, \Phi_n^{JM}\, dS, \tag{5}$$

on the surface of the internal region, where f_{sl_s} is defined by

$$f_{sl_s} = (du_{l_s}^{(+)}(k_s a_s)/dr_s)/u_{l_s}^{(+)}(k_s a_s), \tag{6}$$

a_s being the channel radius. It is also the eigenfunction of the total angular momentum $\boldsymbol{J}^2$ and its z-component J_z with the eigenvalues $J(J+1)$ and M, respectively. $\tilde{\Phi}_n^{JM}$ is also the eigenfunction of H,

$$(H - \tilde{W}_n^J)\,\tilde{\Phi}_n^{JM} = 0 \tag{7}$$

but it satisfies the boundary condition complex conjugate to (5),

$$\int \psi_{s\nu_s}^* Y_{l_s m_s}^* \,\mathrm{grad}_n\, \tilde{\Phi}_n^{JM}\, dS = (f_{sl_s}^* - a_s^{-1}) \int \psi_{s\nu_s}^* Y_{l_s m_s}^*\, \tilde{\Phi}_n^{JM}\, dS, \tag{8}$$

so that the eigenvalue $\tilde{W}_n^J$ is complex conjugate to W_n^J

$$\tilde{W}_n^J = W_n^{J*} \tag{9}$$

Φ_n^{JM} and $\tilde{\Phi}_n^{JM}$ form a biorthogonal complete set of functions in the internal region,

$$\int_I \tilde{\Phi}_n^{JM*}\, \Phi_{n'}^{J'M'}\, d\tau = N_n^J \delta_{nn'}\,\delta_{JJ'}\,\delta_{MM'}, \tag{10}$$

where N_n^J is a certain constant which is fixed by the normalization of the wave functions,

$$\int_I |\Phi_n^{JM}|^2\, d\tau = \int_I |\tilde{\Phi}_n^{JM}|^2\, d\tau = 1, \tag{11}$$

where the suffix I attached to the integration signs signifies the integration over the internal region.

The energy E_n^J and the width Γ_n^J of the n-th compound state are given by

$$W_n^J = E_n^J - i\Gamma_n^J/2, \tag{12}$$

and the width $u_{sl_s\nu_s m_s}^{n,JM}$ and $\tilde{u}_{sl_s\nu_s m_s}^{n,JM}$ in (3) are given by

$$u_{sl_s\nu_s m_s}^{n,JM} = (\hbar^2/M_s)^{1/2} k_s^{1/2} [u_{l_s}^{(+)}(k_s a_s)]^{-1}\, \phi_{sl_s\nu_s m_s}^{n,JM} \tag{13a}$$

and

$$\tilde{u}^{nJM}_{sl_s\nu_s m_s} = (\hbar^2/M_s)^{1/2} k_s^{1/2} [u^{(-)}_{l_s}(k_s a_s)]^{-1} \tilde{\phi}^{nJM}_{sl_s\nu_s m_s}, \tag{13b}$$

respectively, where

$$\phi^{nJM}_{sl_s\nu_s m_s} = a_s^{-1} \int \psi^*_{s\nu_s} Y^*_{l_s m_s} \Phi^{JM}_n dS \tag{14a}$$

and

$$\tilde{\phi}^{nJM}_{sl_s\nu_s m_s} = a_s^{-1} \int \psi^*_{s\nu_s} Y^*_{l_s m_s} \tilde{\Phi}^{JM}_n dS. \tag{14b}$$

Γ^J_n can also be expressed in terms of u through the equation,

$$\Gamma^J_n = \sum_{sl_s\nu_s m_s} |u^{nJM}_{sl_s\nu_s m_s}|^2 = \sum_{sl_s\nu_s m_s} |\tilde{u}^{nJM}_{sl_s\nu_s w_s}|^2. \tag{15}$$

On the other hand, in the R-matrix formalism of W-E, the compound state X^{JM}_λ, which is the solution of the Schrödinger equation in the internal region,

$$(H - E^J_\lambda) X^{JM}_\lambda = 0, \tag{16}$$

is defined to satisfy the real and energy independent boundary condition,

$$\int \psi^*_{s\nu_s} Y^*_{l_s m_s} \operatorname{grad}_n X^{JM}_\lambda dS = (-l_s/a_s - a_s^{-1}) \int \psi^*_{s\nu_s} Y^*_{l_s m_s} X^{JM}_\lambda dS, \tag{17}$$

on the surface of the internal region, instead of (5) for the decaying state. X^{JM}_λ is, again, taken to be the eigenfunction $\boldsymbol{J}^2$ and J_z with the eigenvalue $J(J+1)$ and M, respectively. X^{JM}_λ's form an orthonormal complete set of functions in the internal region,

$$\int_I X^{JM*}_\lambda X^{J'M'}_{\lambda'} d\tau = \delta_{\lambda\lambda'} \delta_{JJ'} \delta_{MM'}. \tag{18}$$

In accordance with the boundary condition (17), we define a set of standing wave solutions of the Schrödinger equation in each of the channels by the equations,

$$D^{JM}_{sl_s} = \sum_{\nu_s m_s} (j_s l_s \nu_s m_s | j_s l_s JM) \psi_{s\nu_s}(i_s) r_s^{-1} \mathcal{S}_{sl_s}(r_s) Y_{l_s m_s}(\Omega_s) \tag{19a}$$

and

$$V^{JM}_{sl_s} = \sum_{\nu_s m_s} (j_s l_s \nu_s m_s | j_s l_s JM) \psi_{s\nu_s}(i_s) r_s^{-1} \mathcal{C}_{sl_s}(r_s) Y_{l_s m_s}(\Omega_s), \tag{19b}$$

which correspond to (2) of the K-P theory. Here, $\mathcal{S}_{sl_s}$ and $\mathcal{C}_{sl_s}$ are the radial wave functions for the relative motion of the pair s which satisfy the following boundary conditions at the channel radius;

$$\begin{aligned} &\mathcal{S}_{sl_s}(a_s) = 0, && \mathcal{C}_{sl_s}(a_s) = (M_s/\hbar)^{1/2} \\ &d\mathcal{S}_{sl_s}(a_s)/dr_s = (M_s/\hbar)^{1/2}, && d\mathcal{C}_{sl_s}(a_s)/dr_s = (-l_s/a_s)(M_s/\hbar)^{1/2} \end{aligned} \tag{19c}$$

It should be noted, however, that $D^{JM}_{sl_s}$ and $V^{JM}_{sl_s}$ are eigenfunctions of $\boldsymbol{J}^2$ and J_z, while $I_{sl_s\nu_s m_s}$ and $E_{sl_s\nu_s m_s}$ are not.

Corresnonding to $\Psi_{sl_s\nu_s m_s}$, we define the wave function of the total system $\varphi^{JM}_{sl_s}$ to have the asymptotic form in the external region,

$$\varphi^{JM}_{sl_s} = D^{JM}_{sl_s} + \sum_{tl_t} R^{J}_{sl_s;\,tl_t} V^{JM}_{tl_t}, \tag{20}$$

where $R^{J}_{sl_s;\,tl_t}$ is the element of the R-matrix for a definite J (which is independent of M). According to W-E, $R^{J}_{sl_s;\,tl_t}$ is given by equation,

$$R^{J}_{sl_s;\,tl_s} = \sum_{\lambda} \frac{\gamma^{J}_{\lambda sl_s}\,\gamma^{J}_{\lambda tl_t}}{E_\lambda - E}, \tag{21}$$

where the reduced width $\gamma^{J}_{\lambda sl_s}$ is defined by

$$\gamma^{J}_{\lambda sl_s} = (\hbar^2/2M_s)^{1/2}(M_s/\hbar)^{-1/2}\int V^{JM*}_{sl_s} X^{JM}_{\lambda}\, dS, \tag{22}$$

corresponding to (13). The collision matrix u^J for a definite J is given by

$$u^{J}_{sl_s;\,tl_t} = \left[\omega\frac{1+iBR^JB+iC}{1-iBR^JB-iC}\,\omega\right]_{sl_s;\,tl_t}, \tag{23}$$

and the ordinary collision matrix U can be obtained from u^J through the necessary decomposition of relevant angular momenta,

$$U_{sl_s\nu_s m_s;\,tl_t\nu_t m_t} = \sum_{J} (j_s l_s \nu_s m_s | j_s l_s JM)\, u^{J}_{sl_s;\,tl_t} (j_t l_t JM | j_t l_t \nu_t m_t). \tag{24}$$

In (23), B, C and ω are diagonal matrices associated with the amplitude and phase of the asymptotic form of the radial wave functions $\mathcal{S}_{sl_s}$ and $\mathcal{C}_{sl_s}$, and are defined through the following equations:

$$\mathcal{S}_{sl_s}(r_s) = (i/2)\,(\hbar k_s/M_s)^{-1/2}[A_{sl_s} u^{(-)}_{l_s}(k_s r_s) - A^{*}_{sl_s} u^{(+)}_{l_s}(k_s r_s)], \tag{25a}$$

$$\mathcal{C}_{sl_s}(r_s) = (1/2)\,(\hbar k_s/M_s)^{-1/2}[B_{sl_s}\omega^{*}_{sl_s} u^{(-)}_{l_s}(k_s r_s) + B_{sl_s}\omega_{sl_s} u^{(+)}_{l_s}(k_s r_s)], \tag{25b}$$

$$B_{sl_s}\omega_{sl_s}A_{sl_s} = 1 - iC_{sl_s}, \tag{25c}$$

$$B_{sl_s},\ C_{sl_s} = \text{real, and } |\omega_{sl_s}|^2 = 1. \tag{25d}$$

§ 3. The proof of identity of the collision matrices*

A. The relation between the compound states Ψ_n and X_λ

As has been described in the preceding section, the compound state Ψ_n of K-P satisfies different boundary conditions (5) from that of X_λ of W-E, (17). Since they are both eigenfunctions of the same Hamiltonian, the relation between them can be obtained by solving an eigenvalue problem as will be shown in the following.

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* In this section, we shall often drop the superscripts J and M, and abbreviate the set of subscripts sl_s simply as s, so that X^{JM}_{λ}, $V^{JM}_{sl_s}$, $\gamma^{J}_{\lambda sl_s}$, *etc.*, will often be written as X_λ, V_s, $\gamma_{\lambda s}$ *etc.*.

Let us expand Φ_n and $\widetilde{\Phi}_n$ in the series of the complete set, $\{X_\lambda\}$, in the internal region,*

$$\Phi_n = \sum_\lambda c_\lambda^{\,n} X_\lambda , \tag{26a}$$

$$\widetilde{\Phi}_n = \sum_\lambda \tilde{c}_\lambda^{\,n} X_\lambda . \tag{26b}$$

The expansion coefficient, $c_\lambda^{\,n}$, can be obtained from the equation,

$$(W_n - E_\lambda) c_\lambda^{\,n} = \int_I \{X_\lambda^* H\Phi_n - (HX_\lambda^*)\Phi_n\} d\tau , \tag{27}$$

which follows from (4) and (16). Through the use of the Green's theorem, the right-hand side of (27) is reduced to,

$$\sum_s (-\hbar^2/2M_s) \int_s \{X_\lambda^* \operatorname{grad}_n \Phi_n - (\operatorname{grad}_n X_\lambda^*)\Phi_n\} dS , \tag{28}$$

where the subscript s attached to the integration sign is meant to denote the integration over the part of the surface of the internal region which corresponds to the channel s. Taking into account the boundary conditions, (5) and (17), for Φ_n and X_λ, respectively, (28) can be rewritten as

$$\sum_s (-\hbar^2/2M_s) \Delta_s \int_s X_\lambda^* \Phi_n dS , \tag{29}$$

where Δ_s stems from the difference between the boundary condition (5) and (17) and is defined by

$$\Delta_s (\equiv \Delta_{sl_s}) = f_{sl_s} - (-l_s/a_s) . \tag{30}$$

Now, on the surface S the functions V_s form a complete orthogonal set,

$$\int V_s^* V_t dS = \delta_{st}\delta_{l_s l_t} (M_s/\hbar) , \tag{31}$$

so that the function X_λ can be expressed in terms of V_s on S, as

$$X_\lambda (\text{on } S) = \sum_S (\hbar^2/2M_s)^{-1/2} (M_s/\hbar)^{-1/2} \gamma_{\lambda s} V_s . \tag{32}$$

Equation (32) follows immediately from the definition of $\gamma_{\lambda s}$, *i. e.*, (22). Inserting (32) for X_λ in (29), we have

$$-\sum_S (\hbar^2/2M_s)^{1/2} (M_s/\hbar)^{-1/2} \gamma_{\lambda s} \Delta_s \int V_s^* \Phi_n dS . \tag{33}$$

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* It is assumed that the expansion (26) is actually possible in the region including the *surface* of the internal region. On account of the difference of the boundary conditions, mentioned in the text, the convergence of the series, (26), is not uniform in the mentioned region. Thus, for example, the term-by-term differentiation of the series is not generally permissible. The procedure similar to that adopted by *K-P* concerning the expansion of the wave function in terms of Φ_n yields the same result as that obtained in this paper.

Defining the amplitude g_{ns} of Φ_n at the entrance of the channel s by the equation,

$$g_{ns}(\equiv g^J_{nsl_s}) = (\hbar^2/2M_s)^{1/2}(M_s/\hbar)^{-1/2}\int V^{JM*}_{sl_s}\Psi^{JM}_n dS, \tag{34}$$

(33) reduces to

$$-\sum_s \gamma_{\lambda s}\Delta_s g_{ns}, \tag{35}$$

and we have from (27),

$$(W_n - E_\lambda)c_\lambda^n = -\sum_s \gamma_{\lambda s}\Delta_s g_{ns}. \tag{36}$$

But in the integrand on the right-hand side of (34), we can express Φ_n as the superposition of X_λ by (26a), and we have

$$g_{ns} = \sum_\mu c_\mu^n \gamma_{\mu s} \tag{37}$$

where use has been made of (22).

Thus, from (36) and (37), we finally obtain the equation for c_λ^n,

$$(E_\lambda - W_n)c_\lambda^n - \sum_\mu h_{\lambda\mu}c_\mu^n = 0 \tag{38a}$$

where

$$h_{\lambda\mu}(\equiv h^J_{\lambda\mu}) = \sum_{sl_s}\gamma^J_{\lambda sl_s}\Delta_{sl_s}\gamma^J_{\mu sl_s}. \tag{39}$$

(38a) is an eigenvalue equation for W_n and c_λ^n. The matrix of coefficients $((h_{\lambda\mu}))$ is symmetric,

$$h_{\lambda\mu} = h_{\mu\lambda} \tag{40}$$

but not hermitic, as can be seen from (39), since Δ_s is a complex quantity. Similarly, it can be shown that $\tilde{c}_\lambda^n$'s satisfy the eigenvalue equation,

$$(E_\lambda - \widetilde{W}_n)\tilde{c}_\lambda^n - \sum_\mu h^*_{\mu\lambda}\tilde{c}_\mu^n = 0 \tag{38b}$$

which is the hermitian conjugate equation of (38a).

The eigenvalues, W_n and $\widetilde{W}_n$, are complex and dependent on energy, E, just as $h_{\lambda\mu}$. Moreover, it is seen from (38a), (38b) and (40) that

$$\widetilde{W}_n = W_n^*. \tag{41}$$

The expansion coefficients, c_λ^n's and $\tilde{c}_\lambda^n$'s satisfy the biorthogonal relation,

$$\sum_\lambda \tilde{c}_\lambda^{n*} c_\lambda^{n'} = N_n\delta_{nn'} \tag{42}$$

and their normalizations are fixed by,

$$\sum_\lambda |c_\lambda^n|^2 = \sum_\lambda |\tilde{c}_\lambda^n|^2 = 1, \tag{43}$$

which are the direct consequences of (10), (11), (18) and (26).

B. *The transformation of the R-matrix into the collision matrix*

The transformation, (23), of the R-matrix into the collision matrix, u, implies an inversion of the matrix, $1-iC-iBRB$, which has infinite number of matrix elements. This situation gives rise to a difficulty in explicitly carrying out the transformation.

Following Wigner[3] and Thomas,[10] we first separate the potential scattering term from u;

$$u=\omega\frac{1+iC}{1-iC}\omega+2i\frac{\omega}{1-iC}BRB\frac{1}{1-i(1-iC)^{-1}BRB}\frac{\omega}{1-iC}, \tag{44}$$

and, then, the matrix inversion of the resonance term, i. e., the second term on the right-hand side of the above equation, can be carried out through the introduction of a set of infinite number of parameters, $\eta_{\lambda\mu}(\equiv\eta_{\lambda\mu}^{J})$, each of which is scalar with respect to the channel index, as

$$BRB\{1-i(1-iC)^{-1}BRB\}^{-1}=\sum_{\lambda\mu}\eta_{\lambda\mu}\beta_{\lambda}{}^{J}\times\beta_{\mu}{}^{J}, \tag{45}$$

where β_{λ} is a vector with respect to channel index, s-component of which is $B_s\gamma_{\lambda s}$ and $\beta_{\lambda}\times\beta_{\mu}$ is a matrix with the s, t-element $\beta_{\lambda s}\beta_{\mu t}$. From (45) it follows that $\eta_{\lambda\mu}$'s satisfy a set of linear inhomogeneous simultaneous equations

$$(E_{\lambda}-E)\eta_{\lambda\mu}-\sum_{\nu}h'_{\lambda\nu}\eta_{\nu\mu}=\delta_{\lambda\mu}, \tag{46}$$

where

$$h'_{\mu\nu}(\equiv h'^{J}_{\lambda\nu})=\sum_{s}\frac{iB_s^{2}}{1-iC}\gamma_{\lambda s}\gamma_{\nu s}. \tag{47}$$

The solution to the linear inhomogeneous simultaneous equations, (46), can be achieved by first solving the set of linear homogeneous simultaneous equations,

$$(E_{\lambda}-E)\eta_{\lambda}-\sum_{\nu}h'_{\lambda\nu}\eta_{\nu}=0 \tag{48a}$$

and its hermitian conjugate equations,

$$(E_{\lambda}-\widetilde{E})\tilde{\eta}_{\lambda}-\sum_{\nu}h'^{*}_{\nu\lambda}\tilde{\eta}_{\nu}=0. \tag{48b}$$

(48a) and (48b) are actually eigenvalue equations for E and η_{λ}, and $\widetilde{E}$ and $\tilde{\eta}_{\lambda}$, respectively. Denoting the eigenvalues and the associated eigenvectors for (48a) and (48b) by $W'_{n}(\equiv W'^{J}_{n})$, $\eta_{\lambda}{}^{n}(\equiv\eta_{\lambda}^{nJ})$, and $\widetilde{W}'_{n}(\equiv\widetilde{W}'^{J}_{n})$, $\tilde{\eta}_{\lambda}{}^{n}(\equiv\tilde{\eta}_{\lambda}^{nJ})$, respectively, we have,

$$(E_{\lambda}-W'_{n})\eta_{\lambda}{}^{n}-\sum_{\nu}h'_{\lambda\nu}\eta_{\nu}{}^{n}=0 \tag{49a}$$

and

$$(E_{\lambda}-\widetilde{W}'^{n}_{\lambda})\tilde{\eta}_{\lambda}{}^{n}-\sum_{\nu}h'^{*}_{\nu\lambda}\tilde{\eta}_{\lambda}{}^{n}=0. \tag{49b}$$

On account of the symmetry of the matrix $h'_{\lambda\nu}$,

$$h'_{\lambda\nu}=h'_{\nu\lambda}, \tag{50}$$

which directly follows from the definition of $b'_{\lambda\nu}$, it follows from (49a) and (49b) that

$$W'_n = \widetilde{W}'_n{}^* \tag{51}$$

and

$$\eta_\lambda{}^n = \tilde{\eta}_\lambda{}^{n*}. \tag{52}$$

The eigenvectors $\eta_\lambda{}^n$ and $\tilde{\eta}_\lambda{}^n$ are fixed by the normalization,

$$\sum_\lambda |\eta_\lambda{}^n|^2 = 1 \tag{53a}$$

and

$$\sum_\lambda |\tilde{\eta}_\lambda{}^n|^2 = 1 . \tag{53b}$$

Further, we define constants N'_n's through the equation,

$$\sum_\lambda \tilde{\eta}_\lambda{}^{n*} \eta_\lambda{}^{n'} = N'_n \delta_{nn'} \; ; \; (N'_n \equiv N'^J_n) . \tag{54}$$

Now the solution of the original equation, (46), is expressed by the solution of (49) as

$$\eta_{\lambda\mu} = \sum_n \frac{\tilde{\eta}_\lambda{}^{n*} \eta_\mu{}^n}{N'_n (W'_n - E)} . \tag{55}$$

Inserting (55) into (45) and the latter into (44), we finally obtain the expression for u

$$u = \omega \frac{1+iC}{1-iC} \omega + 2i \frac{B\omega}{1-iC} \sum_n \frac{(\sum_\lambda \tilde{\eta}_\lambda{}^{n*} \gamma_\lambda) \times (\sum_\mu \eta_\mu{}^n \gamma_\mu)}{N'_n (W'_n - E)} \frac{B\omega}{1-iC} . \tag{56}$$

The s, t-element of the ordinary collision matrix, U, is therefore given by (with the full description of channel indices),

$$U_{sl_s \nu_s m_s ; tl_t \nu_t m_t} = \omega^2_{sl_s} \frac{1+iC_{sl_s}}{1-iC_{sl_s}} \delta_{st} \delta_{l_s l_t} \delta_{\nu_s \nu_t} \delta_{m_s m_t}$$

$$+ i \sum_{nJ} \frac{\tilde{q}^{nJM}_{sl_s \nu_s m_s} \; q^{nJM}_{tl_t \nu_t m_t}}{N'^J_n (W'^J_n - E)} \delta_{\nu_s + m_s, \, M} \delta_{M, \, \nu_t + m_t} , \tag{57}$$

where

$$q^{nJM}_{sl_s \nu_s m_s} = \sqrt{2} \left(\frac{B\omega}{1-iC} \right)_{sl_s} (j_s l_s \nu_s m_s | j_s l_s JM) \sum_\lambda \eta_\lambda^{nJ} \gamma^\lambda_{sl_s} \tag{58a}$$

and

$$\tilde{q}^{nJM}_{sl_s \nu_s m_s} = \sqrt{2} \left(\frac{B\omega}{1-iC} \right)^*_{sl_s} (j_s l_s \nu_s m_s | j_s l_s JM) \sum_\lambda \tilde{\eta}_\lambda^{nJ} \gamma'^J_{\lambda sl_s} \tag{58b}$$

where use has been made of (24). The equation (57) is of the form of a pole expansion of U in terms of energy, i. e., of a dispersion formula.

From the procedure described above, it is clear that the transformation of R into u reduces to the solution of an eigenvalue equation, (49). The eigenvalues, W'_n and $\widetilde{W}'_n$,

and the eigenvectors, $\eta_\lambda{}^n$ and $\tilde{\eta}_\lambda{}^n$, are complex quantities and are dependent on the energy of the total system, E, because of the energy dependence and the non-hermitic property of the matrix $((h'_{\lambda\mu}))$.

C. *The proof of identity of the collision matrices*

We see that the expression, (57), for the collision matrix derived from the *R*-matrix formalism is of quite a similar form to that, (3), given by *K-P.* In fact we shall show in the following that they are actually identical term by term with each other.

We first observe that the singularities W_n and W'_n, of the energy expansion of the collision matrices (3) and (57), respectively, are the eigenvalues of the equations (38a) and (49a), respectively. But, we can show that

$$h_{\lambda\mu}=h'_{\lambda\mu}. \tag{59}$$

In fact, from (25c) it follows that

$$B_s\omega_s/1-iC_s=A_s^{-1}, \tag{60}$$

and, from (25a), its differentiated equation with respect to r_s and the Wronskian relation for $u_s^{(+)}$ and $u_s^{(-)}$, together with (19c), it is easy to show that

$$A_s^{-1}=k_s^{1/2}[u_{l_s}^{(+)}(k_sa_s)]^{-1}.$$

From the above equation and (60), we have

$$B_s\omega_s/1-iC_s=k_s^{1/2}[u_{l_s}^{(+)}(k_sa_s)]^{-1}. \tag{61}$$

In a similar way as above, it follows immediately from (25b) and (19c),

$$B_s\omega_s{}^*=-ik_s^{-1/2}u_{l_s}^{(+)}(k_sa_s)\,\Delta_s. \tag{62}$$

Hence, from (61) and (62), we have

$$iB_s{}^2/1-iC_s=iB_s\omega_s{}^*(B_s\omega_s/1-iC_s)=\Delta_s. \tag{63}$$

Comparison of (39) and (47), i. e., the definitions of $h_{\lambda\mu}$ and $h'_{\lambda\mu}$, respectively, with the aid of (63), establishes the desired equation, (59).

Therefore, the eigenvalue equations, (38a) and (49a), obtained in $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, are actually indentical with each other. Thus, it follows,

$$W_n=W'_n, \tag{64}$$

and

$$c_\lambda{}^n=\eta_\lambda{}^n, \tag{65}$$

because of the same normalizations for $c_\lambda{}^n$ and $\eta_\lambda{}^n$, i. e., (43) and (53). In particular,

$$N_n=N'_n. \tag{66}$$

Next, we compare the parameters, $q^{nJM}_{sl_s\nu_sm_s}$ and $\tilde{q}^{nJM}_{sl_s\nu_sm_s}$, defined by (58) with the ones, $u^{nJM}_{sl_s\nu_sm_s}$ and $\tilde{u}^{nJM}_{sl_s\nu_sm_s}$, in (13). By virtue of (37) together with (65), we have

$$\sum_\lambda \eta_\lambda{}^n\gamma_{\lambda s}=\sum_\lambda c_\lambda{}^n\gamma_{\lambda s}=g_{ns}. \tag{67}$$

g_{ns} is easily observed to be related to $\phi^{nJM}_{sl_s\nu_s m_s}$ defined by (14a). In fact, from (19b) and (19c) we have

$$\psi_{s\nu_s} Y_{l_s m_s} = a_s (M_s/\hbar)^{-1/2} \sum_{J'M'} (j_s l_s \nu_s m_s | j_s l_s J' M') V^{J'M'}_{sl_s} \text{ (on } S).$$

Inserting this into the integral in the right-hand side of (14a) and using (34), it follows that

$$\phi^{nJM}_{sl_s\nu_s m_s} = (\hbar^2/2M_s)^{-1/2} (j_s l_s \nu_s m_s | j_s l_s JM) g^J_{nsl_s}. \tag{68}$$

Hence, we have

$$q^{nJM}_{sl_s\nu_s m_s} = (\hbar^2/M_s)^{1/2} k_s^{1/2} [u^{(+)}_{l_s}(k_s a_s)]^{-1} \phi^{nJM}_{sl_s\nu_s m_s},$$

because of (58a) together with (67) and (61). The right-hand side of the above equation, however, is identical with that of (13a), so that we have

$$q^{nJM}_{sl_s\nu_s m_s} = u^{nJM}_{sl_s\nu_s m_s}. \tag{69a}$$

In a similar way, it can be shown that

$$\tilde{q}^{nJM}_{sl_s\nu_s m_s} = \tilde{u}^{nJM}_{sl_s\nu_s m_s}. \tag{69b}$$

From equations (64), (66) and (69) we see that each of the resonance terms of (57) is identical with the corresponding term in (3).

The identity of the potential scattering terms in (57) and (3) is easily observed, since we have,

$$\omega_s^2 \frac{1+iC_s}{1-iC_s} = \frac{\omega_s}{\omega_s^*} \frac{1+iC_s}{1-iC_s} = \frac{A_s^*}{A_s} = \frac{u^{(-)}_{l_s}(k_s a_s)}{v^{(-)}_{l_s}(k_s a_s)} \tag{70}$$

from (25d), (25c), (25a) and (19c).

Thus, it has been proved that the expressions for the collision matrix, $U_{sl_s\nu_s m_s; tl_t\nu_t m_t}$, derived from the R-matrix formalism and by K-P formalism are identical term by term with one another.

§ 4. The interpretation of the transformation, $R \to U$

In this section we attempt a rather pictorial interpretation of the transformation, $R \to U$, described in § 3 ***B***. In order to avoid unnecessary complexities arising from the angular momenta, it will be assumed that all the particles participating in the reaction have no spin and only those channels need be considered for which the relative orbital angular momenta are zero. The extention to a more general case is straightforward.

We consider a reaction initiated by the incident plane wave, $(\hbar k_s/M_s)^{-1/2} e^{ik_s z_s} \psi_s$. We denote by S_{st} the amplitude of the outgoing wave E_t, in the channel t, resulting from the reaction which will hereafter be referred to as S-matrix. Then S_{st} bears the relation to U_{st} defined by (1),

$$S_{st} = \sqrt{4\pi} (2ik_s)^{-1} (U_{st} - \delta_{st}). \tag{71}$$

Inserting, for U_{st} in (71), the expression (44) together with (45) for the special case of zero angular momenta, we have

$$S_{st}=\frac{\sqrt{4\pi}}{2ik_s}\left(\omega_s^2\frac{1+iC_s}{1-iC_s}-1\right)\delta_{st}+\frac{\sqrt{4\pi}}{k_s}\sum_{\lambda\mu}\frac{B_s\omega_s}{1-iC_s}\gamma_{\lambda s}\eta_{\lambda\mu}\gamma_{\lambda t}\frac{B_t\omega_t}{1-iC_t}. \tag{72}$$

The first term of (72) is the well-known potential scattering term for $l_s=0$, i. e.,

$$\frac{\sqrt{4\pi}}{2ik_s}\left(\omega_s^2\frac{1+iC_s}{1-iC_s}-1\right)=-\frac{\sqrt{4\pi}}{k_s}\sin k_s a_s\, e^{-ik_s a_s}, \tag{73}$$

as has been shown for a general case in (70), § 3. The resonance term of (72),

$$S_{st}^{(\mathrm{res})}=\frac{\sqrt{4\pi}}{k_s}\sum_{\lambda\mu}\frac{B_s\omega_s}{1-iC_s}\gamma_{\lambda s}\eta_{\lambda\mu}\gamma_{\mu t}\frac{B_t\omega_t}{1-iC_t} \tag{74}$$

is of our main concern here.

Solving the equation (46) for $\eta_{\lambda\mu}$ by means of the iteration procedure, we have

$$\eta_{\lambda\mu}=\frac{\delta_{\lambda\mu}}{E_\lambda-E}+\frac{h_{\lambda\mu}}{(E_\lambda-E)(E_\mu-E)}+\sum_\nu\frac{h_{\lambda\nu}h_{\nu\mu}}{(E_\lambda-E)(E_\nu-E)(E_\mu-E)}+\cdots. \tag{75}$$

Inserting (75) into (74), we have

$$S_{st}^{(\mathrm{res})}=S_{st}^{(1)}+S_{st}^{(2)}+\cdots, \tag{76}$$

where

$$S_{st}^{(1)}=\frac{\sqrt{4\pi}}{k_s}\sum_\lambda\frac{B_s\omega_s}{1-iC_s}\gamma_{\lambda s}\frac{1}{E_\lambda-E}\gamma_{\lambda t}\frac{B_t\omega_t}{1-iC_t}, \tag{77a}$$

$$S_{st}^{(2)}=\frac{\sqrt{4\pi}}{k_s}\sum_{\lambda\mu}\frac{B_s\omega_s}{1-iC_s}\gamma_{\lambda s}\frac{h_{\lambda\mu}}{(E_\lambda-E)(E_\mu-E)}\gamma_{\mu t}\frac{B_t\omega_t}{1-iC_t}, \tag{77b}$$

and so on.

We shall interpret the expressions (76), (77a), (77b), and so on, in the following way. We first construct a D-wave from the incident plane wave by superposing the outgoing wave E_s with suitable amplitude,

$$\frac{\sqrt{4\pi}}{k_s}\frac{B_s\omega_s}{1-iC_s}D_s=\left(\frac{\hbar k_s}{M_s}\right)^{-1/2}\psi_s\frac{\sin k_s r_s}{k_s r_s}+\frac{\sqrt{4\pi}}{2ik_s}\left(\omega_s^2\frac{1+iC_s}{1-iC_s}-1\right)E_s, \tag{78}$$

where the first term on the right-hand side is the $l_s=0$ part of the incident plane wave. On comparison of the coefficient of E_s in (78) with (73) we see that it is just the potential scattering term of the S-matrix. In other words, the continuation of the incident plane wave onto the D-wave, which excites the compound states X_λ's, necessarily gives rise to the reflection of wave the amplitude of which is equal to the potential scattering amplitude.

Now, a D-wave in a channel s with unit amplitude gives rise to the excitation of each compound state X_λ, with the amplitude,

$$(\hbar/2)^{1/2}\gamma_{\lambda s}/(E_\lambda-E), \tag{79}$$

as can be shown by expanding the wave function φ_s, which has the asymptotic form (20), in terms of X_λ in the internal region:

$$\varphi_s=\left(\frac{\hbar}{2}\right)^{1/2}\sum_\lambda \frac{\gamma_{\lambda s}}{E_\lambda-E}X_\lambda. \tag{80}$$

Furthermore, as can be seen from (22) and (31), the excitation of X_λ with the amplitude $(\hbar/2)\gamma_{\lambda s}/(E_\lambda-E)$, mentioned above, results in the excitation of V-wave in every channel t, with the amplitude,

$$\gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda-E), \tag{81}$$

which is a part of the R-matrix element, R_{st}, due to the compound state X_λ.

However, as has been mentioned in § 1, the V-wave does not satisfy the boundary condition suitable for the scattering problem. In order to adjust the boundary condition to the scattering problem, i. e., to that of the outgoing wave, a D-wave with appropriate amplitude should be superposed on V,

$$V_t+\Delta_t D_t=(B_t\omega_t/1-iC_t)E_t. \tag{82}$$

In other words, the V-wave with unit amplitude is partly transmitted as E_t-wave with the amplitude $B_t\omega_t/1-iC_t$ and partly reflected as D_t-wave with the amplitude Δ_t on passing through the entrance of the channel t from the internal region to the external region. Thus, the amplitude of the outgoing wave emitted in the first process described above is, from (78), (81) and (82),

$$\frac{\sqrt{4\pi}}{k_s}\frac{B_s\omega_s}{1-iC_s}\sum_\lambda\frac{\gamma_{\lambda s}\gamma_{\lambda t}}{E_\lambda-E}\frac{B_t\omega_t}{1-iC_t}. \tag{83}$$

(83) is seen to be just equal to $S_{st}^{(1)}$, (77a).

Now the D-wave, in (82), which is generated by the reflexion of the V-wave at each of the channel entrances, becomes the source of the secondary process; the excitation of X_λ's, following excitation of V_t's and its transmission and reflexion at the channel entrances. The amplitude of the secondary transmitted outgoing wave in the channel t is given by

$$\sum_{\lambda\mu}\sum_{t'}\left(\frac{\sqrt{4\pi}}{k_s}\frac{B_s\omega_s}{1-iC_s}\frac{\gamma_{\lambda s}\gamma_{\lambda t'}}{E_\lambda-E}\Delta_{t'}\right)\left(\frac{\gamma_{\mu t'}\gamma_{\mu t}}{E_\mu-E}\right)\left(\frac{B_t\omega_t}{1-iC_t}\right). \tag{84}$$

In (84), the first factor represents the amplitude of the secondary source, the second the amplitude of the resulting V_t-wave per unit amplitude of $D_{t'}$, and the last factor the transmission coefficient of E_t at the channel entrance. Using (39), (84) can be re-written as

$$\frac{\sqrt{4\pi}}{k_s}\sum_{\lambda\mu}\frac{B_s\omega_s}{1-iC_s}\gamma_{\lambda s}\frac{h_{\lambda\mu}}{(E_\lambda-E)(E_\mu-E)}\gamma_{\mu t}\frac{B_t\omega_t}{1-iC_t}. \tag{85}$$

We see that it is just equal to $S_{st}^{(2)}$, (77b). The amplitude of the reflected D_t-wave in this second process is given by

$$\sum_{\lambda\mu}\sum_{t'}\left(\frac{\sqrt{4\pi}}{k_s}\frac{B_s\omega_s}{1-iC_s}\frac{\gamma_{\lambda s}\gamma_{\lambda t'}}{E_\lambda-E}\Delta_{t'}\right)\left(\frac{\gamma_{\mu t'}\gamma_{\mu t}}{E_\mu-E}\right)\Delta_t, \tag{86}$$

in which the first two factors are the same as those in (84) and the last factor, Δ_t, represents the "reflection coefficient" of V_t-wave at the channel entrance. All the terms in the series (76) can be interpreted as the repetition of the processes described above.

The interpretation described above may be visualised as follows. We regard the nucleus as an assembly of resonators each of which has single resonance energy, E_λ. The hard sphere scattering takes place as the reflection of the incident plane wave on entering into the internal region. The wave which enters into the internal region excites the resonators and the latter send out the secondary wave. The secondary wave emitted from one resonator is partly transmitted out from the internal region as the scattered wave and partly reflected to another resonator and excites the latter. The resonator thus excited sends out the tertiary wave, and so on. The S-matrix is the sum of the amplitudes of the transmitted outgoing waves from the channel entrances in each of the multiple processes mentioned above.

Now, from (84) it is clear that the reflection coefficient from one level, λ, to another, μ, is just equal to the quantity $h_{\lambda\mu}$ defined before,

$$h_{\lambda\mu}=\sum_s \gamma_{\lambda s}\Delta_s\gamma_{\mu s},$$

and it is the sum of the partial reflection coefficients each of which corresponds to a channel s,

$$h_{\lambda\mu}=\sum_s h_{\lambda\mu,s}\,;\quad h_{\lambda\mu,s}\equiv\gamma_{\lambda s}\Delta_s\gamma_{\mu s}. \tag{87}$$

The matrix $((h_{\lambda\mu}))$ is actually the same thing as what Thomas[10] called the level matrix, but we see from the above consideration that the "reflection coefficient matrix" may be more appropriate for it from its role in the process of propagation of the wave function.

§ 5. Concluding remarks

We have shown that the collision matrices obtained from the formalism of Wigner and Eisenbud is term by term identical with that of Kapur and Peierls. At the same time it became clear that the transformation of the R-matrix into the collision matrix can be intuitively interpreted by the "multiple reflexion interpretation".

The generalization of the considerations given in this paper to a more general boundary condition of the R-matrix formalism will be possible.

The possibility of the surface direct reaction[14),15),16)] has not been considered throughout this paper, although it is certainly of great importance in some mode of nuclear reactions. When the surface direct reaction plays an unnegligible role, however, the both formalisms of Kapur-Peierls and Wigner-Eisenbud themselves will become impractical, and the formalism which takes this effect into account should be constructed anew.[5),10),17),18),19)]

The multiple reflexion interpretation introduced in this paper concerning the transformation, $R \to U$, is useful for understanding and constructing various approximations for collision matrix. For example, Breit-Wigner's one level formula,[6] Bethe-Placzek's many level formula,[7] Wigner-Eisenbud's generalized one level formula,[1] and so on, can be interpreted very intuitively by this interpretation. Also the interpretation of the channel elimination procedure of Teichman and Wigner[5] is easy. The discussion on these matters will be given in the following paper.

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The Dispersion Relations for the Scattering of Photons from Proton

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The dispersion relations are derived for the non-forward amplitudes for photon-proton scattering. By expanding these relations in momentum transfer and retaining first few terms, the dispersion relations for six independent combinations of the partial wave amplitudes are obtained. Contributions of the unphysical region are calculated to the first order in the fine structure constant, and are found to be consistent with the low energy theorem. Using these relations and photopion production data, the differential cross sections are calculated. In doing this, the amplitudes for the mixtures of dipole and quadrupole waves are taken into account in addition to electric dipole, electric quadrupole and magnetic dipole amplitudes. The results turn out to be fairly improved compared with the dipole model, and roughly explain experiments.

§ 1. Introduction

Gell-Mann, Goldberger and Thirring[1] have derived two dispersion relations for spin-flip and non-spin-flip parts of the forward amplitude for photon-proton scattering. If one assumes that only the partial wave amplitudes for electric and magnetic dipole radiations with total angular momenta 1/2 and 3/2, respectively, are enhanced by photopion production, these two dispersion relations are sufficient to determine the real parts of these amplitudes from their imaginary parts. As the imaginary parts can be related to the partial wave amplitudes for photopion production owing to the unitarity of S matrix, one can predict the differential cross sections for photon-proton scatterings using data on photopion production. We refer to this method as the dipole model. Recently, calculations by this method have been made by Capps,[2] predicting too large cross sections compared with experiments below 200 Mev.

In the present paper we will derive the dispersion relations for the non-forward amplitudes. By expanding these relations in powers of momentum transfer, we can obtain more than two dispersion relations. Then one can inquire whether all these dispersion relations can or cannot be satisfied by the above two dipole amplitudes alone. Thus we can examine the validity of the dipole model. We will show in § 4 that the dipole model is sufficient to describe general aspects of the phenomena, but is not so accurate numerically.

Although derivation of the non-forward dispersion relations has been studied in detail for pion-nucleon scattering,[3] it is not so trivial for the present case because the polarization vectors depend on energy and momentum transfer for the non-forward scattering.

However, we can prove necessary symmetry properties and analyticity of the amplitudes by an appropriate choice of four independent combinations of initial and final polarization vectors. As is well known, the unphysical region appears in the non-forward dispersion relations, in spite of the vanishing mass of a photon. For sufficiently small momentum transfer, and to the first order in the fine structure constant, contributions of the unphysical region come only from the single proton intermediate states, and therefore can be easily calculated. They can, however, be determined by another method. By virtue of the low energy theorem,[4] expression of the scattering amplitude is known to the first order in photon energy without entering into details of the theory, and comparison of this expression with the dispersion relations determines the contributions of the unphysical region completely. It is of interest to examine whether these two determinations give the same result, since the direct calculation is not rigorous on account of the fact that it compels us to deal with an unphysical state with imaginary momentum. The affirmative answer to this question will be given in § 2.

Expanding the non-forward dispersion relations in powers of momentum transfer, we obtain from the zeroth order parts two dispersion relations which naturally coincide with those derived in reference 1) and two new relations from the first order parts. Relations obtained from the higher order parts contain, in general, undetermined constants which cannot be expressed in terms of charge and magnetic moment of proton alone. As the present experimental knowledge does not seem accurate enough to determine so many parameters, we derive only two relations from the second order part. The two undetermined constants appearing in these relations are found to be equal to each other to the approximation adopted in calculating the cross sections. We thus obtain six dispersion relations in all, derivation of which will be made in § 2.

In § 3 the cross sections are calculated with the aid of the above dispersion relations, restricting the participating partial wave amplitudes appropriately. The results are found to be fairly improved compared with the dipole model. But a disagreement with experiments still remains in the energy dependence of the cross section. Possibilities of further improvements are discussed in § 4.

§ 2. Dispersion relations

We consider the process in which a photon of momentum $\boldsymbol{k}$ and polarization $\boldsymbol{e}_\tau$ collides with a proton of momentum $\boldsymbol{p}$, and then the photon is scattered into the state of momentum $\boldsymbol{k}'$ and polarization $\boldsymbol{e}_\sigma$, leaving the proton in the state with momentum $\boldsymbol{p}'$. It is convenient for our purpose to take the coordinate system in which the momenta of the colliding particles obey the relations $\boldsymbol{p}=-\boldsymbol{p}'=-\boldsymbol{q}$, $\boldsymbol{k}=\boldsymbol{Q}+\boldsymbol{q}$, $\boldsymbol{k}'=\boldsymbol{Q}-\boldsymbol{q}$, where $\boldsymbol{Q}$ is the total momentum. In this system we define the causal amplitude for the process as in the case of pion-nucleon scattering, which will be denoted by $M(\boldsymbol{k}', \boldsymbol{e}_\sigma, \boldsymbol{k}, \boldsymbol{e}_\tau)$ as a matrix in spin space. The causal amplitude is identical with the Feynman matrix element for the same process when $\nu \equiv |\boldsymbol{k}| = |\boldsymbol{k}'|$ is larger than q, and moreover, owing to the requirement of causality, it has the important property of being analytic in the

upper half part of the ν-complex plane, provided that we properly define the ν-dependence of polarization vectors $\mathbf{e}_\tau$, $\mathbf{e}_s$, as will be shown below.

Table I. Sets of polarization which show simple ν-dependence.

polarization \ λ	I	II	III	IV
initial $\mathbf{e}_\tau$	$\mathbf{n}\times\boldsymbol{\kappa}$	$\mathbf{n}$	$\mathbf{n}\times\boldsymbol{\kappa}$	$\mathbf{n}$
final $\mathbf{e}_s$	$\mathbf{n}\times\boldsymbol{\kappa}'$	$-\mathbf{n}$	$-\mathbf{n}$	$\mathbf{n}\times\boldsymbol{\kappa}'$

The polarization vectors $\mathbf{e}_\tau$, $\mathbf{e}_s$ generally change their direction as ν varies, showing indeed that they must be regarded as functions of ν. We give four sets of polarization in Table I, which seem to exhibit the simplest ν-dependences. Here $\mathbf{n}$ is a unit vector perpendicular to the scattered plane, $\boldsymbol{\kappa}=\boldsymbol{k}, k$, and $\boldsymbol{\kappa}'=\boldsymbol{k}', k'$. We denote the causal amplitude characterized by the λ-th set of polarizations in Table I by $M^{(\lambda)}$, and confine our consideration to the four amplitudes $M^{(1)},\cdots,M^{(4)}$, since a general causal amplitude can be represented in a linear combination of them. To define the ν-dependence of these amplitudes explicitly, we first put

$$\mathbf{Q}=Q(\nu)\boldsymbol{\pi}\ ;\ Q(\nu)=(\nu^2-q^2)^{1/2}, \tag{2·1}$$

where $\boldsymbol{\pi}$ is a unit vector. We introduce cuts along the real axis from $-\infty$ to $-q$ and q to $+\infty$ to make $Q(\nu)$ a one-valued function, and suppose that ν always lies slightly above the real axis; hence $Q(-\nu)=-Q(\nu)$ for $\nu>q$. From (2·1) the polarization vectors in Table I, as functions of ν, take the form

$$\begin{gathered}\mathbf{n}\times\boldsymbol{\kappa}=q^{-1}\mathbf{e}_1(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q}),\ \mathbf{n}\times\boldsymbol{\kappa}'=q^{-1}\mathbf{e}_1(\nu,\ -\boldsymbol{\pi},\ \boldsymbol{q})\\ \mathbf{n}=q^{-1}\mathbf{e}_2(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q}),\end{gathered} \tag{2·2}$$

where $\mathbf{e}_1(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\nu^{-1}(q^2\boldsymbol{\pi}-Q(\nu)\boldsymbol{q})$, $\mathbf{e}_2(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\boldsymbol{q}\times\boldsymbol{\pi}$. From (2·1) and (2·2) the quantities which occur in the causal amplitudes can be expressed in terms of ν, $\boldsymbol{\pi}$, $\boldsymbol{q}$. Hence the above four amplitudes may be considered as functions of ν, $\boldsymbol{\pi}$ and $\boldsymbol{q}$. They are invariant under spatial rotations, but under reflection $(\boldsymbol{\pi},\ \boldsymbol{q}\ )\to(-\boldsymbol{\pi},\ -\boldsymbol{q})$ $M^{(3,4)}$ change the sign, while $M^{(1,2)}$ remain invariant. Consequently $M^{(\lambda)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ can be written in the form

$$M^{(1,2)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)+i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}(\nu)\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2), \tag{2·3a}$$

$$M^{(3,4)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=i\boldsymbol{\sigma}\cdot\mathbf{Q}(\nu)\mathcal{M}_+{}^{(3,4)}(\nu,\ q^2)+i\boldsymbol{\sigma}\cdot\boldsymbol{q}\mathcal{M}_-{}^{(3,4)}(\nu,\ q^2). \tag{2·3b}$$

Now we state the notable properties of $M^{(\lambda)}$ which follow from the requirement of the causality[3] and the invariance under time reflection[5]: (i) $\mathcal{M}^{(\lambda)}(\nu,\ q^2)$ are analytic in the upper half part of the ν-complex plane, (ii) $\mathrm{Re}\,\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)$, $\mathrm{Im}\,\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2)$, and $\mathrm{Re}\,\mathcal{M}_{+,-}^{(3,4)}(\nu,\ q^2)$ are even, while $\mathrm{Im}\,\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)$, $\mathrm{Re}\,\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2)$ and $\mathrm{Im}\,\mathcal{M}_{+,-}^{(3,4)}(\nu,\ q^2)$ are odd, as functions of ν, (iii) $\mathcal{M}_+{}^{(3)}(\nu,\ q^2)=\mathcal{M}_-{}^{(4)}(\nu,\ q^2)$ and $\mathcal{M}_-{}^{(3)}(\nu,\ q^2)=-\mathcal{M}_-{}^{(4)}(\nu,\ q^2)$. The last condition means that we may exclude $M^{(4)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ from our consideration. The proof of these statements can be done following the familiar pattern, and, perhaps, needs scarcely to be discussed here.

There remains one thing to be done prior to the derivation of dispersion relations, i.e., to examine the behavior of the imaginary part of amplitudes in the unphysical region $\nu<q$. For this purpose we decompose $M^{(\lambda)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ into the dispersive and absorptive

part, which are real and imaginary part of $\mathcal{M}^{(\lambda)}(\nu, q^2)$, respectively:

$$M^{(\lambda)}(\nu, \pi, \boldsymbol{q}) = D^{(\lambda)}(\nu, \pi, \boldsymbol{q}) + iA^{(\lambda)}(\nu, \pi, \boldsymbol{q}). \tag{2·4}$$

The absorptive parts can be writren in the form:

$$\begin{aligned} A^{(1,2)}(\nu, \pi, \boldsymbol{q}) = \frac{(2\pi)^6 q^{-2}}{4}\left(\frac{E_q}{M}\right)\sum_{M_n}\{&(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=\mathbf{Q}|\mathbf{e}_{1,2}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q+\nu-E_n) \\ &-(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=-\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_{1,2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q-\nu-E_n)\}, \end{aligned} \tag{2·5a}$$

$$\begin{aligned} A^{(3)}(\nu, \pi, \boldsymbol{q}) = \frac{(2\pi)^6 q^{-2}}{4}\left(\frac{E_q}{M}\right)\sum_{M_n}\{&(\boldsymbol{q}|\mathbf{e}_2(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=\mathbf{Q}|\mathbf{e}_1(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q+\nu-E_n) \\ &-(\boldsymbol{q}|\mathbf{e}_1(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=-\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_2(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q-\nu-E_n)\}, \end{aligned} \tag{2·5b}$$

where $\boldsymbol{j}(x)$ is the current operator: $\Box\boldsymbol{A}_\mu(x) = -\boldsymbol{j}_\mu(x)$, $|\pm\boldsymbol{q})$ is the state vector for the single proton of momentum $\pm\boldsymbol{q}$, and E_q the proton energy. Now the intermediate states in the sum contribute to $A^{(\lambda)}(\nu, \pi, \boldsymbol{q})$ in the unphysical region as follows (we keep $\nu \geq 0$). *We always neglect all but the leading term in powers of* e^2. Under this approximation we can exclude the possibility of photons being in the intermediate states, which means that M_n consists of two spectra: a point spectrum $M_n = M$, which corresponds to the free proton states, and a continuous spectrum $M+\mu \leq M_n < \infty$, μ being the pion mass, which originates from the states of one real nucleon together with a certain number of pions, nucleon pairs and other particles. Inquiring, for a given ν, those values of M_n that make the argument of $\delta(E_q \pm \nu - E_n)$ vanish, we find that the point spectrum gives contributions at the point $\nu = \nu_b \equiv q^2/E_q$ through the terms multiplied by $\delta(E_q - \nu - E_n)$. The continuous spectrum does not contribute in the region $0 < \nu < q$ if $q < \left(\frac{M+\mu/2}{M+\mu}\right)\mu$, but does over the interval $\nu_a \leq \nu < q$ through the terms multiplied by $\delta(E_q+\nu-E_n)$ if $\left(\frac{M+\mu/2}{M+\mu}\right)\mu < q \leq (M\mu+\mu^2/2)^{1/2}$, where $\nu_a = (M\mu+\mu^2/2-q^2)/E_q$. If $q > (M\mu+\mu^2/2)^{1/2}$, the continuous spectrum yields further contributions over the interval $0 < \nu \leq -\nu_a$ through the terms multiplied by $\delta(E_q-\nu-E_n)$. For simplicity, however, we shall leave the last case out of our consideration.

Expressing the contribution of the free proton states, as distinguished from that of the other intermediate states, as

$$A^{(1,2)}(\nu, \pi, \boldsymbol{q}) = \pi\{\Gamma_N^{(1,2)}(q^2) + i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}(\nu_b)\,\Gamma_S^{(1,2)}(q^2)\}\,\delta(\nu-\nu_b), \tag{2·6a}$$

$$A^{(3)}(\nu, \pi, \boldsymbol{q}) = \pi\{i\boldsymbol{\sigma}\cdot\mathbf{Q}(\nu_b)\,\Gamma_+^{(3)}(q^2) + i\boldsymbol{\sigma}\cdot\boldsymbol{q}\Gamma_-^{(3)}(q^2)\}\,\delta(\nu-\nu_b), \tag{2·6b}$$

we obtain the dispersion relations under certain assumptions on the boundedness of the causal amplitudes :

$$\mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(\nu, q^2) - \mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(q, q^2) = \frac{2\Gamma_N'^{(\alpha)}(q^2) E_q (\nu^2 - q^2)}{M^2(\nu^2 - \nu_b^2)} + \frac{2(\nu^2 - q^2)}{\pi} \int_{\nu_a}^{\infty} \frac{\nu' \mathrm{Im}\,\mathcal{M}_N^{(\alpha)}(\nu', q^2)}{(\nu'^2 - q^2)(\nu'^2 - \nu^2)} d\nu', \tag{2·7}$$

$$\mathrm{Re}\,\mathcal{M}_S^{(\alpha)}(\nu, q^2) = -\frac{2\nu \Gamma_S'^{(\alpha)}(q^2)}{\nu^2 - \nu_b^2} + \frac{2\nu}{\pi} \int_{\nu_a}^{\infty} \frac{\mathrm{Im}\,\mathcal{M}_S^{(\alpha)}(\nu', q^2)}{\nu'^2 - \nu^2} d\nu' \tag{2·8}$$

$$\text{with} \quad \alpha = 1, 2.$$

$\mathcal{M}_{+,-}^{(3)}(\nu, q^2)$ obeys the same dispersion relations as (2·7), except that $\Gamma_{+,-}^{(3)}(q^2)$ occurs now instead of $\Gamma_N^{(\alpha)}(q^2)$. Explicit calculation of $\Gamma^{(\lambda)}(q^2)$ ($\lambda = 1, 2, 3$) is given in Appendix I, where it is shown that these can be completely expressed in terms of the proton mass, charge and static magnetic moment.

We employ the notations $M_c^{(\lambda)}$ ($\lambda = 1, 2, 3, 4$) for the four Feynman amplitudes in the barycentric system characterized by the sets of polarizations in Table I, and inquire the transformation formulae between $M_c^{(\lambda)}$ and $M^{(\lambda)}$. It can be proved that photons may be treated in this transformation as if they were spinless particles. Therefore the relation between $M_c^{(\lambda)}$ and $M^{(\lambda)}$ is independent of λ, and is same as in the pion-nucleon scattering[3]. The proof is as follows : Referring to the coordinate system so far used as " q-system ", we introduce four-vectors e_μ and e_μ', whose components in the q-system are respectively given by

$$\begin{aligned} e_p &= (\mathbf{e}_r + \alpha \boldsymbol{k},\ \alpha k), \\ e_\mu' &= (\mathbf{e}_s + \alpha' \boldsymbol{k}',\ \alpha' k') \end{aligned} \tag{2·9}$$

with

$$\alpha = \frac{(\mathbf{e}_r \cdot \boldsymbol{v})}{k - (\boldsymbol{v} \cdot \boldsymbol{k})} \quad \text{and} \quad \alpha' = \frac{(\mathbf{e}_s \cdot \boldsymbol{v})}{k' - (\boldsymbol{v} \cdot \boldsymbol{k}')},$$

where $\boldsymbol{v}$ is the velocity of the CMS in the q-system. Then, $M^{(\lambda)}$ can be *symbolically* written as

$$M^{(\lambda)} \sim (\boldsymbol{q} | (e_\mu' j_\mu')\,(e_\mu j_\mu) | -\boldsymbol{q}) \tag{2·10}$$

in virtue of the gauge invariance, which asserts that $e_\mu j_\mu = \mathbf{e}_r \cdot \boldsymbol{j}$, $e_\mu' j_\mu = \mathbf{e}_s \cdot \boldsymbol{j}'$ effectively. Here j_μ, j_μ', $\boldsymbol{j}$, $\boldsymbol{j}'$ stand for the current operator. Applying a unitary transformation R corresponding to the Lorentz transformation from q-system to the CMS, (2·10) becomes

$$M^{(\lambda)} \sim (\boldsymbol{q} | R^{-1} (e_{c\mu}' j_{c\mu}')\,(e_{c\mu} j_{c\mu})\, R | -\boldsymbol{q}), \tag{2·11}$$

where the subscript c means that quantities with it refer to the C.M.S. From (2·9) we can easily check that $e_{c4} = 0$, and $\mathbf{e}_c$ is a unit vector parallel or perpendicular to the plane, through which the photon is scattered, according as $\mathbf{e}_r$ is parallel or perpendicular to the same plane, and that the same is true for $e_{c\mu}'$. Consequently $M_c^{(\lambda)}$ can be written as

$$M_c^{(\lambda)} \sim (\boldsymbol{p}_c'|\,(e_{c\mu}'j_{c\mu}')\,(e_{c\mu}j_{c\mu})\,|\boldsymbol{p}_c), \tag{2·12}$$

where $|\boldsymbol{p}_c)$ and $|\boldsymbol{p}_c')$ represent the initial and recoiled proton state, respectively. On comparsion of (2·12) with (2·11), it is inferred that $M^{(\lambda)}$ can be obtained from $M_c^{(\lambda)}$ by means of a Lorentz transformation of the free proton states to which the matrix element $M_c^{(\lambda)}$ is referred, irrespective of the photon polarization. This completes the proof.

For practical applications it is convenient to express $M^{(\lambda)}$ in terms of the scattering amplitude $f^{(\lambda)}$ in the C.M.S., which relates to $M_c^{(\lambda)}$ through $M_c^{(\lambda)}=M^{-1}W_c f^{(\lambda)}$, W_c being a total energy in C.M.S. The relation between $M^{(\lambda)}$ and $M_c^{(\lambda)}$ disputed above immediately gives the ones between $M^{(\lambda)}$ and $f^{(\lambda)}$. We use the notation $\boldsymbol{k}_c$, $\boldsymbol{k}_c'$, ν_c, θ_c, and E_c for the momenta of the incident and scattered photon, the photon energy, the scattering angle, and the proton energy respectively, all referring to the CMS. Then $f^{(\lambda)}$ can be written in the form

$$f^{(1,2)}(\nu_c,\,\theta_c)=f_N^{(1,2)}(\nu_c,\,\theta_c)+i\boldsymbol{\sigma}\cdot\boldsymbol{k}_c\times\boldsymbol{k}_c'\nu_c^{-2}f_S^{(1,2)}(\nu_c,\,\theta_c), \tag{2·13a}$$

$$f^{(3,4)}(\nu_c,\,\theta_c)=\nu_c^{-1}\left\{f_+^{(3,4)}(\nu_c,\,\theta_c)\,i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{k}_c+\boldsymbol{k}_c'}{2}+f_-^{(3,4)}(\nu_c,\,\theta_c)\,i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{k}_c-\boldsymbol{k}_c'}{2}\right\}. \tag{2·13b}$$

The relations of $M^{(\lambda)}$ to $f^{(\lambda)}$ are given by

$$\mathcal{M}_N^{(1,2)}(\nu,\,q^2)=\frac{W_c}{M}\left\{\frac{E_q^2+ME_c}{E_q(M+E_c)}\,f_N^{(1,2)}(\nu_c,\,\theta_c)-2\left(\frac{q}{\nu_c}\right)^2\frac{E_q^2-E_c^2}{E_q(M+E_c)}f_S^{(1,2)}(\nu_c,\,\theta_c)\right\}, \tag{2·14a}$$

$$\mathcal{M}_S^{(1,2)}(\nu,\,q^2)=\frac{1}{M(M+E_c)}\left\{-f_N^{(1,2)}(\nu_c,\,\theta_c)+2\frac{E_q^2+ME_c}{\nu_c^2}f_S^{(1,2)}(\nu_c,\,\theta_c)\right\}, \tag{2·14b}$$

$$\mathcal{M}_+^{(3,4)}(\nu,\,q^2)=\frac{E_q}{M\nu_c}f_+^{(3,4)}(\nu_c,\,\theta_c), \tag{2·14c}$$

$$\mathcal{M}_-^{(3,4)}(\nu,\,q^2)=\frac{W_c}{M\nu_c}f_-^{(3,4)}(\nu_c,\,\theta_c), \tag{2·14d}$$

where ν_c, θ_c are connected with ν, q^2 through

$$\nu=W_cE_cE_q^{-1}-E_q, \tag{2·15}$$

$$2q^2=\nu_c^2(1-\cos\theta_c). \tag{2·16}$$

Eqs. (2·14 c, d) show that $f_+^{(3)}=f_+^{(4)}$, $f_-^{(3)}=-f_-^{(4)}$. Besides, we note that $f_N^{(1)}$, etc., depend on θ_c through $\cos\theta_c$ alone.

By virtue of eqs. (2·13)—(2·16) dispersion relations (2·7), (2·8) and the ones for $\mathcal{M}_{+,-}^{(3)}$ are all expressed in terms of the scattering amplitudes $f^{(\lambda)}$; the form is most convenient for the practical application. For the sake of getting comparatively simple expressions, we do this only in two ways, one simply taking the limit $q\to 0$, and another taking the limit $q\to 0$ after once differentiating dispersion relations with respect to q^2.

In the limit $q\to 0$, $\mathrm{Re}\,\mathcal{M}^{(\lambda)}(q,\,q^2)$ are fixed from the low energy theorem, and there occurs nothing unknown in the inhomogenous terms of dispersion relations. We have

$$\mathrm{Re}\, f_N^{(1)}(\nu) = -\frac{e^2}{4\pi M}\frac{1}{\sqrt{1+2\nu/M}} + \frac{2\nu^2}{\pi\sqrt{1+2\nu/M}}\int_{\nu_t}^{\infty} d\nu' \frac{\sqrt{1+2\nu'/M}}{\nu'(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_N^{(1)}(\nu'), \tag{DI}$$

$$\mathrm{Re}\{2f_S^{(1)}(\nu) - \epsilon(\nu) f_N^{(1)}(\nu)\} = \frac{e^2}{4\pi M^2}\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\{2f_S^{(1)}(\nu') - \epsilon(\nu') f_N^{(1)}(\nu')\}, \tag{DII}$$

$$\mathrm{Re}\{2f_S^{(2)}(\nu) - \epsilon(\nu) f_N^{(2)}(\nu)\} = -\frac{e^2 g_p^{\,2}}{4\pi M^2}\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\{2f_S^{(2)}(\nu') - \epsilon(\nu') f_S^{(2)}(\nu')\}, \tag{DIII}$$

$$\mathrm{Re}\, f_+^{(3)}(\nu) = -\frac{e^2}{8\pi M^2}(g_p-1)^2\frac{\nu}{\sqrt{1+2\nu/M}} + \frac{2\nu^3}{\pi\sqrt{1+2\nu/M}}\int_{\nu_t}^{\infty} \frac{\sqrt{1+2\nu'/M}}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_+^{(3)}(\nu')\, d\nu', \tag{DIV}$$

$$\mathrm{Re}\, f_-^{(3)}(\nu) = \frac{e^2}{8\pi M^2}(g_p^{\,2}-1)\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_-^{(3)}(\nu'), \tag{DV}$$

where

$$\nu_t = \mu + \mu^2/2M,$$

$$\epsilon(\nu) = (\nu/M)^2(1+2\nu/M)^{-1/2}[1+\nu/M+\sqrt{1+2\nu/M}]^{-1},$$

$$f^{(\lambda)}(\nu) \equiv f^{(\lambda)}(\nu_c=\nu/\sqrt{1+2\nu/M},\ \theta_c=0).$$

Evidently $f_N^{(1)} = -f_N^{(2)}$ in the limit $q\to 0$, and hence the dispersion relation for $f_N^{(2)}$ is equivalent to (DI). It is easily seen that (DI) and (DIV) are identical with the two dispersion relations derived by *GGT*, respectively for the non-spin-flip and spin-flip part of the forward scattering amplitude. It should be noted further that (DII), (DIII) and (DV) are not independent because of the relation

$$f_S^{(1)}(\nu) + f_S^{(2)}(\nu) = -f_-^{(3)}(\nu),$$

which we have proved by means of the partial wave expansion without any approximation.

The differentiations with respect to q^2 can be done most conveniently, if the independent variables of (2·7, 8) are changed from ν, ν', q^2 to ν_c, ν_c', q^2 by making use of (2·15, 16). Then, by successive differentiations with respect to q^2, we get dispersion

relations for $[\partial f^{(\lambda)}(\nu_c, \theta_c)/\partial \cos\theta_c]_{\cos\theta_c=1}$, $[\partial^2 f^{(\lambda)}(\nu_c, \theta_c)/\partial(\cos\theta_c)^2]_{\cos\theta_c=1}$, etc. The procedure is rather cumbersome, so we treat $\mathcal{M}_N^{(1,2)}(\nu, q^2)$ alone, expanding all quantities in powers of ν/M, ν'/M, and neglecting all but the lowest and second order terms. The result can be put in the form:

$$\begin{aligned}\mathrm{Re}\,\dot{f}_N^{(\alpha)}(\nu) = &\left(1-\frac{3\nu}{M}\right)\left\{\frac{\Gamma_N^{(\alpha)}(0)}{M}-\nu^2\left(\frac{\Delta_N^{(\alpha)}}{2}+\frac{\dot{\Gamma}_N^{(\alpha)}}{M}\right)\right\}\\ &+\frac{2\nu^4}{\pi}\int_{\nu_t}^{\infty}d\nu'\left(1+3\frac{\nu'-\nu}{M}\right)\frac{\mathrm{Im}\,\dot{f}_N^{(\alpha)}(\nu')}{\nu'^3(\nu'^2-\nu^2)}\\ &+\frac{\nu^2}{\pi}\int_{\nu_t}^{\infty}\frac{d\nu'}{\nu'^3}\left\{1+\frac{2\nu'-3\nu}{M}-\frac{\nu'^3}{M(\nu'+\nu)^2}\right\}\mathrm{Im}\,f_N^{(\alpha)}(\nu')\end{aligned} \tag{DVI}$$

$$\text{with}\quad \alpha=1,\ 2,$$

where

$$\dot{f}_N^{(\alpha)}(\nu)=[\partial f_N^{(\alpha)}(\nu_c=\nu-\nu^2/M,\ \theta_c)/\partial\cos\theta_c]_{\cos\theta_c=1},$$

$$\Delta_N^{(\alpha)}=[\partial\,\mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(q, q^2)/\partial q^2]_{q=0},$$

$$\dot{\Gamma}_N^{(\alpha)}=[\partial\Gamma_N^{(\alpha)}(q^2)/\partial q^2]_{q=0}.$$

$\Gamma_N^{(\alpha)}(0)$, $\dot{\Gamma}_N^{(\alpha)}$ are immediately obtained from Appendix I. The result is that

$$\Gamma_N^{(1)}(0)=-e^2/4\pi, \tag{2·17}$$

and the others are all negligible to the present approximation. However, $\Delta_N^{(\alpha)}$ remain unknown. For the present we have no reliable method to evaluate them theoretically. It is impossible to continue the procedure of q^2-differentiation to obtain new dispersion relations without being worried by further appearance of such unknown constants. This is the main reason why we have ceased to repeat the procedure more than once.

The right-hand side of each of the dispersion relations thus obtained is fixed up to the terms linear in ν, independent of $\Delta_N^{(\alpha)}$ and the imaginary part of amplitudes, while the left-hand side is fixed to the same order from the low energy theorem.[4] Therefore, we can examine the consistency of our dispersion relations to the first order in ν, and thereby get a justification for our method of the evaluation of the unphysical region contributions. In fact, according to the low-energy theorem, we have, to the first order in ν,

$$\begin{aligned}&f_N^{(1)}(\nu_c, \theta_c)=\frac{e^2}{4\pi M}\left\{-\cos\theta_c+\frac{\nu_c}{M}(-1+\cos\theta_c+\cos^2\theta_c)\right\},\\ &f_N^{(2)}(\nu_c, \theta_c)=\frac{e^2}{4\pi M}\left(1-\frac{\nu_c}{M}\right),\\ &f_S^{(1)}(\nu_c, \theta_c)=\frac{e^2}{8\pi M}\frac{\nu_c}{M},\quad f_S^{(2)}(\nu_c, \theta_c)=-\frac{e^2}{8\pi M}\frac{\nu_c}{M}g_{\mathrm{p}}^2,\end{aligned} \tag{2·18}$$

$$f_{+}^{(3)}(\nu_c, \theta_c) = -\frac{e^2}{8\pi M}\frac{\nu_c}{M}(g_p{}^2 - g_p + 1 - g_p \cos\theta_c),$$

$$f_{-}^{(3)}(\nu_c, \theta_c) = \frac{e^2}{8\pi M}\frac{\nu_c}{M}(g_p{}^2 + g_p - 1 - g_p \cos\theta_c).$$

The consistency of these with the dispersion relations can be easily checked.

This section will be concluded with the partial-wave expansion of $f^{(\lambda)}(\nu_c, \theta_c)$. The exact expressions can be obtained from (A II–3) in Appendix II. Here we neglect, however, all the electric multipole amplitudes of the order higher than 2, all the magnetic multipole amplitudes of the order higher than 1, and all the amplitudes mixing electric and magnetic multipole waves except for $a_{3/2}$ $(E1, M1)$ and $a_{3/2}(E2, M1)$, because this approximation will be sufficient for the application in the next section. Then we have from (A II–3)

$$\begin{aligned}
f_N^{(1)}(\nu_c, \theta_c) &= e_N \cos\theta_c + m_N + e_N'(2\cos^2\theta_c - 1),\\
f_N^{(2)}(\nu_c, \theta_c) &= -e_N - (m_N + e_N')\cos\theta_c,\\
f_S^{(1)}(\nu_c, \theta_c) &= (1/2)(e_S + 4e_S' \cos\theta_c - c_1),\\
f_S^{(2)}(\nu_c, \theta_c) &= -(1/2)(m_S + e_S' - c_2),\\
f_{+}^{(3)}(\nu_c, \theta_c) &= -(1/2)[e_S + m_S - e_S' + (2e_S' + c_1 + c_2)\cos\theta_c],\\
f_{-}^{(3)}(\nu_c, \theta_c) &= -(1/2)[e_S - m_S + e_S' + (2e_S' - c_1 + c_2)\cos\theta_c].
\end{aligned} \tag{2·19}$$

The notations used here will be explained in Appendix II. It is found that, to this approximation, the undetermined constants $A_N^{(\alpha)}$ coincide with each other:

$$A_N^{(1)} = A_N^{(2)} \equiv A. \tag{2·20}$$

In fact, on account of (2·15) and (2·16), (2·14a) gives

$$\mathcal{M}_N^{(\alpha)}(q, q^2) = M^{-1}(E_q + q) f_N^{(\alpha)}(q, \pi).$$

Since $f_N^{(1)} = f_N^{(2)}$ for $\theta_c = \pi$ because of (2·19), it is evident from the definition of $A_N^{(\alpha)}$ that (2·20) holds.

§ 3. Calculation of the cross sections

In this section we calculate the cross sections at energies between 180 and 320 Mev, applying the dispersion relations derived in the preceding section. We assume that the imaginary parts of the amplitudes at such high energies, where multiple photopion production is significant, do not contribute to the integrals appearing in the dispersion relations. In virtue of the unitarity of S matrix, the imaginary part of a partial wave amplitude with definite angular momentum and parity is related to the partial wave amplitudes with the same angular momentum and parity for photopion production, if the multiple photopion production is neglected. The relations will be derived in Appendix II. We further assume that photopions are produced only in S and P states, and that S-wave pion is

produced by electric dipole radiation into positive charge state with amplitude E_{δ}^{+}, and P-wave pion by magnetic dipole and electric quadrupole radiations into the $(3-3)$ state with amplitudes M_{33} and E_{33} respectively. Then, as is shown in Appendix II, we have

$$\begin{aligned} &\text{Im } e_{1/2}=2\nu_c|E_{\delta}^{+}|^2, \quad \text{Im } m_{3/2}=12\nu_c M_{33}^{\ 2}, \\ &\text{Im } e'_{3/2}=\nu_c E_{33}^{\ 2}, \qquad \text{Im } c_2=12\nu_c M_{33}E_{33}, \end{aligned} \tag{3·1}$$

all the remaining imaginary parts being zero. For the experimental values of E_{δ}^{+} we use those given by Watson et al.[6] These authors have given values of M_{33} and E_{33} also. According to recent investigations,[7] however, it seems that the values of E_{33} are still not decisive. We therefore put

$$E_{33}=cM_{33} \tag{3·2}$$

and make calculations for two choices $c=1$ and $c=0$. If the values in reference 6) were adopted, we would have $c\approx 1$. We determine values of M_{33} by means of

$$\sigma_0=32\pi M_{33}^{\ 2}, \tag{3·3}$$

where σ_0 denotes the total cross section for neutral photopion production. Strictly speaking, a contribution of E_{33} should be added to the right-hand side, but this is so small that we have neglected it. We neglect Im $e'_{3/2}$ since this is much smaller than either of Im $m_{3/2}$ and Im c_2.

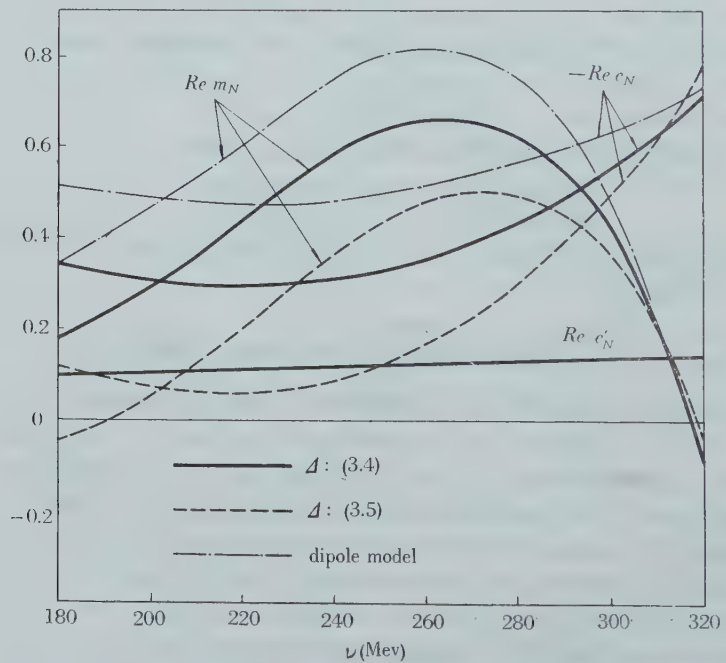

Fig. 1. Real parts of e_N, m_N and e_N' in units of $e^2/4\pi M$

To determine the real parts through the dispersion relations, we neglect the higher partial waves as was mentioned at the end of the preceding section. Then real parts of the non-spin-flip amplitudes e_N, m_N and e_N' are determined by (DI) and (DVI), if the constant Δ is known. It is easily seen that $\mathrm{Re}\,(e_N+m_N)$ and $\mathrm{Re}\,e_N'$ do not depend on Δ, but only $\mathrm{Re}\,(e_N-m_N)$ does. On the other hand, calculations show that $\mathrm{Re}\,(e_N+m_N)$ is very small owing to cancellation of its value at low energy limit by the imaginary part integral. Therefore, for appropriate values of Δ, both $\mathrm{Re}\,e_N$ and $\mathrm{Re}\,m_N$ become very small. This is important, since it will be found later that the smallness of $\mathrm{Re}\,e_N$ and $\mathrm{Re}\,m_N$ is at any rate necessary for getting agreement with experiments. We have estimated Δ by means of perturbation theory,[8] obtaining

$$\Delta=-0.42\,\mu^{-2}(e^2/4\pi M). \tag{3·4}$$

If we determine Δ so as to make the 90° cross section at 190 Mev as small as possible, we obtain

$$\Delta=-1.07\,\mu^{-2}(e^2/4\pi M). \tag{3·5}$$

We calculate $\mathrm{Re}\,e_N$, $\mathrm{Re}\,m_N$ and $\mathrm{Re}\,e_N'$ for these two values of Δ. The results are shown in Fig. 1. We note that contribution of the imaginary part integral to $\mathrm{Re}\,e_N'$ is very small.

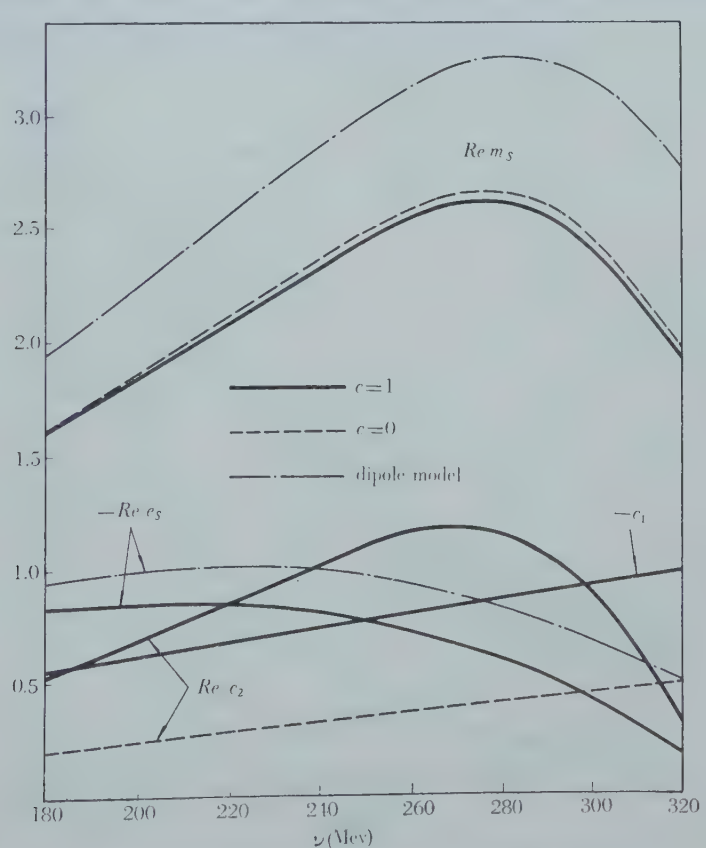

Fig. 2. Real parts of c_1, e_S, m_S and c_2 in units $e^2/4\pi M$

Next we calculate real parts of the spin-flip amplitudes. In view of the smallness of contribution of the imaginary part integral to Re e_S', it is natural to assume that the amplitudes with vanishing imaginary parts are unaffected by the meson effect. Therefore we replace c_1 and e_S' by their respective expressions given by the low energy theorem:

$$c_1 = -g_p(\nu/M)(e^2/4\pi M), \quad e_S' = 0. \tag{3·6}$$

Then Re e_S, Re m_S and Re c_2 are calculated by the use of (DII), (DIII), and (DIV). The results are shown in Fig. 2.

The largeness of Re m_S is mainly due to the large magnetic moment of proton. The peak in Re m_S at 270 Mev is due to the imaginary part integral. The imaginary part integral containing Im $m_{3/2}$ changes its sign at about the resonance energy for photo-pion production, and multiplication of this by ν^3 produces a peak blow the resonance energy. It is likely that experiments on the cross sections at 90° C.M.S. show a peak slightly below the resonance energy, and to explain this the peak in Re m_S is important. From comparison of Fig. 1 with Fig. 2, we observe that the non-spin-flip amplitudes are much smaller than the spin flip ones in the real parts.

In terms of the partial wave amplitudes, the differential cross section in C.M.S. is written as

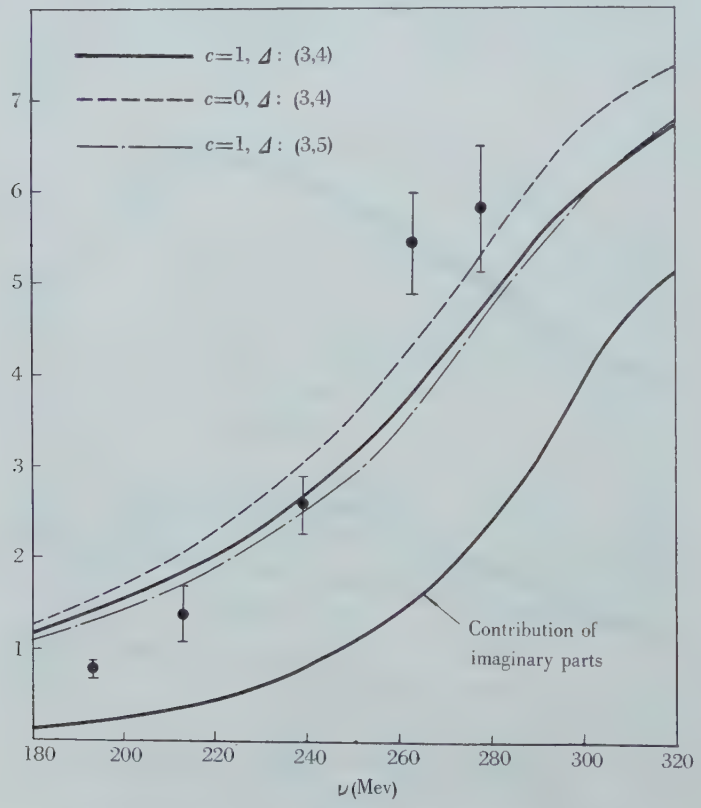

Fig. 3. Cross sections at 90° C.M.S. in units of $(e^2/4\pi M)^2$/sterad

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{2}|e_N|^2+\tfrac{1}{2}|m_N-e_N'|^2+\tfrac{1}{4}|e_S|^2+\tfrac{1}{4}|m_S|^2+\tfrac{1}{8}|e_S-c_1|^2+\tfrac{1}{8}|m_S-c_2|^2$$
$$+\tfrac{1}{2}\mathrm{Re}\{m_S^*(e_S+c_1)+c_2^*e_S+4e_N^*m_N\}\cos\theta_c$$
$$+\{\tfrac{1}{8}|e_S+c_1|^2+\tfrac{1}{8}|m_S+c_2|^2+\tfrac{1}{4}|c_1|^2+\tfrac{1}{4}|c_2|^2-\tfrac{1}{4}|e_S-c_1|^2 \qquad (3\cdot7)$$
$$-\tfrac{1}{4}|m_S-c_2|^2+\tfrac{1}{2}|e_N|^2+\tfrac{1}{2}|m_N|^2-\tfrac{3}{2}|e_N'|^2+3\mathrm{Re}\,m_N^*e_N'\}\cos^2\theta_c$$
$$+\tfrac{1}{2}\mathrm{Re}(c_1^*c_2+4e_N^*e_N')\cos^3\theta_c+2|e_N'|^2\cos^4\theta_c\,.$$

Using this, we calculate the differential cross section at 90° C.M.S. as a function of energy, and angular distributions at 200, 260 and 320 Mev. The results are shown in Fig. 3 and Fig. 4. In Fig. 3 we also show experimental data given by Yamagata et al.[9] From Fig. 3 we observe that the two values of Δ (3·4) and (3·5) yield almost same results. This means that real parts of the non-spin-flip amplitudes have been made so small already by the choice of (3·4) that their contribution to the cross section is very small. Indeed main contributions to the results in Fig. 3 come from $\mathrm{Re}\,m_S$ and the imaginary parts. In Fig. 4, main contributions to the results come from the real parts of the amplitudes at 200 and 260 Mev. We observe that, at these energies, the cross section is larger in backward directions than in forward. This is due to the opposite signs of $\mathrm{Re}\,m_S$ and $\mathrm{Re}(e_S+c_1)$. For $c=1$, the largeness of $\mathrm{Re}\,c_2$ makes the term in $\cos^2\theta_c$

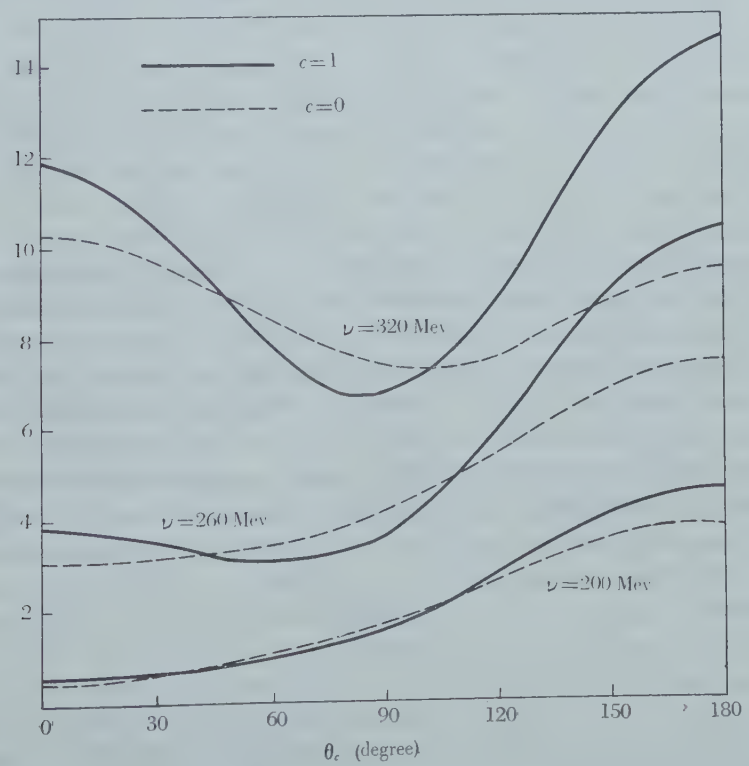

Fig. 4. Angular distributions in units of $(e^2/4\pi M)^2$/sterad calculated with $\Delta=-0.42\ \mu^{-2}(e^2/4\pi M)$

in (3·7) large, and results in the angular distributions rather indented in the neighbourhood of 90°. At 320 Mev, main contributions come from the imaginary parts. Dropping the real parts entirely, (3·7) becomes

$$d\sigma/d\Omega = \begin{cases} \frac{1}{2}\mathrm{Im}\, e_{1/2}^{\,2} + \frac{3}{4}\mathrm{Im}\, m_{1/2}^{\,2} + \frac{5}{4}\mathrm{Im}\, m_{3/2}^{\,2} \cos^2\theta_c & \text{for } c=1, \\ \frac{1}{2}\mathrm{Im}\, e_{1/2}^{\,2} + \frac{7}{8}\mathrm{Im}\, m_{3/2}^{\,2} + \frac{1}{2}\mathrm{Im}\, e_{1/2} m_{3/2} \cos\theta_c + \frac{3}{8}\mathrm{Im}\, m_{3/2}^{\,2} \cos^2\theta_c & \text{for } c=0. \end{cases}$$

These expressions explain general features of the results at 320 Mev in Fig. 4.

Yamagata et al. have given a few data on the cross sections at 70° and 130° also. From these data, the ratio of the cross section at 130° to the one at 90° is estimated to be about 1.7 at 200 Mev and about 2 at 240 Mev. Our results are in good agreement with these values for $c=1$, but the agreement is rather bad for $c=0$. The data at 70° shows a cross section at this angle larger than at 90° at 226 Mev. This cannot be understood by our calculation.

§ 4. Discussions

We consider the difference between the cross section at 90° and contribution of the imaginary parts to it. Experimentally, this is considerably smaller than $(e^2/4\pi M)^2$ at energies near 200 Mev, then increases with energy, and reaches a maximum or a plateau at about 270 Mev. These features are explained roughly by the smallness of $\mathrm{Re}\, e_N$ and $\mathrm{Re}\, m_N$ and the behavior of $\mathrm{Re}\, m_N$ as a function of energy mentioned in the preceding section. However, there remain considerable discrepancies between theory and experiment, namely, theory predicts too large cross sections in the neighborhood of 200 Mev, and rather small ones at energies near 270 Mev, as compared with experiments. In the following, we will look for a possible origin of these discrepancies. First, errors in the data on photopion production come into mind. Indeed, if we consider the cross section at a particular energy, the errors produce a considerable ambiguity in the calculated value of it. However, in view of the rather systematic character of the above discrepancies, it is unlikely that these discrepancies are entirely attributable to these errors over the whole energy range considered. Second, we have neglected imaginary parts of the partial wave amplitudes in the high energy region. Nowadays, data on photon-proton collisions in Bev region are not available, and even if they become available, the partial wave analysis of them will be extremely difficult. Therefore there is no possibility of obtaining experimental values of the imaginary parts at such high energies. However, contributions of such a high energy region to the imaginary part integrals will be almost independent of energy ν if $\nu \lesssim 300$ Mev. Therefore the effect of including the high energy contributions is that a constant is added to each of the imaginary part integrals. We have attempted to fit the energy dependence of the 90° cross section to experiments by adjusting the above constants, and found that the fit cannot be achieved unless some of the constants take unreasonably large values. Thirdly, we have neglected the amplitudes for photopion other than E_{α}^{+}, M_{33} and E_{33}. We have calculated the amplitude $M_{\alpha}(1/2)$ (See (A II–12).), using the meson theory to the lowest order perturbation approximation, and estimated

Im $m_{1/2}$. We have, however, found that the inclusion of this only makes the 90° cross section slightly smaller over the whole energy range considered, making the discrepancies in the neighborhood of 270 Mev larger, and that, even at energies near 200 Mev, the decrement is too small to explain the discrepancies. Finally, we have used the approximate expressions (3·6) for c_1 and e_S'. It should be noted that these expressions are correct only to the first order in ν/M, and that changes in c_1 and e_S' produce the one in Re m_S, the largest real part. A quantity of the order $(\nu/M)^2$ cannot be disregarded even in the neighborhood of 200 Mev, when this is multiplied by a rather large numerical factor, and interferes with Re m_S. To obtain the expressions for c_1 and e_S' exact within the approximation to which the partial wave amplitudes other than those considered so far are all neglected, we must derive two new dispersion relations for the spin-flip amplitudes. As was mentioned in § 2, these dispersion relations will contain new adjustable parameters of the same kind as Δ. It is not unexpected that the use of the expressions for c_1 and e_S' thus obtained, on adjusting the parameters appropriately, remove the discrepancies mentioned above.

Finally we will examine the validity of the dipole model. Following the assumptions of this model, we replace the partial wave amplitudes other than $e_{1/2}$ and $m_{3/2}$ by their respective expressions given by the low energy theorem. Thus we have, in addition to (3·6),

$$2e_N+e_S=\frac{e^2}{4\pi M}\left\{-2+\frac{\nu}{M}(3-g_p)\right\},$$

$$m_N-m_S=-\frac{e^2}{4\pi M}\frac{\nu}{M}(g_p^2+\tfrac{1}{2}),$$

$$e_N'=\frac{e^2}{4\pi M}\cdot\frac{\nu}{2M}, \quad c_2=0.$$

Then we calculate real parts of e_N, e_S, m_N and m_S, using only two dispersion relations (DI) and (DIV), neglecting Im c_2. The results are plotted in Fig. 1 and Fig. 2. We observe that individual amplitude in the dipole model is considerably larger than that calculated in § 3, while relative magnitudes of the amplitudes, as well as their energy dependences, are not so different from the results in § 3. Therefore the dipole model will predict energy and angular dependences of the cross section similar to the results for $c=0$ in the preceding section. But it predicts cross sections much larger than the results in the preceding section, making the agreement with experiment worse.

In conclusion, the authors express their thanks to Professor K. Nakabayasi for his kind interest taken in this work.

Appendix I

Calculation of contributions of the unphysical region

Here we concern ourselves with the derivation of explicit expressions for $I^{(\lambda)}(q^2)$ defined in (2·6a, b). The contributions to $A^{(\lambda)}(\nu, \pi, \boldsymbol{q})$ of the intermediate free proton

state, which are separated off from the sum (2·5a, b), are given by

$$\begin{aligned} A^{(1,2)}(\nu,\pi,\boldsymbol{q}) &= -\frac{(2\pi)^6 q^{-2}}{4}\left(\frac{M}{E_q}\right)(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu,\pi,\boldsymbol{q})\cdot\boldsymbol{j}(0)\,|M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}) \\ &\quad\times(M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_{1,2}(\nu,-\pi,\boldsymbol{q})\cdot\boldsymbol{j}(0)\,|-\boldsymbol{q})\,\delta(\nu-\nu_b), \\ A^{(3)}(\nu,\pi,\boldsymbol{q}) &= -\frac{(2\pi)^6 q^{-2}}{4}\left(\frac{M}{E_q}\right)(\boldsymbol{q}|\mathbf{e}_1(\nu,\pi,\boldsymbol{q})\cdot\boldsymbol{j}(0)\,|M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}) \\ &\quad\times(M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_2(\nu,-\pi,\boldsymbol{q})\cdot\boldsymbol{j}(0)\,|-\boldsymbol{q})\,\delta(\nu-\nu_b), \end{aligned} \tag{AI–I}$$

where $\mathbf{Q}$ depends on ν through eq. (2·1). Calculation can be performed with the aid of the well-known expression for the matrix elements of the current operator referred to the free proton states:

$$(\boldsymbol{p}'|j_\mu(0)\,|\boldsymbol{p}) = (2\pi)^{-3}\bar{u}(\boldsymbol{p}')\left\{ie\gamma_\mu F[(\Delta p)^2]-\frac{ie}{2M}G[(\Delta p)^2]\sigma_{\mu\nu}(\Delta p)_\nu\right\}u(\boldsymbol{p}). \tag{AI–2}$$

Here $\bar{u}(\boldsymbol{p}')$ and $u(\boldsymbol{p})$ denote the Dirac spinors which are normalized by the condition $\bar{u}(\boldsymbol{p}')\beta u(\boldsymbol{p})=\boldsymbol{l}$, where $\boldsymbol{l}$ is a unit matrix in spin space. Besides, $(\Delta p)_\nu=p_\nu'-p_\nu$; $p_\nu=(\boldsymbol{p},E_p)$, $p_\nu'=(\boldsymbol{p}',E_{p'})$, and $F(0)=1$, $G(0)=g_p$, the proton magnetic moment in units of the nuclear magneton.

(AI–2) is now substituted in (AI–I) with $\{\boldsymbol{p}',\boldsymbol{p}\}$ identified in turn with $\{\boldsymbol{q},-\mathbf{Q}_b\}$, $\{-\mathbf{Q}_b,-\boldsymbol{q}\}$, $\{\boldsymbol{q},-\mathbf{Q}_b\}$, $\{-\mathbf{Q}_b,-\boldsymbol{q}\}$, where $\mathbf{Q}_b=i(qM/E_q)\pi$, *an imaginary momentum*. Regardless of the assignment of $\{\boldsymbol{p}',\boldsymbol{p}\}$, it is found that

$$(\Delta p)^2=0. \tag{AI–3}$$

Therefore, $F[(\Delta p)^2]$, $G[(\Delta p)^2]$ in (AI–2) may be replaced by 1, g_p respectively. This shows that the contributions under consideration can be expressed in terms of the proton mass charge and static magnetic moment only.

After rather lengthy calculations we find

$$\begin{aligned} A^{(1)}(\nu,\pi,\boldsymbol{q}) &= -\frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)\left\{1+\left(\frac{M}{E_q}\right)^2+\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}_b}{ME_q}\right\}\delta(\nu-\nu_b), \\ A^{(2)}(\nu,\pi,\boldsymbol{q}) &= -\frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)g_p{}^2\left\{\left(\frac{q}{E_q}\right)^2-\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}_b}{ME_q}\right\}\delta(\nu-\nu_b), \\ A^{(3)}(\nu,\pi,\boldsymbol{q}) &= \frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)\frac{g_p}{M}\left\{i\boldsymbol{\sigma}\cdot\mathbf{Q}_b-\frac{M}{E_q}i\boldsymbol{\sigma}\cdot\boldsymbol{q}\right\}\delta(\nu-\nu_b). \end{aligned} \tag{AI–4}$$

On comparison with (2·6a, b) we arrive at the final results:

$$\begin{aligned} &\left\{\begin{aligned} \Gamma_N^{(1)}(q^2) &= -\frac{e^2}{8\pi}\frac{E_q{}^2+M^2}{E_q{}^2}, \\ \Gamma_S^{(1)}(q^2) &= -\frac{e^2}{8\pi}\frac{1}{ME_q}, \end{aligned}\right. \qquad \left\{\begin{aligned} \Gamma_N^{(2)}(q^2) &= -\frac{(eg_p)^2}{8\pi}\left(\frac{q}{E_q}\right)^2, \\ \Gamma_S^{(2)}(q^2) &= \frac{(eg_p)^2}{8\pi}\frac{1}{ME_q}, \end{aligned}\right. \\ &\Gamma_+^{(3)}(q^2)=\frac{e^2}{8\pi}\frac{g_p}{M}, \qquad \Gamma_-^{(3)}(q^2)=-\frac{e^2}{8\pi}\frac{g_p}{E_q}. \end{aligned} \tag{AI–5}$$

Appendix II

Properties of the partial wave amplitudes resulting from the invariance under time reflection and the unitarity of S matrix

Throughout this appendix, we take the barycentric system, and consider the total energy to be fixed to a definite value. Let $|\boldsymbol{\kappa}, \mathbf{e}, \alpha)$ denote the state vector representing the linear momentum state of a free photon propagating in the direction of unit vector $\boldsymbol{\kappa}$ with polarization $\mathbf{e}$ and a free proton in the spin state α, and let $|j, m, El)$ $(|j, m, Ml))$ represent the state with total angular momentum j and its z-component m in which the photon is in the electric (magnetic) multipole state of order l. Inner products of these state vectors are written as

$$\begin{aligned}(j, m, El|\boldsymbol{\kappa}, \mathbf{e}, \alpha) &= \sum_{m'=\pm 1/2} C_{l,\,1/2}(j, m\,;\, m-m', m') \\ &\quad \times (\boldsymbol{\kappa}\times\mathbf{e})\cdot X^*_{l,m-m'}(\boldsymbol{\kappa})\,(\chi^*_{m'}\chi_\alpha), \\ (j, m, Ml|\boldsymbol{\kappa}, \mathbf{e}, \alpha) &= i\sum_{m'=\pm 1/2} C_{l,\,1/2}(j, m\,;\, m-m', m') \\ &\quad \times \mathbf{e}\cdot X^*_{l,m-m'}(\boldsymbol{\kappa})\,(\chi^*_{m'}\chi_\alpha),\end{aligned} \tag{AII–1}$$

where χ_m denotes the spin eigenstate with magnetic quantum number m, $C_{l,\,1/2}(j, m\,;\, m-m', m')$ the Clebsch-Gordon coefficient, and $X_{l,m}(\boldsymbol{\kappa})$ the vector spherical harmonics.[10] One can easily verify the orthogonality condition

$$\begin{aligned}\int d\Omega_{\boldsymbol{\kappa}} \sum_{\alpha,\mathbf{e}} (j, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha)\,(\boldsymbol{\kappa}, \mathbf{e}, \alpha|j', m', P'l') \\ = \delta_{jj'}\delta_{mm'}\delta_{ll'}\delta_{PP'},\end{aligned} \tag{AII–2}$$

where the integration is to be performed over all directions of κ, and P and P' represent either of E and M. Owing to the invariance under spatial rotation and reflection, the scattering amplitude for the transition from state $|\boldsymbol{\kappa}, \mathbf{e}, \alpha)$ to $|\boldsymbol{\kappa}', \mathbf{e}', \beta)$ is written, in terms of the partial wave amplitudes $a_j(P'l', Pl)$ in the following form:

$$\begin{aligned}&f(\boldsymbol{\kappa}', \mathbf{e}', \beta\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &= 4\pi\sum_{l=1}^{\infty}\Big\{\sum_{j=l\pm 1/2}\sum_{m=-j}^{j}\sum_{l'} a_j(Pl, Pl) \\ &\quad \times (\boldsymbol{\kappa}', \mathbf{e}', \beta|j, m, Pl)\,(j, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &\quad + \sum_{l'=l\pm 1}\sum_{m=-(l+l')/2}^{(l+l')/2}\sum_{P''=P'} a_{(l+l')/2}(P'l', Pl) \\ &\quad \times (\boldsymbol{\kappa}', \mathbf{e}', \beta|(l+l')/2, m, P'l')\,((l+l')/2, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha)\Big\}.\end{aligned} \tag{AII–3}$$

Now, from the invariance under time reflection, it follows that

$$\begin{aligned}&f(\boldsymbol{\kappa}', \mathbf{e}', \beta\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &= \sum_{\alpha'\beta'}\sum_{\mathbf{e}''\mathbf{e}'''} (\chi^*_\beta\sigma_y\chi^*_{\beta'})\,(\mathbf{e}'\cdot\mathbf{e}''') \\ &\quad \times f(-\boldsymbol{\kappa}, \mathbf{e}'', \alpha'\,;\, -\boldsymbol{\kappa}', \mathbf{e}''', \beta')\,(\mathbf{e}''\cdot\mathbf{e})\,(\chi_{\alpha'}\sigma_y\chi_\alpha).\end{aligned} \tag{AII–4}$$

On the other hand, it is shown by an elementary calculation that from (AII–1) follows

$$\sum_{\beta' e'''} (\chi_\beta^* \sigma_y \chi_{\beta'}^*)\,(\mathbf{e}'\cdot\mathbf{e}''')\,(j,\ m,\ Pl|-\boldsymbol{\kappa}',\ \mathbf{e}''',\ \beta') = -i(-1)^{j+m}(\boldsymbol{\kappa}',\ \mathbf{e}',\ \beta|j,\ m,\ Pl)\,. \tag{AII–5}$$

Substituting (AII–3) in (AII–4), and using (AII–5) and its complex conjugate, we obtain

$$a_j(Ej\pm 1/2,\ Mj\mp 1/2) = a_j(Mj\mp 1/2,\ Ej\pm 1/2)\,. \tag{AII–6}$$

Next, we consider a system of a free pion and a free nucleon with total charge unity. Let the state of this system, in which the pion propagates in the direction of unit vector $\boldsymbol{l}$ and the nucleon is in the spin state α, be represented by the state vector $|\boldsymbol{l},\ Q,\ \alpha)$, and let the state in which the pion has orbital angular momentum l and total angular momentum and its x-component are j and m respectively be represented by $|j,\ m,\ Ql)$, where Q is charge of the pion. Then, similarly to (AII–1), we have

$$(j,\ m,\ Ql|\boldsymbol{l},\ Q,\ \alpha) = \sum_{m'=\pm 1/2} C_{l,\ 1/2}(j,\ m:\ m-m',\ m')\times Y_{l,m-m'}^*(\boldsymbol{l})\,(\chi_{m'}^*,\ \chi_\alpha)\,, \tag{AII–7}$$

which satisfies an orthogonality condition analogous to (AII–2). The photopion production amplitude for the transition from $|\boldsymbol{\kappa},\ \mathbf{e},\ \alpha)$ to $|\boldsymbol{l},\ Q,\ \beta)$ is written in the form

$$\begin{aligned} &g(\boldsymbol{l},\ Q,\ \beta\,;\ \boldsymbol{\kappa},\ \mathbf{e},\ \alpha) \\ &\quad = 4\pi \sum_{l=1}^{\infty} \Big\{ \sum_{j=l\pm 1/2} \sum_{m=-j}^{j} b_j(Ql,\ Ml) \\ &\qquad \times (\boldsymbol{l},\ Q,\ \beta|j,\ m,\ Ql)\,(j,\ m,\ Ml|\boldsymbol{\kappa},\ \mathbf{e},\ \alpha) \\ &\quad + \sum_{l'=l\pm 1} \sum_{m=-(l+l')/2}^{(l+l')/2} b_{(l+l')/2}(Ql',\ El) \\ &\qquad \times (\boldsymbol{l},\ Q,\ \beta|(l+l')/2,\ m,\ Ql')\,((l+l')/2,\ m,\ El|\boldsymbol{\kappa},\ \mathbf{e},\ \alpha)\Big\}\,. \end{aligned} \tag{AII–8}$$

Now, it follows from the unitarity of S matrix that

$$\begin{aligned} &f(\boldsymbol{\kappa}',\ \mathbf{e}',\ \beta:\ \boldsymbol{\kappa},\ \mathbf{e},\ \alpha) - f^*(\boldsymbol{\kappa},\ \mathbf{e},\ \alpha\,;\ \boldsymbol{\kappa}',\ \mathbf{e}',\ \beta) \\ &\quad = \frac{i\nu_c}{2\pi}\int d\Omega_l \sum_{\beta' Q} g^*(\boldsymbol{l},\ Q,\ \beta'\,;\ \boldsymbol{\kappa}',\ \mathbf{e}'\ \beta)\,g(\boldsymbol{l},\ Q,\ \beta'\,;\ \boldsymbol{\kappa},\ \mathbf{e},\ \alpha) \end{aligned} \tag{AII–9}$$

provided that multiple photopion production is disregarded and terms of the order higher than e^2 are neglected. Substituting (AII–3) and (AII–8) in this relation, and carrying out the integration with the aid of the orthogonality condition, we have

$$\begin{aligned} &\mathrm{Im}\, a_{l\pm 1/2}(El,\ El) = \nu_c \sum_Q |b_{l\pm 1/2}(Ql\pm 1,\ El)|^2, \\ &\mathrm{Im}\, a_{l\pm 1/2}(Ml,\ Ml) = \nu_c \sum_Q |b_{l\pm 1/2}(Ql,\ Ml)|^2, \end{aligned} \tag{AII–10}$$

$$\operatorname{Im} a_{l\pm1/2}(Ml\pm1,\ El) = \nu_c \sum_Q b_{l\pm1/2}(Ql\pm1,\ El) \times b^*_{l\pm1/2}(Ql\pm1,\ Ml\pm1).$$

In the text we have made the following abbreviations:

$$\begin{aligned}
&e_j = a_j(E1,\ E1), \quad m_j = a_j(M1,\ M1), \quad e_j' = a_j(E2,\ E2),\\
&c_1 = 2\sqrt{3}\, a_{3/2}(E1,\ M2) = 2\sqrt{3}\, a_{3/2}(M2,\ E1),\\
&c_2 = 2\sqrt{3}\, a_{3/2}(E2,\ M1) = 2\sqrt{3}\, a_{3/2}(M1,\ E2),\\
&e_N = e_{3/2} + \tfrac{1}{2} e_{1/2}, \qquad e_S = e_{3/2} - e_{1/2},\\
&m_N = m_{3/2} + \tfrac{1}{2} m_{1/2}, \qquad m_S = m_{3/2} - m_{1/2},\\
&e_N' = e_{3/2}' + \tfrac{3}{2} e_{5/2}', \qquad e_S' = e_{5/2}' - e_{3/2}'.
\end{aligned} \tag{AII–11}$$

If all other amplitudes are neglected, (AII–3) yields (2·19) in the text. As for photo-pion production, the partial wave amplitudes used by Watson et al.[6] are related to ours through

$$\begin{aligned}
&E_d^q = -\frac{1}{\sqrt{2}} b_{1/2}(Q0,\ E1), \quad E_q^q = -\sqrt{6}\, b_{3/2}(Q1,\ E2),\\
&M_d^q(j) = -\frac{1}{\sqrt{2}} b_j(Q1,\ M1) \quad (j = 1/2,\ 3/2).
\end{aligned} \tag{AII–12}$$

If production of P-wave pion into all states other than the (3–3) state and production of neutral S-wave pion are neglected, eqs. (3·1) in the text follow from (AII–10), definitions of E_{33} and M_{33} being found in reference 6).

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On the Decay Interactions of Strange Particles

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Under the assumption that the Konopinski-Uhlenbeck interaction works as an origin of weak interactions in which the existence of strange particles should be taken into account, the decay events $K^{\pm}_{\mu 3}$, $K^{\pm}_{e3}$, $K^{0}_{\mu 3}$ and K^{0}_{e3} are investigated. The available data seem to be favourable to our predictions, though it is premature to draw any definite conclusions. Some remarks are also made about the β-decay of Hyperons and the K-meson decay into two leptons.

§ 1. Introduction

Since the discovery of the first V-event by Rochester and Butler in 1947, the existence of a number of particles with curious properties, the so-called strange particles, has been established. Various properties of these particles, especially their copious productions and their long lifetimes were described consistently in terms of the beautiful scheme of Nishijima-Gell-Mann-Pais[1),2)] in a somewhat qualitative manner, and now it is generally accepted that, besides the strong and the electromagnetic interactions, there exists a group of interactions called weak interactions. In spite of a wide variety of the nature of particles which participate in these interactions, we find a remarkable fact that the strength of all these couplings is almost of the same order of magnitude[3)] ($\sim 10^{-13}$ in the unit $\hbar=c=r_0$ ($\sim 10^{-13}$ cm) $=1$). It is reasonable, therefore, to assume that all these weak interactions originate in a single interaction which has a universal character to some extent. From such a point of view the weak Fermi interaction (F-int.) seems to be of special interest, since it is the interaction between two pairs of Fermions and we may possibly derive from it, in collaboration with the strong interactions, the other kinds of interactions, such as the weak Boson-Fermion and the weak Boson-Boson interactions.*

Many investigations have been made under the assumption that the weak F-int. acts as an origin of various weak interactions.[4)–7)] It seems to the present author, however, that such a hypothesis neither has any *a priori* reason nor is so efficient in explaining various phenomena. Indeed, it has been pointed out by Ōneda and Wakasa[5)] that, in order to get a unified description of all phenomena involving strange particles as well as familiar nucleons, pions and leptons in terms of the strong, the electromagnetic and the

* In this picture of the weak interactions, the concept of the F-int. must be extended so as to include four-baryon processes as well as the well-known neutrino processes.

weak *F*-int., it is necessary to add a more complicated character to the *F*-int., e. g., the process dependence of the coupling type. Such a restriction is rather artificial and *ad hoc*, and is unsatisfactory from the view-point that the primary interaction must be of a universal character.

In the analysis of the decay processes of charged K_{e3} and $K_{\mu 3}$ by Furuichi et al.[4),5)] similar situations did occur. According to their calculations the energy spectrum of the secondary electrons (muons) has a large distribution in the high energy region for most of the coupling types (S, V, A, P) of the *F*-int. both for scalar and vector *K*-mesons. They also found that, for the tensor coupling in the case of spin 0 *K*-meson, there are two peaks in the electron (muon) distribution, one being in the low energy region and another in the high energy region. The experimental results[9)] give a large distribution of electrons (muons) in rather low energy regions. Hence, in order to get the best fit to the Rochester data, appropriate combinations of $S(T)$- and $S(P)$-couplings for K_{e3} and $S(T)$- and $S(A)$-couplings (or $V(P)$- and $V(T)$-couplings) for $K_{\mu 3}$ have been required.[5)]

Here it is quite interesting to observe the following points. Among various events that are realized by the weak interactions, we have, on the one hand, many phenomena which can be settled without considering the existence of strange particles. The β-decay of nucleons and the μ^--capture, for example, belong to this class. On the other hand, there are also phenomena in which the existence of strange particles should inevitably be taken into account. The decays of Hyperons and *K*-mesons are contained in this class.* In the first class of phenomena the weak *F*-int. has gained a brilliant success. Yet, in view of the circumstances mentioned above, it seems highly probable that the interactions which have a large applicability to the first class of phenomena are not necessarily responsible to, or at least do not play an important role in, the second class of phenomena. We may assume as well that some new kind of universal interactions other than the usual weak *F*-int. plays an essential role in the weak interactions in which particles with strange properties participate. Along this line of thought we are led to expect that the experimental features of phenomena belonging to the second class will be explained in terms of some universal interaction of a new kind without any further artificial restriction.

In this connection the interaction proposed by Konopinski and Uhlenbeck[10)] (*KU*-int.) about twenty years ago is of particular interest. It is the one-point interaction as is the *F*-int. Furthermore the *KU*-int. contains a time derivative of the wave function of the neutrino, hence it is, as is well known, more favourable to the emission of high energy neutrinos than the *F*-int. and this situation seems to be of practical value for describing a large distribution of low energy electrons (muons) in charged K_{e3} ($K_{\mu 3}$)-decay.

In order to ascertain the above conjecture, we shall examine in this paper the validity of the *KU*-int. as an origin of weak interactions in which the existence of strange particles

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* Strictly speaking, such a classification is not possible because we must take into account all particles that exist in nature. The distinction between these two classes of phenomena may be somewhat obscure, but it has to be understood only in the phenomenological sense.

should be taken into account. In § 2 we look for the most adequate phenomena to test our proposal and give general considerations of their treatment. Using the procedure there explained, we examine in § 3 the $K_{\mu 3}^{\pm}$- and $K_{e3}^{\pm}$-decay, and the energy spectra of the secondary muons and electrons are compared with those based on the *F*-int. In order to clearly show the difference between the *KU*-int. and the *F*-int., we next consider in § 4 the energy distribution of neutrinos in the K_{e3}^{0} ($K_{\mu 3}^{0}$)-decay. § 5 is devoted to a discussion of the *K*-meson decay into two leptons and the β-decay of Hyperons.

§ 2. General considerations

As has been mentioned in the previous section, we have assumed that the *KU*-int. plays an essential role in the second class of phenomena. In those events which contain no strange particles both in the initial and final states, the *KU*-int., if it works at all, operates only indirectly, strange particles appearing only in intermediate states. Further, we give a preferential treatment for neutrinos in the *KU*-int. Therefore, the most adequate phenomena to test our proposal are those in which we have strange particles and neutrinos either in the initial or in the final state or in both of them. In this paper we are mainly interested in the following phenomena :

$$K_{\mu 3}^{\pm} \to \mu^{\pm} + \nu + \pi^{\circ}, \quad K_{e3}^{\pm} \to e^{\pm} + \nu + \pi^{\circ}, \tag{2·1}$$

$$K_{\mu 3}^{0} \to \mu^{\pm} + \nu + \pi^{\mp}, \quad K_{e3}^{0} \to e^{\pm} + \nu + \pi^{\mp}, \tag{2·2}$$

$$K_{\mu 2}^{\pm} \to \mu^{\pm} + \nu, \tag{2·3}$$

$$\Lambda^{\circ}(\Sigma^{\circ}) \to P + e + \nu. \tag{2·4}$$

Processes (2·1)—(2·3) are realized by the cooperation of the strong and the weak *KU*-int., while the process (2·4) can be realized by the direct *KU*-int. among four Fermions. For strong interactions it is desirable to avoid the perturbational treatment as far as possible, hence we adopt the procedure used by Ōneda and Wakasa[8] and by Furuichi et al.[4] in the analysis based on the *F*-int.

The starting points of our analysis are :

1. The transition matrix elements are Lorentz invariant.
2. The decay of the *K*-meson with lepton secondaries is realized by the collaboration of the strong and the weak *KU*-int. which acts only once in the whole course of the decay process (Fig. 1).

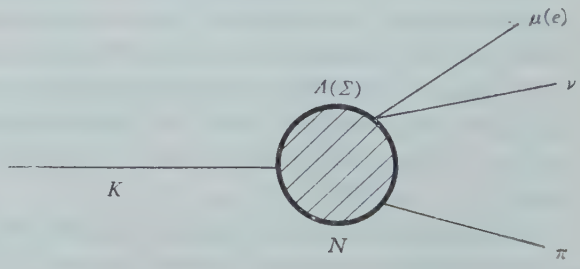

Fig. 1a. Diagram for the process (2·1) or (2·2).

In contrast to the five coupling types of the weak *F*-int., there are three coupling types in the *KU*-int. which involve the time derivative of the neutrino wave function,

namely*

vector : $(\bar{u}\gamma_\mu w)\,(\bar{\psi}\partial_\mu\varphi)$, (2·5)

tensor : $(\bar{u}\sigma_{\mu\nu} w)\,(\bar{\psi}\gamma_\mu\partial_\nu\varphi)$, (2·6)

pseudovector : $(\bar{u}\gamma_5\gamma_\mu w)\,(\bar{\psi}\gamma_5\partial_\mu\varphi)$, (2·7)

where u, w and ψ are wave functions of three Fermions** and φ is the wave function of the neutrino. Then, from the requirement of the Lorentz invariance, the transition matrix for the processes (2·1) and (2·2) can be written in a very general form :

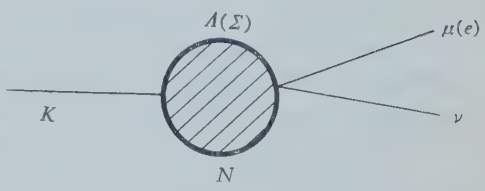

Fig. 1b. Diagram for the process (2·3).

$$M=(2\pi)^4\delta^4(k-p-q-l)\,\bar{\psi}(p)\,O_j\varphi(q)\,\phi^*(l)\,\Phi_i(k)\,A^{i(j)}, \qquad (2\cdot8)$$

where $\psi(p)$, $\varphi(q)$, $\phi(l)$ and $\Phi_i(k)$ are the wave functions of electron (muon), neutrino, pion and K-meson respectively. The O_j's are operators corresponding to the coupling types of (2·5), (2·6) and (2·7). The $A^{i(j)}$'s are some functions of the four-momenta p, q and k with an appropriate transformation property corresponding to O_j and the spin of the K-meson. They are the contributions from all the possible virtual states, which involve the four-momenta of electron (muon) and neutrino only in the combination $p+q$ since the *KU*-int. is a one-point interaction as is the *F*-int.*** For the explicit expressions of $A^{i(j)}$, we make the same assumption as in reference 4), i. e., the momentum dependence of A is restricted only to the lowest order. At present we have no adequate method of discussing the validity of this approximation. But it must be kept in mind that in this paper we are not interested in the absolute value of the matrix elements, but only in the

Table 1. Forms of $A^{i(j)}$ for processes (2·1) and (2·2)

K-meson \ coupling type	vector q_μ	tensor $\gamma_\mu q_\nu-\gamma_\nu q_\mu$	pseudovector $\gamma_5 q_\mu$
scalar Φ	forbidden	$G^{S(T)}\,k_\mu(p+q)_\nu$	$G_1{}^{S(A)}\,k_\mu{}^\dagger$ $G_2{}^{S(A)}\,(p+q)_\mu{}^\dagger$ $G_3{}^{S(A)}\,l_\mu{}^\dagger$
vector Φ_λ	$G^{V(V)}\epsilon_{\lambda\mu\nu\rho}k_\nu(p+q)_\rho$	$G_1{}^{V(T)}\,\delta_{\mu\lambda}\,k_\nu{}^\dagger$ $G_2{}^{V(T)}\,\delta_{\mu\lambda}(p+q)_\nu{}^\dagger$ $G_3{}^{V(T)}\,\delta_{\mu\lambda}\,l_\nu{}^\dagger$	$G^{V(A)}\,\delta_{\lambda\mu}$

† Any of these three can be expressed as a linear combination of the other two, but we give all of them explicitly for convenience' sake.

* We use the *KU*-int. in its original sense,[10] that is, it involves a time derivative of the neutrino wave function only. Of course, any other coupling type cannot be excluded. But we do not intend to investigate into all cases but confine ourselves to (2·5)—(2·7) as a possible example.

** We assume that all Fermions are particles with spin 1/2.

*** We omit some types of higher order radiative corrections (see reference 4)).

shape of the distribution of secondary electrons (muons) or neutrinos. In Table 1 we record the possible forms of $A^{i(j)}$'s for the processes (2·1) and (2·2). Owing to the vanishing of the neutrino mass we cannot discuss the parity of the K-meson, so we here consider the K-meson as a scalar or vector particle. For the pseudoscalar or pseudovector K-meson we have only to take the A with opposite parity.

It should be noted that, owing to the appearance of q in O_j and to the fact that $q_\mu{}^2=0$, the four-momenta p and q do not necessarily appear in the symmetrical combination $p+q$ in the matrix elements (2·8). This differs from the case of the F-int., where the matrix elements always contain p and q in such a combination.

By using these matrix elements the transition probability is easily obtained and will be given in the following two sections.

§ 3. Energy spectra of electrons (muons) in charged K_{e3} ($K_{\mu 3}$)-decay

In this section we investigate the energy spectra of the secondary electron and muon in the decay processes

$$K^{\pm}_{e3}\to e^{\pm}+\nu+\pi^0, \tag{3·1}$$

$$K^{\pm}_{\mu 3}\to \mu^{\pm}+\nu+\pi^0. \tag{3·2}$$

In general the transition matrix element is given by an arbitrary linear combination of various coupling types tabulated in Table 1, but we here give the result only for each coupling for convenience' sake.

The calculated spectra are: ($\hbar=c=1$)

for scalar K-meson

$$\left.\begin{aligned}
&P_1^{S(A)}(E)=(G_1^{S(A)2}/16\pi^3)M^5(W-E)^4(MI_5-I_4),\\
&P_2^{S(A)}(E)=(G_2^{S(A)2}/16\pi^3)M^3(W-E)^4(M^3I_5-3M^2I_4+3MI_3-I_2),\\
&P_3^{S(A)}(E)=(G_3^{S(A)2}/16\pi^3)M^3(W-E)^4(MI_3-I_2),\\
&P^{S(T)}(E)=(G^{S(T)2}/16\pi^3)M^5(W-E)^4[I_2-(3M-2E)I_3\\
&\qquad+(3M^2+m^2-4ME)I_4-(M^2+m^2-2ME)MI_5],
\end{aligned}\right\} \tag{3·3}$$

and for vector K-meson

$$\left.\begin{aligned}
&P^{V(A)}(E)=(G^{V(A)2}/48\pi^3)M^3(W-E)^4(MI_5-I_4),\\
&P^{V(V)}(E)=(G^{V(V)2}/48\pi^3)M^5(W-E)^4[I_2-(3M-2E)I_3\\
&\qquad+(3M^2+m^2-4ME)I_4-(M^2+m^2-2ME)MI_5],\\
&P_1^{V(T)}(E)=(G_1^{V(T)2}/24\pi^3)M^5(W-E)^4(MI_3-I_4),\\
&P_2^{V(T)}(E)=(G_2^{V(T)2}/48\pi^3)M^3(W-E)^4[(M^2+2ME-m^2)MI_5\\
&\qquad-(3M^2+4ME-m^2)I_4+(3M+2E)I_3-I_2],
\end{aligned}\right\} \tag{3·4}$$

$$P_3^{V(T)}(E) = (G_3^{V(T)2}/48\pi^3)\, M^3 (W-E)^4 [-I_2 - (M-2E)\, I_3 + (3M^2+m^2-4ME)\, I_4 - (M^2+m^2-2ME)\, MI_5],$$

where M, m and κ are the masses of K-meson, electron (muon) and pion respectively. W is given by

$$W = (M^2+m^2-\kappa^2)/(2M),$$

and I_2, $I_3 \cdots$ are the functions defined by

$$I_n = \int_0^{\pi} (p \sin\theta\, d\theta/(M-E+p\cos\theta)^n).$$

The curves of these spectra are given in Fig. 2 and Fig. 3. In what follows we shall discuss these results by comparing them with experimental data and with the results obtained by Furuichi et al.[5] on the basis of the primary F-int.

1. *Charged* $K_{\mu 3}$

Among various coupling types of the KU-int. the $G_3^{S(A)}$-coupling for scalar K-meson and $G_3^{V(T)}$-coupling for vector K-meson give the results which fit the experimental data rather well. Our results almost coincide with the best result obtained by Furuichi et al.[5] (an appropriate combination of the $S(T)$- and $S(A)$-coupling) except a slight difference in the high energy end of the distribution. Here it should be noted that all couplings in Table 1 have only one peak in the energy distributions and there is no single coupling type which has two peaks, contrary to the $S(T)$-coupling of the F-int. In the Rochester data for charged $K_{\mu 3}$ the second

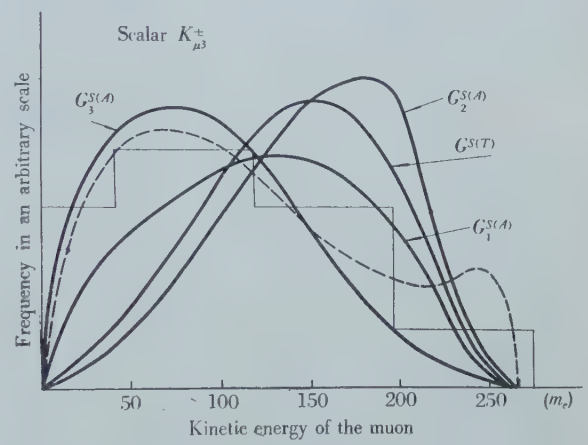

Fig. 2a. Energy spectra of the secondary muons for the scalar $K^{\pm}_{\mu 3}$. The dotted curve indicates the best result obtained by Furuichi et al.

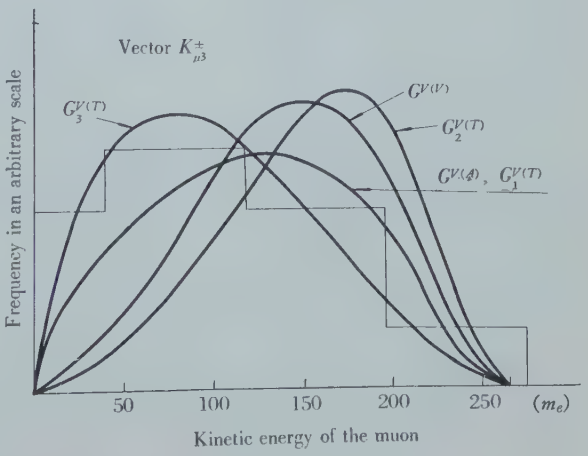

Fig. 2b. Energy spectra of the secondary muons for the vector $K^{\pm}_{\mu 3}$.

maximum in the high energy region of the muon distribution is not observed, therefore we do not necessarily require the $S(T)$-coupling if we only have some coupling type which gives a large distribution of the low energy muons. Practically the $G_3^{S(A)}$-coupling of the KU-int. differs from the $S(P)$-coupling of the F-int. only by a factor (lq), which is reduced in the rest system of the K-meson to

$$(lq) = -M(W-E).$$

This factor favours the predominant emission of muons of lower energy than in the case of the F-int. and shifts the distribution to the low energy regions.

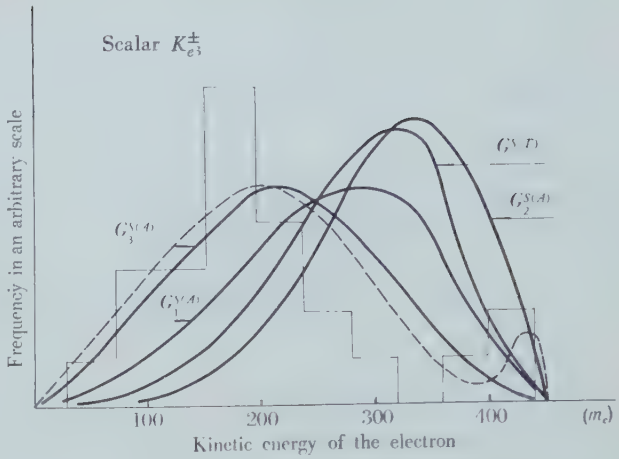

Fig. 3a. Energy spectra of the secondary electrons for the scalar $K_{e3}^{\pm}$. The dotted curve indicates the best result obtained by Furuichi et al.

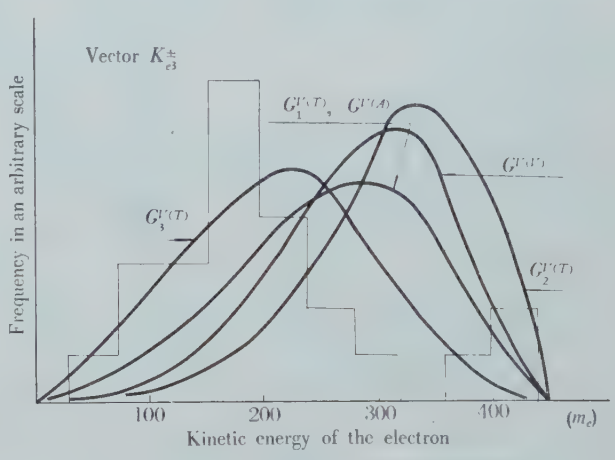

Fig. 3b. Energy spectra of the secondary electrons for the vector $K_{e3}^{\pm}$.

2. *Charged* K_{e3}

The agreement with the Rochester data for charged K_{e3} is less satisfactory than in the case of $K_{\mu 3}^{\pm}$, if we use only one coupling type of the KU-int. Nevertheless, the $G_3^{S(A)}$-coupling or the $G_3^{V(T)}$-coupling again yields almost the same result as that obtained in reference 5) (an appropriate combination of the $S(T)$- and $S(P)$-couplings) except the second maximum at the high energy end. The absence of the second maximum in our theory may become more significant. Of course, we can realize the bimodal distribution of electrons by an appropriate admixture with the other coupling types. It should be noticed, however, that if there exists the two-body decay mode $K \to e + \nu$ the resulting electron will have an energy ~ 246 Mev. Within the accuracy of the present experimental technique it will be very hard to remove such a contamination. At present it seems to the author that we cannot entirely rule out the possibility that the second maximum at the high energy end is due to the K_{e2} mode.

Thus, it is very likely that a single coupling type of the KU-int. ($G_3^{S(A)}$- or $G_3^{V(T)}$-coupling) can describe both the electron and the muon distributions in the $K_{e3}^{\pm}$- and $K_{\mu 3}^{\pm}$-

decays, while in the case of the F-int. two different selections of the coupling types have been required ($S(T)+S(P)$ for $K^{\pm}_{e3}$ and $S(T)+S(A)$ for $K^{\pm}_{\mu 3}$).

§ 4. Energy spectra of neutrinos in neutral $K_{e3}(K_{\mu 3})$-decay

As has been noticed in the introduction, the characteristics of the KU-int. are that they are more favourable to the emission of high energy neutrinos than the F-int. Consequently, it will be the energy distribution of neutrino that decides whether the F-int. or the KU-int. is participating in the process. Unfortunately, however, the neutrino energy in charged $K_{\mu 3}$- and K_{e3}-decays cannot be measured experimentally, since there is only one visible secondary particle. Yet recent discoveries[11),12),13)] of the possible decay mode of neutral K-particles $K^0_{e3}(K^0_{\mu 3}) \to e^{\pm}(\mu^{\pm})+\nu+\pi^{\mp}$ may give a direct test of this question. In these processes we can get experimental data in the laboratory system about the energy-momentum of the electron (muon) $(p_0, \boldsymbol{p})$, that of the pion $(l_0, \boldsymbol{l})$ and the angle between them, from which the neutrino energy q_0 in the rest system of the K-meson can be evaluated, i. e.,

$$q_0=[M^2-(p_0+l_0)^2+(\boldsymbol{p}+\boldsymbol{l})^2]/(2M). \tag{4·1}$$

Thus, we can easily compare theoretical predictions with experiments in spite of the vague information about the momentum or the direction of the incident neutral K-meson which decays in flight. In this section we shall deal with this problem.

Some investigations of the K^0-events on the basis of the weak F-int. have been made by Pais and Treiman[6)] and also by Furuichi.[14)] Since K^0-events are mainly decays in flight, the theoretical predictions about the energy distribution of pions and the angular correlation between the final pion and electron (muon) cannot be compared with experimental data until the direction and the momentum of the incident K^0-meson are determined. Thus, the only practical data which are available for a comparison between theory and experiment are the energy distributions of the neutrino, with which we are concerned here. (The energy spectra of electrons (muons) are almost identical with those in charged $K_{e3}(K_{\mu 3})$-decay except a small correction due to the mass difference between charged and neutral pions.) Recent experiments on θ- and τ-mesons strongly suggest that they are particles with spin 0. Moreover, from the fact that K-particles, which have various decay modes, have almost the same mass and lifetime, we here consider only the K-meson with spin 0, although we cannot exclude the possibility of spin 1 K-meson as is seen in the previous section.

The energy distributions of neutrino for the various coupling types of the KU-int. are :

$$\left.\begin{aligned}
P_1^{S(A)}(q_0) &= (G_1^{S(A)2}/16\pi^3)\,M^2 q_0^2\left[(1/2)\,(E_{\max}^2-E_{\min}^2)-(W-q_0)\,(E_{\max}-E_{\min})\right],\\
P_2^{S(A)}(q_0) &= (G_2^{S(A)2}/16\pi^3)\,(M^2/4)\left[(W-q_0-E_{\max})^4-(W-q_0-E_{\min})^4\right],\\
P_3^{S(A)}(q_0) &= (G_3^{S(A)2}/16\pi^3)\,M^2\left[(1/4)\left\{(W-E_{\max})^4-(W-E_{\min})^4\right\}\right.\\
&\quad\left.-(q_0/3)\left\{(W-E_{\max})^3-(W-E_{\min})^3\right\}\right],
\end{aligned}\right\} \tag{4·2}$$

$$P^{S(T)}(q_0) = (G^{S(T)2}/16\pi^3) M^2 [-(M^2/4)\{(W-q_0-E_{\max})^4-(W-q_0-E_{\min})^4\}$$
$$+(m^2 q_0^2/2)\{-(W-q_0-E_{\max})^2+(W-q_0-E_{\min})^2\}$$
$$+Mq_0(W-q_0)^2(E^2_{\max}-E^2_{\min})-(4Mq_0/3)(W-q_0)(E^3_{\max}-E^3_{\min})$$
$$+(Mq_0/2)(E^4_{\max}-E^4_{\min})], \qquad (4\cdot 2)$$

where

$$E^{\max}_{\min}=[(M-q_0)(W-q_0) \pm q_0\sqrt{(Q-q_0)(Q'-q_0)}]/(M-2q_0),$$
$$Q=[M^2-(m+\kappa)^2]/(2M), \quad Q'=[M^2-(k-m)^2]/(2M).$$

These curves are given in Fig. 4 (assuming K^0-mass $=K^{\pm}$-mass $=$ 966 m_e) and, as are expected, have large distributions in high energy regions both for K^0_{e3}- and $K^0_{\mu 3}$-decays. Here we are especially interested in the energy distribution of neutrino in the K^0_{e3}-decay. In the theory based on the F-int. the transition matrix elements contain the four-momenta of electrons and neutrinos always in the combination $p+q$ for every coupling type. Further the electron mass is negligibly small. Owing to these circumstances, the neutrino distribution almost coincide with the electron distribution for any combination of various coupling types of the F-int. Therefore, if we use the linear combination of $S(T)$- and $S(P)$-couplings of the F-int., by which the fittest result to the experimental distribution of electrons in charged K_{e3}-decay can be obtained, the neutrino distribution in the K^0_{e3}-decay will have its maximum at $\sim$100 Mev and have a large distribution in the region 50 Mev$\sim$150 Mev. In contrast to this, the result for the $G_3^{S(A)}$-coupling of the KU-int. behaves quite differently and has

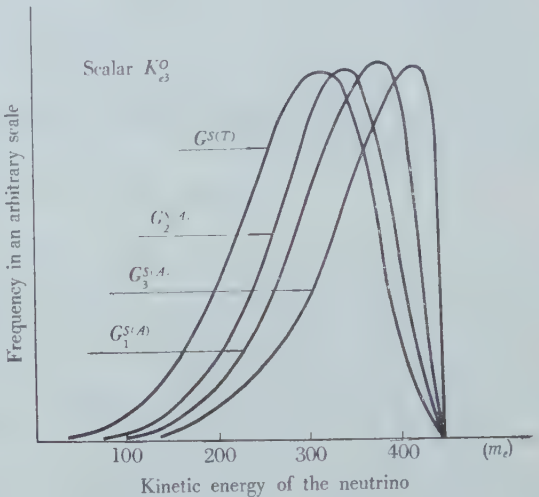

Fig. 4a. Energy spectra of the neutrinos for the scalar K^0_{e3}.

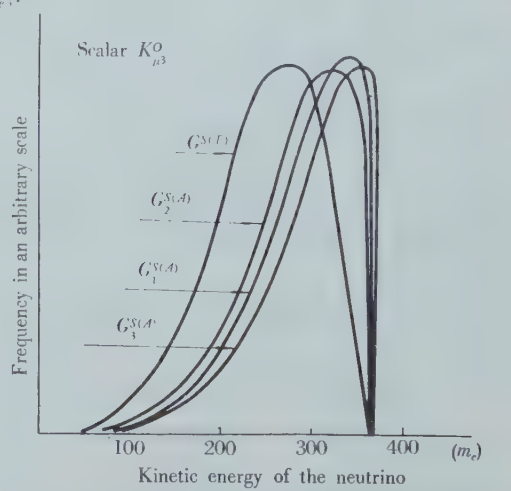

Fig. 4b. Energy spectra of the neutrinos for the scalar $K^0_{\mu 3}$.

a large distribution in the energy region 160 Mev∼210 Mev. Thus, the neutrino distribution in the K^0_{e3}-decay has a decisive role whether the *KU*-int. or the *F*-int. is participating in these decays.

At present we have few experimental data on the process $K^0_{e3}(K^0_{\mu 3}) \to e^{\pm}(\mu^{\pm}) + \nu + \pi^{\mp}$ and fewer data in which both pions and electrons (muons) are identified. We record in Table 2 the experimental data concerning events which are probably K^0_{e3}-decays, together with the neutrino energy q_0 calculated by (4·1). It seems that the neutrino is likely to be emitted with rather high energy and this tendency is consistent with our predictions. But, in view of poor statistics of the experimental data and much uncertainty in the momentum measurements, we cannot draw any definite conclusions. Further experimental information on the K^0_{e3}-decay process is very important and desirable.

Table 2. Data of the probable K^0_{e3}-decay

event No.	+momentum (Mev/c)	−momentum (Mev/c)	angle (degree)	q_0 (Mev)	
4	224±5	58.5±4	91.5±3	195	(probable e^-)[12]
9	241±33	67±20	142±3	163	(probable e^-)[12]
15	>150	62	99±3	198*	(probable e^-)[12]
59647	409∼535	39.5±1.5	83±1	188∼196	(π^+, e^-)[11]
69328	11.9±0.9	200^{+36}_{-27}	46.1	202	(π^-, e^+)[13]

* q_0 for event 15 is calculated for $p_+ = 150$ Mev/c.

§ 5. Remarks on the hypothesis of the *KU*-int.

In this paper we have assumed that the *KU*-int. plays an important role in the weak interactions in which the existence of strange particles must be taken into account, and decay events of charged and neutral K_{e3} and $K_{\mu 3}$ have been investigated. If, however, our assumption is close to reality, some modifications will be necessary of discussions about the other phenomena, such as the $K_{\mu 2}$-decay and the β-decay of Hyperons. In this section we make some remarks on these problems.

1. *Decay of the K-meson into two leptons*

Concerning the decay of the *K*-meson into two leptons we are informed of the decay mode $K^{\pm}_{\mu 2} \to \mu^{\pm} + \nu$, but the corresponding decay mode $K^{\pm}_{e2} \to e^{\pm} + \nu$ has not been found.* The situation is quite similar to the well-known $\pi - \mu$ and $\pi - e$ puzzle. So far the absence of the $\pi - e$- or K_{e2}-decay has been interpreted by assuming the interaction among pion (*K*-meson), neutrino and electron or muon of the following form:

$$\sim \bar{\psi} \gamma_5 \gamma_\mu \varphi \cdot \partial_\mu \phi . \qquad (5 \cdot 1)$$

((5·1) can be regarded as the pseudovector interaction among two Fermions and a Boson,[16] or alternatively as the transition matrix element of the Boson decay via baryon loops by

* There are two events which may belong to the K_{e2} mode.[9),15]

the pseudovector coupling of the F-int.[8])

In our theory in which the KU-int. is assumed for strange particles, the possible coupling which contributes to the $K_{\mu 2}(K_{e2})$-decay is the vector coupling for scalar K-meson (pseudovector coupling for pseudoscalar K-meson). Hence we have

$$\tau_{K-\mu}/\tau_{K-e}=(M^2-m_e^2)^4/(M^2-m_\mu^2)^4, \tag{5·2}$$

which is close to unity and is inconsistent with the experiments. This may be a defect of our theory. If, however, we tentatively assume, besides (2·5)—(2·7), the following interaction

$$\sim(\bar{u}\gamma_\mu\partial_\nu w+\bar{u}\gamma_\nu\partial_\mu w)\,(\bar{\psi}\gamma_\mu\partial_\nu\varphi), \tag{5·3}$$

which contains an additional derivative of the baryon wave function* and is symmetric with respect to indices μ and ν, the transition matrix element of the following type is possible for the $K_{\mu 2}(K_{e2})$-decay:

$$\sim\bar{\psi}\gamma_\mu\partial_\nu\varphi\cdot\partial_\mu\partial_\nu\Phi. \tag{5·4}$$

This leads to

$$\tau_{K-\mu}/\tau_{K-e}=(m_e^2/m_\mu^2)\cdot(M^2-m_e^2)^4/(M^2-m_\mu^2)^4\sim 10^{-5}, \tag{5·5}$$

which is consistent with experimental observations.

The interaction (5·3) will also give rise to K_{e3}- and $K_{\mu 3}$-decays. There are, however, some difficulties in treating such an interaction, where the notion of free particles and their mutual interaction becomes somewhat ambiguous. Moreover, as was mentioned in § 3, we cannot distinguish the electrons at the high energy end of the K_{e3} mode from those due to the K_{e2} mode within the accuracy of the present experimental technique, and the relative ratio $K_{e2}/K_{\mu 2}$ may be altered in future experiments. Therefore, we merely suggest (5·3) as a tentative assumption and do not consider this problem any further. The problem of the K_{e2}-decay must be further investigated both theoretically and experimentally.

2. *β-decay of Hyperons*

The existence of the KU-int. will bring about the possible β-decay of Hyperons, e. g.,

$$\Lambda^0\to P+e+\nu. \tag{5·6}$$

It has been pointed out by Furuichi[14] that the assumption of the primary F-int. also leads to this process. If, however, (5·6) is caused by the F-int. the energy will be equally distributed among electron and neutrino,[10] and the most probable electron energy is $\sim E_0/2$, where E_0 is the maximum electron energy.[14] In contrast to this, if the KU-int. is responsible to the process (5·6), the electron receives on the average much less

* So far we have not met with phenomena in which derivatives of the wave function of heavy particles are required. It is interesting, however, to notice that the very weak interaction between the gravitational and the matter field may contain such a derivative.[17]

than half the maximum energy E_0 and the most probable energy will be shifted to $\sim E_0/3$.[10] At present we have only one event which may be a β-decay of Σ[18] and further experimental investigations are eagerly awaited.

§ 6. Concluding summary

Under the assumption that some universal interaction of a new kind may play an essential role in weak interactions in which particles with strange properties take part, we have analysed the decays of K-mesons and Hyperons with lepton secondaries. Our starting points are :

1. Transition matrix elements are Lorentz invariant.

2. The decay of K-mesons is caused by the collaboration of the strong and the weak KU-int. which acts only once in the whole course of the decay process.

In the theory based on the F-int., different selections of coupling types are required for the energy spectra of electrons and muons in the $K^{\pm}_{e3}$- and $K^{\pm}_{\mu 3}$-decay respectively, while in our theory a single coupling type ($G_3^{S(A)}$- or $G_3^{V(T)}$-coupling) gives the results which agree with the experimental data to the same extent as those obtained in the case of the F-int. (§ 3). In order to make clear the contrast between the F-int. and the KU-int., we have investigated the neutrino distribution in the neutral $K_{e3}(K_{\mu 3})$-decay and have found that the KU-int. is more favourable to the emission of high energy neutrinos than the F-int. Let it be emphasized that the neutrino distribution in the K^0_{e3}-decay gives a crucial test which of the two is responsible to these processes. Though we have very few experiments on the K^0_{e3}-decay, a preliminary investigation on available data seems to be favourable to, at least not inconsistent with, our predictions (§ 4). In the last section the $K_{\mu 2}$-decay and the possible β-decay of Hyperons are briefly discussed.

It is indeed very interesting from the view-point of aesthetics as well as economy of thought to describe all phenomena through the collaboration of the strong and the familiar Fermi interaction. Nevertheless, those phenomena in which strange particles play an active part are, as it were, the new and unknown world and it will not be hard to imagine that there are various strange factors which are unfamiliar to our present knowledge. Therefore it may be promising as well to assume that some new kind of interactions, such as the KU-int., may play an essential role when we go a step further inside this world.

We have noticed the existence of two classes of phenomena. Practically speaking, however, the border between these two classes is not so clear and there are phenomena for which it is not apparent to which class they belong. For example, if the $\pi-\mu(e)$ decay is caused through the intermediation of the baryon loop, it will be possible that the hyperons as well as nucleons participate in this process. Although the borderland is rather obscure at present, it is an interesting problem to what extent we can reduce it.

Throughout this paper our considerations have been confined to the weak interactions which involve neutrinos. We also know, however, other weak interactions which are not neutrino processes. It still remains to be an open question whether they can be described

consistently in terms of the *KU*-int.* In view of the results obtained in the present analysis, further investigations into this problem will be of much value and of great importance.

The author wishes to express his sincere thanks to Professor Y. Mimura for his kind interest in as well as for his untiring encouragement throughout this work. His thanks are also due to Mr. Furuichi for sending the preprint prior to its publication.

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* It may be that the concept of the weak *KU*-int. also has to be extended so as to include four-baryon processes.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors Communications should be submitted in duplicate and should be held to within 100 lines (pica type an standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Molecular Orbital Treatment of $1s\sigma\ 2p\sigma\ {}^3\Sigma_u$ State of H_2

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November 4, 1957

The repulsive state $1s\sigma\ 2p\sigma\ {}^3\Sigma_u$ of H_2 is treated by the molecular orbital method within single configuration approximation. The wave function assumes the form:

$$\Psi(1,2) = (1/\sqrt{2})\{\varphi_g(1)\varphi_u(2) - \varphi_u(1)\varphi_g(2)\}.$$

The first choice (I) of φ_g and φ_u is

$$\varphi_g \sim \chi_a + \chi_b;\ \sigma_g,$$

$$\varphi_u \sim \chi_a' - \chi_b';\ \sigma_u,$$

where

$$\chi_a = (Z^3/\pi)^{1/2}\exp(-Zr_a),$$

$$\chi_b = (Z^3/\pi)^{1/2}\exp(-Zr_b),$$

$$\chi_a' = (Z'^3/\pi)^{1/2}\exp(-Z'r_a),$$

$$\chi_b' = (Z'^3/\pi)^{1/2}\exp(-Z'r_b).$$

This is an LCAO MO type approximation. The point is that we take $Z \neq Z'$. The second choice (II) of φ_g and φ_u is

$$\varphi_g \sim e^{-\alpha\xi}(1+c\eta^2),$$

$$\varphi_u \sim e^{-\beta\xi}\eta,$$

where $\xi = (r_a+r_b)/R$ and $\eta = (r_a-r_b)/R$, R being the distance between two nuclei a and b. This may be called a James type approximation.

The calculations of the repulsive energy are performed at three values of R: 1.0, 1.5 and 2.0 a.u. and the results are summarized in Table 1. The values of the parameters in Table 2 are fixed by simple guess work. Notice that we take $Z+Z'=2$ and $Z=2\alpha$, $Z'=2\beta$. c is determined by the Ritz procedure.

Table 1. The repulsive energy in Kcal/mole.

R(a.u.)	(I)	(II)	JCP*	HL**	(I) with $Z=Z'=1$
1.0	249.9	246.8	245***		442.3
1.5	128.6	126.8	119.8	142	200.2
2.0	71.0	69.0	65.9		96.6

* James-Coolidge-Present
** Hirschfelder and Linnett
*** The value due to the formula (12) of HL.

Table 2

R(a.u.)	(I)		(II)		
	ZR	$Z'R$	2α	2β	c
1.0	1.5	0.5	1.5	0.5	0.153941
1.5	2.0	1.0	2.0	1.0	0.350896
2.0	2.5	1.5	2.5	1.5	0.635077

Table 1 shows that the present single

configuration approximation yields rather good numerical results. The most attractive feature is the preservation of the simple image of the molecular orbital point of view. This becomes quite clear if we compare the present treatment with that of Hirschfelder and Linnett.[2] The present treatment indicates how to improve the floating function method of Hurley.[3] The origins of the atomic orbitals should be floated separately, that is, inward direction in the case of σ_g orbital and outward in the case of σ_u orbital. As for the He-He repulsive energy, similar calculations have already been performed by the present author[4] with the function of the first type (I) and by T. Murai[5] with the second type function (II) with considerable success.

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Progress of Theoretical Physics 投稿規定

A 原　　稿

I. 一般的注意

(1) 用紙は普通のタイプ用紙を用い，正副２通作って下さい。なお著者の手元にも写しを必ず保存願います。**正副いずれの原稿にも文字の指定を入れて下さい。**指定のしかたは別項及び見本を参照して下さい。

(2) 数式と特殊タイフの文字の他はすべてタイプ印書して下さい。

(3) 各頁とも，上下，　左右それぞれ2cm あけ，行間は１行ずつあけて下さい。この規定に従うと，１行が大体74字詰くらいになります。但し，脚註もこの中に含めますが，短い脚註は行間なしにつめて下さい。本文と脚註の境の棒は本文の１行分をとって下さい。

(4) 題名，氏名，研究場所，アブストラクト，本文の順に，各１行ずつあけて夫々行を改めて下さい。アブストラクトは１頁以内，氏はキャピタルで記し，名は略さないで全部記して下さい。(例：Kôtarô HONDA)

(5) 節 (§) の標題の上下の行間は，本文の２行分だけ余分にあけて下さい。

(6) 原稿は充分推敲の上清書して（**鉛筆やペンの書込みは(2)項以外は差控えて下さい**），全部をまとめて上の中央部を紙よりでとじ，各紙毎に右肩に頁を入れて下さい。

II. 使 用 文 字

(1) 指定がなければ，本文中の文字はすべて立体，数式中の文字はすべてイタリックで組みます。従って本文中のイタリック，数式中の立体文字（例：log, lim, Trace, sin, *etc.*）とすべてのゴチックは指定願います。(指定記号は見本参照)

(2) ギリシャ文字，ドイツ文字は特に指定して下さい。ベクトルは矢を用いずに，ゴチックで表わして下さい。ギリシャ文字のゴチック体は現在次のものしかありません。

$$\Pi,\ \Delta,\ \Sigma,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \eta,\ \theta,\ \lambda,\ \mu,\ \nu,\ \phi,\ \varphi,\ \pi,\ \rho,\ \sigma,\ \omega$$

(3) **まぎれやすい文字，大文字と小文字の区別の困難な文字は，はっきり書いて下さい。**

i) まぎれやすい文字

0 (ゼロ) と O (オー)，　*a* (エィ) と α (アルファ)，　*v* (ヴィ) と ν (ニュー)，

I (イチ) と *l* (エル)，　*e* (イー) と ε (イプシロン)，　*x* (エックス) と χ (カイ)，

k (ケー) と κ (カッパ)，　*r* (アール) と γ (ガンマ)，　*w* (ダブリュー) と ω (オメガ)

ii) 大文字と小文字の区別が困難な文字

C, K, O, P, S, V, W, X, (または $\mathcal{C}, \mathcal{K}, \mathcal{O}, \mathcal{P}$ の場合)

(4) 添字ははっきり上下のルビにして下さい ($H^{\prime}$ H_{x} etc.)。まぎらわしい時には(上つき)(下つき) と指定して下さい。複雑な添字はなるべくさけるよう表現に工夫して下さい。**二重の添字 (添字の添字)，あるいは添字のゴチックは殆んど組版できません。**

(5) 花文字の場合は，花文字そのものを書かないで（ドイツ花)，(英花) の如く朱書して下さい。

III. 数　　式

(1) 数式の1行は本文の3行に相当するだけあけて，ていねいに書いて下さい。特に長い式は印刷したとき1行に入る長さに切って，何行かに分けて書いて下さい。

(2) **分数はできるかぎり1行に並べて斜線を用いて下さい。**止むをえず分数を含む式は本文の4行の割に見こんで下さい。特に文中に挿入する数式等は $(a/b)\exp(t/r)$ のような記法を用いて下さい。

IV. 引　　用

(1) 本文中の引用番号は 1)，2〜5)，6) の如く右カッコを用いて下さい。

(2) 引用文献は通し番号をつけて論文の終りにまとめ，**次の様式に統一して下さい。**

H. Yukawa, Rev. Mod. Phys. **21** (1949), 474.

(3) 単行本の引用は書名をイタリックで記し，発行された年号，発行所を書いておいて下さい。文献以外のものは番号をつけず，* 印等の印で脚註にして下さい。

(4) 引用文献の著者名は上記(2)のように名の initial letter を必ず入れて下さい。

B 図　　版

(1) 図の原稿は本文用紙と同形の紙か，別紙にていねいに書いたもの（正式のトレースでなくてもよろしい）を本文の最後にまとめ，**著者名，論文名，図番号を必ず明記して下さい。**

(2) 図の位置は本文を適当にあけて指定し，caption はそこへタイプしておいて下さい。

(3) **特に図版中に入る文字の指定**（立体，イタリック，ゴチック，大文字，小文字，etc.）**を忘れないようにし，**著者の画いたトレースをそのまま凸版にする場合は縮尺による文字の大きさを充分考えて下さい。

(4) **凸版のやりなおしは非常に経費と時間を無駄にしますので上記(3)は厳守して下さい。**

C そ の 他

論文の内容が既に「素粒子論研究」，「物性論研究」等に邦文で発表されている場合には投稿の際その旨通知して下さることをおすすめします。**投稿後論文についての問合せの際には論文番号及び受理日付を必ず附記して下さい。**投稿規定に反する論文は返送することがありますからくれぐれも御注意下さい。

◎いまだに大文字と小文字，立体とイタリック，細字とゴチックの区別がしてない論文が数多く見うけられます。著者のほうで十分注意して下さい。なお，大幅な訂正や特殊な記号，文字は絶体にさけて下さい。

原稿の書き方見本

§5. Wave forms of normal vibration

Let us next investigate the spatial wave forms of the normal vibrations for Case M. First consider a regular lattice. In this case the equation determining the allowed initial vector is, in the representation in which $\hat{T}$ is diagonalized,

$$\left\{\left(\underline{I}+\underline{\Lambda}^{(N)}\right)^{\text{エフキ}}-\underline{I}\right\}\underline{x}'_{1}{}^{\text{エフキ}}_{\text{下フキ}}=\begin{pmatrix} e^{2N\mathrm{i}\beta_{\text{ギ}}}-1 & 0 \\ 0 & e^{-2N\mathrm{i}\beta_{\text{ギ}}}-1 \end{pmatrix}\begin{pmatrix} x'_{1} \\ y'_{1} \end{pmatrix}_{\text{下フキ}}=0, \qquad (5.1)$$

where $\underline{x}'_{1}=\left(x'_{1},\ y'_{1}\right)^{(T)}=\underline{U}^{-1}\underline{x}_{1}$. Since for the eigenvalues $\beta_{\text{ギ}}=n\pi_{\text{ギ}}/N$ the matrix in the above formula is null, the initial vector is here arbitrary, except for $\beta_{\text{ギ}}=\pi_{\text{ギ}}/2$, when $\underline{U}^{-1}$ does not exist. Let it be $(x'_{1},y'_{1})^{(T)}$, then the state of the $(n+1)$th atom is given by

$$\begin{pmatrix} e^{2n\mathrm{i}\beta_{\text{ギ}}} & 0 \\ 0 & e^{-2n\mathrm{i}\beta_{\text{ギ}}} \end{pmatrix}\begin{pmatrix} x'_{1} \\ y'_{1} \end{pmatrix}=\begin{pmatrix} e^{2n\mathrm{i}\beta_{\text{ギ}}}x'_{1} \\ e^{-2n\mathrm{i}\beta_{\text{ギ}}}y'_{1} \end{pmatrix}. \qquad (5.2)$$

If we return to the original representation, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \left\{\cos(2n\beta_{\text{ギ}})+\tan\beta_{\text{ギ}}\sin(2n\beta_{\text{ギ}})\right\}x_{1}+\sin(2n\beta_{\text{ギ}})y_{1} \\ -2\tan\beta_{\text{ギ}}\sin(2n\beta_{\text{ギ}})x_{1}+\left\{\cos(2n\beta_{\text{ギ}})-\tan\beta_{\text{ギ}}\sin(2n\beta_{\text{ギ}})\right\}y_{1} \end{pmatrix}. \qquad (5.3)$$

文字指定記号

大文字 (大)又は大文字
小文字 (小)又は小文字
suffix(添字) 上ツキ, 下ツキ
ギリシャ文字 ギ
イタリック(斜体) イタヌは～
ゴチック(太字) ゴチ又は＝
ゴチックイタリック
立体文字 立体又は
表 罫(細い線)
裏 罫(太い線)

校 正 記 号

大文字 (大)又は大文字
小文字 (小)又は小文字
活字悪し
改 行
続 け る
行間つめる
行間あける
文字の凹凸を正す
文字間つめる
文字間あける

上げる
下げる
文字入れかえ
逆字・横字を正す
文字挿入
右へ寄せる
左へ寄せる
suffix(添字) {上ツキ / 下ツキ}
明朝にする 明朝
消し誤り(元のまま) ナシ

校正の仕方見本

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where $x_1'=(x_1', y_1')^T=U^{-1}x_1$. Since for the eigenvalues $\beta=n\pi/N$ the matrix in the above formula is null, The initial vector is here arbitrary, except for $\beta=\pi/2$, when U^{-1} does not exist. Let it be $(x_1', y_1')^T$, then the state of the $(n+1)$-th aton is given by

$$\begin{pmatrix} e^{2ni\beta} & 0 \\ 0 & e^{-2ni\beta} \end{pmatrix}\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}=\begin{pmatrix} e^{2ni\beta}x_1' \\ e^{-2ni\beta}y_1' \end{pmatrix}. \qquad (5\cdot2)$$

if we return to the original representaton, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \{\cos(2n\beta)+\tan\beta\sin(2n\beta)\}x_1+\sin(2n\beta)y_1/\sin 2\beta \\ -2\tan\beta\sin(2n\beta)x_1+\{\cos(2n\beta)-\tan\beta\sin(2n\beta)\}y_1 \end{pmatrix}. \qquad (5\cdot3)$$

刷り上り見本
—上掲の校正を終えて印刷したもの—

§ 5. Wave forms of normal vibrations

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If we return to the original representation, we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}=\begin{pmatrix} \{\cos(2n\beta)+\tan\beta\sin(2n\beta)\}x_1+\sin(2n\beta)y_1/\sin 2\beta \\ -2\tan\beta\sin(2n\beta)x_1+\{\cos(2n\beta)-\tan\beta\sin(2n\beta)\}y_1 \end{pmatrix}. \qquad (5\cdot3)$$

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An Attempt at Reformulating Pion-Nucleon Interaction, I

—Introduction of K-Space—

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An attempt is made at reformulating pion-nucleon interaction in such a form that the large annihilation cross section of antiproton or the recent experiments at Stanford could be understood better. For this purpose, a new space which we name *K*-space is introduced. In terms of *K*-space, charge conjugation can be expressed as a rotation through π around its second axis multiplied by a rotation through π around the second axis in the usual isotopic space. It is shown that if pion-nucleon interaction is assumed as invariant under rotations in *K*-space (it is already invariant under rotations in the usual isotopic space if we assume charge independence), it can not be local, especially the part responsible for the creation and the annihilation of nucleon pairs. Possibilities of checking this assumption experimentally are discussed, and among these the annihilation of antinucleon by nucleon is discussed in some detail.

§ 1. Introduction

Recently, experiments on the scattering of high energy electron by nucleon or by deuteron[1] and the annihilation of antiproton by nucleon[2] have become possible, and several new facts have been found about the structure of the nucleon or the pion-nucleon interaction. Among these, the most striking is that the core of the nucleon seems to extend to about the same order of magnitude with the Compton wave length of the pion. It is very difficult to understand this fact from the usual theory. The purpose of this paper is to describe an attempt at reformulating pion-nucleon interaction in such a form that this fact could be understood better.

The starting point of the discussion is the conservation law of charge. Consider, for example, the simplest case of pion. Its charge is given by

$$Q=eI_3, \tag{1·1}$$

where I_3 is the third component of the isotopic spin. In this case, therefore, the conservation law of charge is nothing but the conservation law of I_3. This fact, and the charge independence suggest a simple picture for the charge space. That is, it is a three

dimensional real orthogonal space, and in it only the direction of the third component is characterized by the electromagnetic field. Therefore, it is symmetric under rotations around its third axis, and can be regarded as effectively isotropic in the approximation in which it is neglected.

As is well known, the gauge transformation (of the first kind)

$$\phi \to e^{i\alpha}\phi, \quad \phi^* \to e^{-i\alpha}\phi^*$$

(α is an arbitrary real constant)

can be regarded as a rotation through α in the charge space around its third axis. The above picture, when combined with this interpretation of the gauge transformation, makes very clear the meaning of gauge invariance or the character of the conservation law of charge. For example, we may say that why the theory is invariant under the gauge transformation is because it is a rotation around the third axis in the charge space, and the charge space is symmetric under it, or why the conservation law of charge follows from the gauge invariance is because it is a conservation law of the third component of an angular momentum in the charge space, and the gauge transformation is a rotation around the third axis,...etc.

Such a picture, however, can not be used in the case of nucleon, where the charge is given by

$$Q = e(I_3 \pm 1/2), \tag{1·2}$$

where $+1/2$ and $-1/2$ correspond to nucleon and antinucleon respectively. Therefore, if we take the above picture seriously, it becomes necessary to assume that this is also the third component of some angular momentum in the charge space.* We shall call this $\boldsymbol{K}$, and assume that

$$|\boldsymbol{K}| = 1/2, \tag{1·3}$$

and

$$K_3 = \begin{cases} +1/2 & \text{for nucleon,} \\ -1/2 & \text{for antinucleon.} \end{cases} \tag{1·4}$$

If these assumptions are made, the charge is given by the third component of an angular momentum in the charge space as a whole, and the interpretation of the gauge invariance or of the conservation law of charge becomes much simpler.

Now, the charge conjugation is the exchange of particle and antiparticle. In our picture, however, they are related to $+1/2$ and $-1/2$ of K_3. Therefore, in our picture, it is expected that the charge conjugation can be replaced by a simple geometrical operation in the charge space, and in fact, as will be shown in § 4, it can be replaced by a rotation through π around the second (first) axis in I-space multiplied by a rotation through π around the second (first) axis in K-space.

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* Corresponding to this, the charge space is assumed as a direct product of I- and K-spaces for nucleon. For pion it is the same as before since $\boldsymbol{K}$ is always assumed as zero for it.

The invariance of the pion-nucleon interaction under rotations in I-space is known as charge independence. Therefore, the requirement of invariance under charge conjugation is equivalent to the requirement of invariance under rotation through π around the second (first) axis in K-space. The assumption we shall make in this paper is that the pion-nucleon interaction must be invariant under the full rotation in K-space. This is exactly analogous to the one which we have made on the passing from charge symmetry to charge independence. As for the electromagnetic interaction, it is assumed as the third component of a vector in I- and K-spaces.

We shall discuss the consequence of these assumptions in detail in the following. It is shown that, if these assumptions are made, the pion-nucleon interaction, especially the part responsible for the creation and the annihilation of nucleon pairs, can not be local so long as it is assumed as bilinear in the nucleon field and linear in the pion field. It would be natural to expect that such non-locality will be responsible for various anomalies mentioned at the beginning.

We shall also discuss the possibility of checking basic assumptions experimentally. According to these, $\boldsymbol{K}$ is conserved in pion-nucleon interaction, and this gives new selection rules in some cases. Therefore, by comparing predictions from these selection rules with experiments, it is possible to check them. This will be done concretely on a few examples.

§ 2. Generator for infinitesimal rotation in K-space

To discuss the transformation properties of various quantities in K-space, we must construct the generator for infinitesimal rotation in K-space.

We shall describe proton and neutron fields by $\psi_p(x)$ and $\psi_n(x)$ respectively.* They contain destruction operators for protons (neutrons) and creation operators for antiprotons (antineutrons). It is convenient, therefore, to start from momentum representation. Thus, expanding them into Fourier series, we have

$$\psi_p(x) = (1/\sqrt{V}) \sum_{k,\sigma} [u(k,\sigma)\, e^{ikx} p(k,\sigma) + v(k,\sigma)\, e^{-ikx} \tilde{p}^*(k,\sigma)],$$

$$\psi_n(x) = (1/\sqrt{V}) \sum_{k,\sigma} [u(k,\sigma)\, e^{ikx} n(k,\sigma) + v(k,\sigma)\, e^{-ikx} \tilde{n}^*(k,\sigma)], \qquad (2\cdot1)$$

and also for antinucleon fields

$$\tilde{\psi}_p(x) = (1/\sqrt{V}) \sum_{k,\sigma} [u(k,\sigma)\, e^{ikx} \tilde{p}(k,\sigma) + v(k,\sigma)\, e^{-ikx} \tilde{p}^*(k,\sigma)],$$

$$\tilde{\psi}_n(x) = (1/\sqrt{V}) \sum_{k,\sigma} [u(k,\sigma)\, e^{ikx} \tilde{n}(k,\sigma) + v(k,\sigma)\, e^{-ikx} n^*(k,\sigma)], \qquad (2\cdot1')$$

where $u(k,\sigma)$ and $v(k,\sigma)$ are usual solutions of the Dirac equation, and $p(k,\sigma)$ $(n(k,\sigma))$, $p^*(k,\sigma)$ $(n^*(k,\sigma))$, $\tilde{p}(k,\sigma)$ $(\tilde{n}(k,\sigma))$ and $\tilde{p}^*(k,\sigma)$ $(\tilde{n}^*(k,\sigma))$ represent the destruc-

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* In the following, interaction representation is used throughout, so that $\psi_p(x)$ and $\psi_n(x)$ satisfy the equations of motion for free fields.

tion and creation operators for proton (neutron) and those for antiproton (antineutron) respectively. According to § 1, nucleon and antinucleon correspond to $+1/2$ and $-1/2$ of K_3. Therefore, following the idea of K-space, let us introduce $\chi(k, \sigma)$ defined by

$$\chi(k, \sigma) = \begin{bmatrix} p(k, \sigma) \\ n(k, \sigma) \\ -\tilde{n}(k, \sigma) \\ \tilde{p}(k, \sigma) \end{bmatrix} \tag{2·2}$$

as our basic quantity. It is quantized according to

$$\{\chi_i^*(k, \sigma),\ \chi_j(k', \sigma')\} = \delta_{ij}\delta_{kk'}\delta_{\sigma\sigma'},$$
$$\{\chi_i(k, \sigma),\ \chi_j(k', \sigma')\} = 0, \tag{2·3}$$

and

$$\{\chi_i^*(k, \sigma),\ \chi_j^*(k', \sigma')\} = 0.$$

In terms of $\chi(k, \sigma)$, the operator for nucleon number N (number of nucleon − number of antinucleon) is expressed as

$$N = \sum_{k,\sigma} \chi^*(k, \sigma)\kappa_3\chi(k, \sigma), \tag{2·4}$$

where κ_3 is one of Pauli matrices in K-space:

$$\kappa_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \kappa_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \kappa_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2·5}$$

Also the isotopic spin operator is given by

$$I_i = (1/2)\int \psi^*(x)\tau_i\psi(x)\,dx$$
$$= (1/2)\sum_{k,\sigma} \chi^*(k, \sigma)\tau_i\chi(k, \sigma), \tag{2·6}*$$

where $\psi(x)$ is an abbreviation for

$$\psi(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix}. \tag{2·7}$$

In terms of (2·6), the infinitesimal rotation in I-sapce through ε_i around its i-th axis can be represented as

$$\mathcal{D} = 1 + i\varepsilon_i I_i. \tag{2·8}$$

In fact, using (2·3), we can show easily that $\chi(k, \sigma)$ is transformed according to

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* In the last line of (2·6), τ_i stands for a four rowed and four columned matrix

$$\begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}.$$

This obvious abbreviation will be used whenever τ_i operates on four rowed matrices such as $\chi(k, \sigma)$.

$$\mathcal{S}^{-1}\chi(k,\sigma)\mathcal{S}=\chi(k,\sigma)+(i/2)\,\varepsilon_i\tau_i\chi(k,\sigma). \tag{2·9}$$

We shall introduce the generator for infinitesimal rotation in K-space in a form analogous to (2·8). Let us define first K_i by

$$K_i=(1/2)\sum_{k,\sigma}\chi^*(k,\sigma)\kappa_i\chi(k,\sigma). \tag{2·10}$$

These K_i's are obviously Lorentz-invariant, and are commutative with I_i. It can further be shown that they satisfy following commutation relations:

$$[K_i,\ K_j]=iK_k,\quad (i,\ j,\ k \text{ cyclic}). \tag{2·11}$$

These commutation relations are just the ones between the generators for infinitesimal rotations in a three dimensional space. Defining $\mathcal{K}$ by

$$\mathcal{K}=1+i\varepsilon_iK_i, \tag{2·12}$$

we can show in a way exactly the same as before that $\chi(k,\sigma)$ is transformed according to

$$\mathcal{K}^{-1}\chi(k,\sigma)\mathcal{K}=\chi(k,\sigma)+(i/2)\,\varepsilon_i\kappa_i\chi(k,\sigma) \tag{2·13}$$

under an infinitesimal rotation through ε_i around the i-th axis in K-space.

It is convenient for later purposes to introduce the second kind spinor defined by

$$\chi'(k,\sigma)=(i\tau_2)\,(i\kappa_2)\chi^*(k,\sigma)=\begin{bmatrix}\tilde{p}^*(k,\sigma)\\ \tilde{n}^*(k,\sigma)\\ -n^*(k,\sigma)\\ p^*(k,\sigma)\end{bmatrix}. \tag{2·14}$$

From (2·9) and (2·13) we see that they are transformed according to

$$\mathcal{S}^{-1}\chi'(k,\sigma)\mathcal{S}=\chi'(k,\sigma)+(i/2)\,\varepsilon_i\tau_i\chi'(k,\sigma), \tag{2·15}$$

and

$$\mathcal{K}^{-1}\chi'(k,\sigma)\mathcal{K}=\chi'(k,\sigma)+(i/2)\,\varepsilon_i\kappa_i\chi'(k,\sigma). \tag{2·16}$$

Returning to the original representation, we have from $\chi(k,\sigma)$ and $\chi'(k,\sigma)$

$$\chi^{(+)}(x)=(1/\sqrt{2V})\sum_{k,\sigma}u(k,\sigma)\,e^{ikx}\chi(k,\sigma), \tag{2·17}$$

and

$$\chi^{(-)}(x)=(1/\sqrt{2V})\sum_{k,\sigma}v(k,\sigma)\,e^{-ikx}\chi'(k,\sigma). \tag{2·18}$$

It should be noted that $\chi^{(+)}(x)$ and $\chi^{(-)}(x)$ are connected by

$$\chi^{(+)}(x)=(i\kappa_2)\,(i\tau_2)\chi^{(-)*}(x),$$

and

$$\chi^{(-)}(x)=(i\kappa_2)\,(i\tau_2)\chi^{(+)*}(x), \tag{2·19}$$

where it is assumed that Majorana representation is used,* and $u(k, \sigma)$ and $v(k, \sigma)$ are connected by

$$u(k, \sigma) = v^*(k, \sigma). \tag{2·20}$$

Combining $\chi^{(+)}(x)$ and $\chi^{(-)}(x)$, we can define $\chi(x)$ by

$$\chi(x) = \chi^{(+)}(x) + \chi^{(-)}(x), \tag{2·21}$$

which is related to $\psi(x)$ by

$$\chi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi(x) \\ -i\tau_2 \tilde{\psi}(x) \end{pmatrix}. \tag{2·22}**$$

Expressed in terms of $\chi(x)$, I_i and K_i are given by

$$I_i = (1/2) \int \chi^*(x) \tau_i \chi(x) dx, \tag{2·23}$$

and

$$K_i = (1/2) \int \chi^*(x) \kappa_i \chi(x) dx. \tag{2·24}$$

It would be obvious from (2·9), (2·13), (2·15) and (2·16) that their transformation properties are given by

$$\mathcal{I}^{-1} \chi^{(\pm)}(x) \mathcal{I} = \chi^{(\pm)}(x) + (i/2) \varepsilon_i \tau_i \chi^{(\pm)}(x), \tag{2·25}$$

$$\mathcal{I}^{-1} \chi(x) \mathcal{I} = \chi(x) + (i/2) \varepsilon_i \tau_i \chi(x), \tag{2·26}$$

$$\mathcal{K}^{-1} \chi^{(\pm)}(x) \mathcal{K} = \chi^{(\pm)}(x) + (i/2) \varepsilon_i \kappa_i \chi^{(\pm)}(x), \tag{2·27}$$

and

$$\mathcal{K}^{-1} \chi(x) \mathcal{K} = \chi(x) + (i/2) \varepsilon_i \kappa_i \chi(x). \tag{2·28}$$

§ 3. Electromagnetic and pion-nucleon interaction

According to assumptions made in § 1, the electromagnetic interaction must be transformed as the third component of a vector, and the pion-nucleon interaction as a scalar in I- and K-spaces. In this section we shall examine whether the interactions usually used satisfy these requirements, and if they do not, what modification must be made.

a) Electromagnetic interaction

The usual electromagnetic interaction is

$$j_\mu(x) A_\mu(x) = (ie/2) \bar{\psi}(x) \gamma_\mu (\tau_3 + 1) \psi(x) A_\mu(x). \tag{3·1}$$

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* Of course, the results of this paper are entirely independent of the choice of representation.

** $\tilde{\psi}(x)$ is defined by

$$\tilde{\psi}(x) = \begin{pmatrix} \tilde{\psi}_p(x) \\ \tilde{\psi}_n(x) \end{pmatrix},$$

where $\tilde{\psi}_p(x)$ and $\tilde{\psi}_n(x)$ are the charge conjugated field of $\psi_p(x)$ and $\psi_n(x)$ respectively.

This can be written in the following form:

$$(ie/2)\,\bar{\chi}(x)\,\gamma_\mu(\tau_3+\kappa_3)\,\chi(x)\,A_\mu(x). \tag{3·2}$$

Therefore, the electromagnetic interaction transforms as the third component of a vector in *I*- and *K*-spaces, which is just the required form.

b) Pion-nucleon interaction

While the electromagnetic interaction satisfies our requirement, this is not the case with the pion-nucleon interaction as will be shown in the following. Consider, for example, the usual pseudoscalar interaction

$$H(x)=ig\sum_{i=1}^{3}\bar{\psi}(x)\,\gamma_5\tau_i\psi(x)\,\phi_i(x). \tag{3·3}*$$

Here $\bar{\psi}(x)\gamma_5\tau_i\psi(x)$ can be written as

$$\bar{\chi}(x)\gamma_5\tau_i\kappa_3\chi(x). \tag{3·4}$$

From (2·28) we see that (3·4) is not a scalar but the third component of a vector in *K*-space.

Before discussing necessary modifications, it is instructive to examine (3·4) in some more detail. Decomposing into positive and negative frequency parts,

$$\begin{aligned}\bar{\chi}(x)\gamma_5\tau_i\kappa_3\chi(x)=&\overline{\chi^{(+)}}(x)\gamma_5\tau_i\kappa_3\chi^{(+)}(x)+\overline{\chi^{(-)}}(x)\gamma_5\tau_i\kappa_3\chi^{(-)}(x)\\&+\overline{\chi^{(+)}}(x)\gamma_5\tau_i\kappa_3\chi^{(-)}(x)+\overline{\chi^{(-)}}(x)\gamma_5\tau_i\kappa_3\chi^{(+)}(x).\end{aligned} \tag{3·5}$$

Using (2·19), the second term of the right-hand side of this equation can be written as

$$\chi^{(+)*}(x)\gamma_4\gamma_5\tau_i\kappa_3\chi^{(+)}(x), \tag{3·6}$$

which is identical with the first term. They describe the emission and the absorption of pion by nucleon or by antinucleon, and correspond to the graphs Fig. 1 and Fig. 2.

Similarly the third and the fourth terms can be written as

$$i\chi^{(+)*}\gamma_4\gamma_5\tau_i\tau_2\kappa_1\chi^{(+)*}(x), \tag{3·7}$$

and

$$-i\chi^{(+)}(x)\gamma_4\gamma_5\tau_2\tau_i\kappa_1\chi^{(+)}(x). \tag{3·8}$$

Fig. 1

Fig. 2

Fig. 3

Fig. 4

* $\phi_i(i=1, 2, 3)$ are real fields, and ϕ_1 and ϕ_2 are related to ϕ and ϕ^* by $\phi=(1/\sqrt{2})\,(\phi_1+i\phi_2)$ and $\phi^*=(1/\sqrt{2})\,(\phi_1-i\phi_2)$.

They describe the creation and the annihilation of nucleon pairs, and correspond to the graphs Fig. 3 and Fig. 4.

We shall next discuss modifications necessary to make (3·4) invariant in K-space. For this, it is important to notice that $\chi(x)$ and $\chi'(x)$ are the only quantities that transform as spinor in K-space. Therefore, to obtain scalar in a form resembling the original one as closely as possible, we must replace κ_3 by unity. Thus, terms in (3·5) must be modified according to

$$\begin{aligned}
&\overline{\chi^{(+)}}\gamma_5\tau_i\kappa_3\chi^{(+)} \to \overline{\chi^{(+)}}\gamma_5\tau_i\chi^{(+)},\\
&\overline{\chi^{(-)}}\gamma_5\tau_i\kappa_3\chi^{(-)} \to \overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(-)},\\
&\overline{\chi^{(+)}}\gamma_5\tau_i\kappa_3\chi^{(-)} \to \overline{\chi^{(+)}}\gamma_5\tau_i\chi^{(-)},
\end{aligned} \tag{3·9}$$

and

$$\overline{\chi^{(-)}}\gamma_5\tau_i\kappa_3\chi^{(+)} \to \overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(+)}.$$

If these modifications are made, however, (3·5) vanishes. To see this, it is sufficient to observe that

$$\begin{aligned}
&\overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(-)} = -\overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(-)},\\
&\overline{\chi^{(+)}}\gamma_5\tau_i\chi^{(-)} = -\overline{\chi^{(+)}}\gamma_5\tau_i\chi^{(-)},
\end{aligned} \tag{3·10}$$

and

$$\overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(+)} = -\overline{\chi^{(-)}}\gamma_5\tau_i\chi^{(+)}.$$

Thus, when modified according to (3·9), the first and second terms of the right-hand side of (3·5) cancel each other, and the third and fourth terms vanish by themselves.

As for the first and the second terms, one way of avoiding cancellation is to alter the sign before the second term, and to use

$$\overline{\chi^{(+)}}(x)\gamma_5\tau_i\chi^{(+)}(x) - \overline{\chi^{(-)}}(x)\gamma_5\tau_i\chi^{(-)}(x). \tag{3·11}$$

If we remove the restriction that the interaction must be local, another possibility arises. For example, we can use a non-local interaction of the type

$$\iint dx_1 dx_2 \rho(x_1-x,\ x_2-x)\left[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2) + \overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)\right], \tag{3·12}$$

where $\rho(x_1-x,\ x_2-x)$ is assumed as scalar for simplicity. Here the second term can be written as

$$-\iint dx_1 dx_2 \rho(x_2-x,\ x_1-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2).$$

Therefore (3·12) is equivalent to

$$2\iint dx_1 dx_2 \rho_a(x_1-x,\ x_2-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2), \tag{3·13}$$

where $\rho_a(x_1-x,\ x_2-x)$ means the antisymmetric part of $\rho(x_1-x,\ x_2-x)$ under the exchange of x_1 and x_2:

$$\rho_a(x_1-x,\ x_2-x)=(1/2)[\rho(x_1-x,\ x_2-x)-\rho(x_2-x,\ x_1-x)]. \qquad (3\cdot14)$$

Taking the hermitean conjugate of (3·13), we get

$$2\iint dx_1 dx_2 \rho_a^*(x_1-x,\ x_2-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2).$$

Therefore, (3·12) is hermitean if $\rho(x_1-x,\ x_2-x)$ is assumed as real, and hermitean interaction hamiltonian corresponding to the graphs Fig. 1 and Fig. 2 is given by

$$g\iint dx_1 dx_2 \rho_a(x_1-x,\ x_2-x)[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2)+\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)]\phi_i(x). \qquad (3\cdot15)$$

As for the third and the fourth terms, the former way can not be used, since the cancellation occurs by themselves. Therefore, we are necessarily led to the second possibility, and the introduction of the non-local interaction becomes inevitable. Thus, the third term must be modified to

$$g\iint dx_1 dx_2 \rho'(x_1-x,\ x_2-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2), \qquad (3\cdot16)$$

where $\rho'(x_1-x,\ x_2-x)$ are assumed as scalar as before. This can easily be written in the form

$$g\iint dx_1 dx_2 \rho_a'(x_1-x,\ x_2-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2), \qquad (3\cdot17)$$

where $\rho_a'(x_1-x,\ x_2-x)$ is the antisymmetric part of $\rho'(x_1-x,\ x_2-x)$.

As for the fourth term, the situation is exactly the same, and we are led to

$$g\iint dx_1 dx_2 \rho_a''(x_1-x,\ x_2-x)\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2). \qquad (3\cdot18)$$

The hermitean conjugate of (3·17) and (3·18) are

$$g\iint dx_1 dx_2 \rho_a'^*(x_1-x,\ x_2-x)\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2),$$

and

$$g\iint dx_1 dx_2 \rho_a''^*(x_1-x,\ x_2-x)\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2). \qquad (3\cdot19)$$

Comparing (3·17) and (3·18) with (3·19), we see that it is necessary to assume that $\rho'(x_1-x,\ x_2-x)$ and $\rho''(x_1-x,\ x_2-x)$ must be real, and further be subjected to the condition

$$\rho'(x_1-x,\ x_2-x)=\rho''(x_1-x,\ x_2-x)$$

to obtain hermitean interaction.

Under this condition, interaction hamiltonian corresponding to the graphs Fig. 3 and Fig. 4 is given by

$$\iint dx_1\, dx_2 \rho_\alpha'(x_1-x,\ x_2-x)\left[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)+\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2)\right]\phi_i(x). \tag{3·20}$$

Summarizing, it is seen that pion-nucleon interaction hamiltonian that is invariant under rotations both in I- and K- spaces is given in either of the following forms:

$$H(x)=g\sum_{i=1}^{3}\{i[\overline{\chi^{(+)}}(x)\gamma_5\tau_i\chi^{(-)}(x)-\overline{\chi^{(-)}}(x)\gamma_5\tau_i\chi^{(-)}(x)] + \iint dx_1 dx_2 \rho_\alpha'(x_1-x,\ x_2-x)\left[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)+\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2)\right]\}\phi_i(x), \tag{3·21}$$

or

$$H(x)=g\sum_{i=1}^{3}\iint dx_1 dx_2 \times \times\{\rho_\alpha(x_1-x,\ x_2-x)\left[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2)+\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)\right] +\rho_\alpha'(x_1-x,\ x_2-x)\left[\overline{\chi^{(+)}}(x_1)\gamma_5\tau_i\chi^{(-)}(x_2)+\overline{\chi^{(-)}}(x_1)\gamma_5\tau_i\chi^{(+)}(x_2)\right]\}\phi_i(x). \tag{3·22}$$

If we assume in the second alternative

$$\rho(x_1-x,\ x_2-x)=\rho'(x_1-x,\ x_2-x), \tag{3·23}$$

(3·22) reduces to the simple form

$$H(x)=g\sum_{i=1}^{3}\iint dx_1 dx_2 \rho_\alpha(x_1-x,\ x_2-x)\bar{\chi}(x_1)\gamma_5\tau_i\chi(x_2)\phi_i(x). \tag{3·24}$$

These are the results obtained by making the minimum extension to the usual one. The essential point is that it must be non-local, especially the part responsible for the creation and the annihilation of nucleon pairs.* In this point, pion-nucleon interaction behaves very differently from electromagnetic interaction, and we see that a qualitative difference between these two interactions is revealed by requiring the invariance under the rotation in K-space.

Written in the form of (3·3), equations (3·21), (3·22) and (3·24) are given by

$$H(x)=g\sum_{i=1}^{3}\{i[\overline{\psi^{(+)}}(x)\gamma_5\tau_i\psi^{(+)}(x)-\overline{\psi^{(-)}}(x)\gamma_5\tau_i\psi^{(-)}(x)] + \iint dx_1 dx_2 \rho_\alpha(x_1-x,\ x_2-x)\left[\overline{\psi^{(+)}}(x_1)\gamma_5\tau_i\psi^{(-)}(x_2)+\overline{\psi^{(-)}}(x_1)\gamma_5\tau_i\psi^{(+)}(x_2)\right]\}\phi_i(x), \tag{3·25}$$

* Exactly the same situation occurs for pseudovector interaction, too. Thus we see that this non-locality is a general feature of pion-nucleon interaction so long as we assume that it is bilinear in nucleon field and linear in pion field.

$$H(x)=g\sum_{i=1}^{3}\iint dx_1 dx_2\times$$
$$\times\{\rho_a(x_1-x,\ x_2-x)[\overline{\psi^{(+)}}(x_1)\gamma_5\tau_i\psi^{(+)}(x_2)+\overline{\psi^{(-)}}(x_1)\gamma_5\tau_i\psi^{(-)}(x_2)]$$
$$+\rho_a'(x_1-x,\ x_2-x)[\overline{\psi^{(+)}}(x_1)\gamma_5\tau_i\psi^{(-)}(x_2)+\overline{\psi^{(-)}}(x_1)\gamma_5\tau_i\psi^{(+)}(x_2)]\}\phi_i(x), \tag{3·26}$$

and

$$H(x)=g\sum_{i=1}^{3}\iint dx_1 dx_2\rho_a(x_1-x,\ x_2-x)\,\overline{\psi}(x_1)\gamma_5\tau_i\psi(x_2)\,\phi_i(x), \tag{3·27}$$

where $\psi^{(-)}(x)$ and $\psi^{(-)}(x)$ denote positive and negative frequency parts of $\psi(x)$ respectively. These interaction hamiltonians involve form factors and moreover (3·25) and (3·26) are written in terms of $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$. So they have very complicated non-locality, and do not satisfy the requirement of causality in general. Into these difficulties, however, we do not proceed at present.

§ 4. Charge conjugation

i) *For nucleon*

Denoting charge conjugation operator by C, we have by definition

$$C^{-1}\psi_p(x)\,C=\tilde{\psi}_p(x),$$
$$C^{-1}\psi_n(x)\,C=\tilde{\psi}_n(x),$$
$$C^{-1}\tilde{\psi}_p(x)\,C=\psi_p(x),$$
and
$$C^{-1}\tilde{\psi}_n(x)\,C=\psi_n(x). \tag{4·1}$$

Therefore,

$$C^{-1}\chi(x)\,C=\frac{1}{\sqrt{2}}C^{-1}\begin{bmatrix}\psi_p(x)\\ \psi_n(x)\\ -\tilde{\psi}_n(x)\\ \tilde{\psi}_p(x)\end{bmatrix}C$$
$$=\frac{1}{\sqrt{2}}\begin{bmatrix}\tilde{\psi}_p(x)\\ \tilde{\psi}_n(x)\\ -\psi_n(x)\\ \psi_p(x)\end{bmatrix}$$
$$=(i\kappa_2)\,(i\tau_2)\,\chi(x). \tag{4·2}$$

Thus, in our picture, charge conjugation is equivalent to the rotation through π around the second axis in K-space multiplied by a rotation through π around the second axis in I-space,* and C is given by

* These axes can be the first axes as well. They are related by simple unitary transformations.

$$C=\exp[i\pi(I_2+K_2)]. \tag{4·3}$$

The equation of motion for $\chi(x)$ is

$$\{\gamma_\mu[\partial_\mu-(ie/2)(\tau_3+\kappa_3)A_\mu]+M\}\chi(x)=0.$$

Under the rotation (4·3), this goes into

$$\{\gamma_\mu[\partial_\mu+(ie/2)(\tau_3+\kappa_3)A_\mu]+M\}\tilde{\chi}(x)=0,$$

where

$$\tilde{\chi}(x)=(i\kappa_2)(i\tau_2)\chi(x).$$

This is just the equation of motion to be satisfied by the charge conjugated $\chi(x)$.

ii) *For pion*

Usually charge conjugation for pion is defined by

$$\begin{aligned} &C^{-1}\phi_1 C=\phi_1,\\ &C^{-1}\phi_2 C=-\phi_2, \end{aligned} \tag{4·4}$$

and

$$C^{-1}\phi_3 C=\phi_3.$$

It must be noted, however, that there is no reason to fix the phase like this, and we can use the following definition as well:

$$\begin{aligned} &C^{-1}\phi_1 C=-\phi_1,\\ &C^{-1}\phi_2 C=\phi_2, \end{aligned}$$

and

$$C^{-1}\phi_3 C=-\phi_3. \tag{4·5}*$$

If we adopt the second alternative, charge conjugation for pion can be expressed by a rotation in I-space, and we have

$$C=\exp(i\pi I_2), \tag{4·6}$$

where for pion I_i is given by

$$I_i=\int[\boldsymbol{\phi}(x)\times\boldsymbol{\pi}(x)]_i\,dx, \tag{4·7}$$

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* In terms of ϕ, (4·5) corresponds to

$$\phi\rightarrow-\phi^*,$$
$$\phi^*\rightarrow-\phi,$$

and

$$\phi_0\rightarrow-\phi_0,$$

whereas (4·4) corresponds to

$$\phi\rightarrow\phi^*,$$
$$\phi^*\rightarrow\phi,$$

and

$$\phi_0\rightarrow\phi_0.$$

where bold face letters mean vector in I-space, and $\pi_i(x)$ is the momentum canonically conjugate to $\phi_i(x)$.

Thus we see that at least for nucleon and pion, charge conjugation can be replaced by rotations in the charge space.

Here it must be noted that if we adopt (4·5), and formulate the interaction in a form invariant under it, some results of the usual theory must be modified. For example, 2γ-decay of π^0 through virtual nucleon pair becomes forbidden. To cause this process, we must introduce the interaction of the form

$$G(\boldsymbol{E}\cdot\boldsymbol{H})\phi_3$$

somewhat arbitrarily, where $\boldsymbol{E}$ and $\boldsymbol{H}$ denote electric and magnetic field strength respectively.

§ 5. Experimental check of the theory

As seen in § 3, our assumption leads to pion-nucleon interaction somewhat different from the usual theory. Therefore, it is possible to check its validity by comparing the prediction from it with experiments. Of course, it is not simple to get unambiguous result, since our hamiltonian is a very complicated non-local one, and moreover the explicit form of the form factor is not known. To get unambiguous result, therefore, we must seek for characteristic features of the theory not depending upon the detail, or process where definite predictions can be given from kinematical consideration alone. In the following will be discussed some examples of them.

i) *Nucleon-antinucleon potential*

As seen in (3·9), it was necessary to replace κ_3 by a unit matrix to get invariants under rotation in K-space. This means that the vertex responsible for the emission and the absorption of pion by antinucleon becomes oppositely signed as compared with the ordinary one. Therefore, the potential between a nucleon and an antinucleon resulting from the exchange of odd number of pions becomes oppositely signed as compared with the ordinary one. In particular, the simplest second order potential becomes oppositely signed. This may be checked by examining the angular distribution of antinucleon scattered by nucleon.

ii) *Annihilation of antinucleon by nucleon*

An example of the process where definite prediction can be given from the kinematical consideration alone is the annihilation of antinucleon by nucleon. Consider, for example, a system composed of a nucleon and an antinucleon. Its nucleon number is zero. According to our picture, however, there can be two states corresponding to $K=0$ and 1 respectively. The former can annihilate into pions as usual, but for the latter this is impossible from the conservation of $\boldsymbol{K}$, and it can annihilate only by emitting γ-ray. (The selection rule for γ-transition is given by $\Delta I=\pm 1, 0$, $\Delta I_3=0$ and $\Delta K=\pm 1, 0$, $\Delta K_3=0$, since electromagnetic interaction transforms as the third component of a vector in I- and K-spaces.) Thus, according to our picture, the neutral annihilation product will become

more abundant than would be expected from the usual theory.*

It is difficult to estimate the fraction of the energy carried by the γ-ray thus produced. In the following will be given a very rough estimate based on the statistical theory.[3]

When the annihilation takes place, the total energy is released to the pion cloud. In the usual theory, it is assumed that a statistical equilibrium is reached in this pion cloud. According to our picture, however, a somewhat different situation exists in the state corresponding to $K=1$. In this state the most rapid process is forbidden, and therefore the lifetime is lengthened. We assume that during this time, the modes for γ-rays are also excited, and reach a statistical equilibrium. If this is assumed, the fraction of the energy carried by γ-ray can be estimated from a simple statistical formula.

According to Belenky and Landau,[4] the particle density and the energy density of pion cloud in a thermal equilibrium n_π and $\mathcal{E}_\pi$ is given by

$$n_\pi = (g_\pi/2\pi^2)\,(T/\hbar c)^3 F_\pi(z),$$

and

$$\mathcal{E}_\pi = T\,(g_\pi/2\pi^2)\,(T/\hbar c)^3 \Phi_\pi(z), \tag{5·1}$$

where $z=\mu c^2/T$, μ the pion mass and g_π is the statistical weight. $F_\pi(z)$ and $\Phi_\pi(z)$ are some functions of z defined by

$$F_\pi(z) = z^3 \int_0^\infty \frac{x^2\,dx}{\exp(z\sqrt{1+x^2})-1},$$

and

$$\Phi_\pi(z) = z^4 \int_0^\infty \frac{\sqrt{1+x^2}\,x^2\,dx}{\exp(z\sqrt{1+x^2})-1}. \tag{5·2}$$

On the other hand, those for γ-ray are given by Planck's formula as follows:

$$n_\gamma = (g_\gamma/2\pi^2)\,(T/\hbar c)^3 F_\gamma, \tag{5·3}$$

and

$$\mathcal{E}_\gamma = T\,(g_\gamma/2\pi^2)\,(T/\hbar c)^3 \Phi_\gamma,$$

where

$$F_\gamma = \int_0^\infty \frac{x^2\,dx}{e^x-1},$$

and

$$\Phi_\gamma = \int_0^\infty \frac{x^3\,dx}{e^x-1}.$$

From (5·1) and (5·3), we get

* Here we discuss annihilation at rest exclusively. To annihilation in flight will contribute only the states corresponding to $K=0$, so that no difference will appear in this case.

$$n_\gamma/n_\pi = (g_\gamma/g_\pi)\cdot(F_\gamma/F_\pi(z)),$$

and

$$\mathcal{E}_\gamma/\mathcal{E}_\pi = (g_\gamma/g_\pi)\cdot(\Phi_\gamma/\Phi_\pi(z)).$$

$F_\gamma/F_\pi(z)$ and $\Phi_\gamma/\Phi_\pi(z)$ are slowly varying functions of z, and, at $z=1$, are 1.3 and 1.1 respectively. g_γ and g_π are 2 and 3 corresponding to two directions of polarization of photon and three charge states of pion. Therefore, if we assume that at equilibrium $z\simeq 1$,

$$n_\gamma/n_\pi \simeq 1.3\,(g_\gamma/g_\pi) = 0.87,$$

and

$$\mathcal{E}_\gamma/\mathcal{E}_\pi \simeq 1.1\,(g_\gamma/g_\pi) = 0.73.$$

Thus, it is estimated that about 40% of the energy will be carried by the direct γ-ray in the state corresponding to $K=1$. Since the states corresponding to $K=0$ and 1 exist equally, the average fraction of the energy carried by γ-ray will be about 20%, and fractions of the energy carried by neutral and charged annihilation products are given by

$$\text{neutral} \simeq 7/15,$$

and

$$\text{charged} \simeq 8/15.$$

Therefore

$$\text{neutral/charged} \simeq 0.9. \qquad (5\cdot4)^*$$

According to the usual theory, this ratio is estimated as

$$\text{neutral/charged} \simeq 0.5 \qquad (5\cdot5)$$

from Watson's theorem.[5] $(5\cdot5)$ is considerably different from $(5\cdot4)$, and the validity of our assumption may be checked by examining which of them fits experiments.**, ***

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* It should be noted that this result should be regarded as the uppper limit, since the assumption that γ-ray also reaches a statistical equilibrium presupposes a strong interaction between γ-ray and pion cloud.

** Recent experimental value for this is 0.53.[6] At first sight (5·4) seems to be inconsistent with this value. It should be noted, however, that these experiments are done with emulsion. Therefore, the annihilation occurs in large nuclei, and various corrections must be made, which makes the result ambiguous. More definite result may be obtained if experiments are made with liquid hydrogen or liquid deuterium bubble chamber. Especially liquid deuterium bubble chamber will be convenient, since in this case Watson's theorem holds exactly, and we need not worry about from which orbit antiproton annihilates.

*** The average number of pion produced in the annihilation of antinucleon (n_π) is believed to be 5.3±0.4 (Barkas et al.[6]) or 4.7±0.4 (Ekspong[7]). These were estimated by two different methods:

i) from the average number of charged pion observed ($n_{\pi^\pm}$). n_π is given by $1.5\times n_{\pi^\pm}$ from charge independence. This gave 5.3±0.6 (Barkas et al.) and 4.7±0.4 (Ekspong).

ii) from the total energy (W) and the average energy of charged pion ($E_{\pi^\pm}$). n_π is given by $W/E_{\pi^\pm}$. This gave 5.2±0.5 (Barkas et al.).

These results seem to be somewhat larger than would be expected from the statistical theory, and also there seems to be a tendency that the later estimate gives always larger value. According to our scheme, in the annihilation at rest, about 20% of the energy is carried by the direct γ-ray. This alters the interpretation of the data incase ii), and n_π is given by (W−energy carried by γ-ray)$/E_{\pi^\pm}$. This gives, if we asume tentatively that the annihilation occurs mainly at rest, 4.2±0.5 (from Barkas et al.'s data) and 4.4 (from Ekspong's data) respectively.

§ 6. Concluding remarks

Thus far, we have shown that by introducing K-space, and assuming invariance under rotations in this space, it is possible to reformulate pion-nucleon interaction in such a form that various anomalies recenty found might be understood better.

In conclusion, a few comments will be added.

i) *The problem of pair suppression*

Our theory relates to this problem in two points. The first point is that our interaction is non-local. Therefore the effect of virtual pair will be cut off at the corresponding critical length, and this will reduce the pair effect.

The second point is that in our picture the Pauli principle operates in a form somewhat different from the usual theory in intermediate states. Consider, for example, the decay of pion through nucleon pairs:

$$\pi \to N + \widetilde{N} \to e + \nu.$$

According to the usual theory, such process occurs inevitably so long as pion-nucleon interactions and β-decay interactions exist. Several attempts have been proposed to overcome this difficulty,[8] but if the pseudoscalar type interaction exist in β-decay interaction, all of them would fail.

According to our picture, the situation is changed. In the rest system of the original pion, the intermediate state composed of a nucleon and an antinucleon must have angular momentum zero and parity odd since pion is pseudoscalar. It must have at the same time $I=1$ and $K=0$ since for pion $I=1$ and $K=0$. It is clear, however, that it is impossible to construct such a wave function in accordance with the Pauli principle. Therefore, in our picture, such intermediate states are completely suppressed, and the decay of pion into $e+\nu$ through nucleon pair is completely forbidden.* At present it is not definite whether the pseudoscalar type interaction exist or not. If its existence is established, it would be worth while to adopt suppression due to this mechanism somewhat seriously, since until now no sensible idea has been proposed for this problem.

ii) *The effect of non-locality on the scattering of high energy pion by nucleon*

At present we have no principle to determine the form of form factor. Therefore, it must be determined phenomenologically. Annihilation cross section of antinucleon by nucleon and experiments at Stanford suggest an interaction range of the order of the Compton wave length of pion.

For pion with kinetic energy larger than about one Bev, its de Broglie wave length becomes shorter than this range, and it is expected that the effect of non-locality will appear explicitly. For example, it is expected that in the scattering of high energy pion by nucleon, the dispersion relation based on microcausality might be destroyed in some cases. In this respect, Puppi's recent result[9] seems to be very interesting.

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* Strictly speaking, this is not more than a conjecture. The reason is that the wave function might lose meaning in intermediate states, since our interaction is non-local, and so the usual hamiltonian formalism will not be applicable there. At present we have no theory to treat non-local interactions correctly, but it would be reasonable to assume that mechanism mentioned here will suppress the pair effect strongly.

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Note added in proof In usual literatures annihilation in flight and at rest are not distinguished. If we assume that both processes occur equally, and take average over both of them, the fraction of the energy carried by the γ-ray become about 10%, and the fraction of the energy between the neutral and charged annihilation product is given by

$$\text{neutral/charged} \simeq 0.7$$

instead of (5·5). Also multiplicity of pion in the annihilation calculated from

$$(Wc^2 - \text{energy carrid by } \gamma\text{-ray})/E_{\pi\pm}$$

is modified to 4.6 ± 0.5 (from Barkas et al's data) and 4.8 (from Ekspong's data) respectively. It should be noted that these values are very close to Ekspong's result (4.7 ± 0.4) obtained from the first method in the footnote of p. 143.

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Invariants in $K_{\mu 3}$ and K_{e3} Decays*

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A discussion is given of the $K_{\mu 3}$ and K_{e3} decay modes, making use of the invariance properties of the theory, under the general hypothesis of parity nonconservation and lepton nonconservation. The assumption of local interaction between the final fermions is examined in detail. Detection of a possible updown asymmetry, with respect to the $K_{\mu 3}$ decay plane, of the electrons from the subsequent μ decay is suggested as a possible test of time reversal.

The $K_{\mu 3}$ and K_{e3} decay modes have recently been discussed by Furuichi et al.[1] and by Pais and Treiman.[2] We present here some general results for such decays which can be directly derived from the invariance properties of the theory. We consider the general case with parity nonconservation and with nonconservation of leptons. The discussion will be limited to local interaction between the two final fermions, and a detailed examination of such a hypothesis will be given.

Following the methods used by Furuichi et al. and by Pais and Treiman, we write the transition-matrix element for a decay process $K \to l + \pi + \nu$, where l denotes either μ or e, in the form

$$T_{fi} = (\bar{l}(C_S + C_S'\gamma_5)\nu) + \frac{i}{m_K}(\bar{l}\gamma_\alpha V_\alpha (C_V + C_V'\gamma_5)\nu) + \frac{1}{m_K^2}(\bar{l}\sigma_{\alpha\beta} V_\alpha U_\beta (C_T + C_T'\gamma_5)\nu) +$$
$$+ (\bar{l}(B_S + B_S'\gamma_5)\nu^C) + \frac{i}{m_K}(\bar{l}\gamma_\alpha V_\alpha' (B_V + B_V'\gamma_5)\nu^C) + \frac{1}{m_K^2}(\bar{l}\sigma_{\alpha\beta} V_\alpha' U_\beta' (B_T + B_T'\gamma_5)\nu^C). \tag{1}$$

In eq. (1) we have denoted the free-particle spinors with the same symbol as used for the corresponding particle; V_α, U_α, V_α' and U_α' are linear combinations of the independent four-momenta in the decay; the scalars C, C', B and B' are also formed from the independent four-momenta; m_K is the mass of the decaying heavy meson. We use the Pauli notations; in particular $\bar{l} = l^*\gamma_4$, $\sigma_{\alpha\beta} = -(i/2)[\gamma_\alpha, \gamma_\beta]$ and $\nu^C = C^{-1}\bar{\nu}$, where C is the charge-conjugation matrix, satisfying $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ and $CC^+ = 1$. The 4 available four-momenta are restricted by conservation of the total four-momentum. In both references 1 and 2 it is assumed that the four-momenta of l and of ν appear in T only in the combination $P^{(l)} + P^{(\nu)}$. This assumption rests on the hypothesis that both leptons are produced at the same local vertex and on the neglect of electromagnetic corrections.

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We shall refer to this assumption as the locality hypothesis. It must be stressed that such locality here refers only to the weak interaction producing the lepton pair. Because of total momentum conservation and of the locality hypothesis, only two four-momenta are independent, and we choose as such the K four-momentum P_α and the π four-momentum p_α.[3] Making use of the locality hypothesis, of the Dirac equation for the leptons, and of the antisymmetry of $\sigma_{\alpha\beta}$, we can write eq. (1) in the form

$$T_{fi}=(\bar{l}(G_S+G_S'\gamma_5)\nu)+\frac{i}{M}(\bar{l}\gamma_\alpha P_\alpha(G_V+G_V'\gamma_5)\nu)-\frac{1}{M^2}(\bar{l}\sigma_{\alpha\beta}p_\alpha P_\beta(G_T+G_T'\gamma_5)\nu)+$$

$$+(\bar{l}(F_S+F_S'\gamma_5)\nu^C)+\frac{i}{M}(\bar{l}\gamma_\alpha P_\alpha(F_V+F_V'\gamma_5)\nu^C)-\frac{1}{M^2}(\bar{l}\sigma_{\alpha\beta}p_\alpha P_\beta(F_T+F_T'\gamma_5)\nu). \quad (2)$$

In eq. (2) G_S is of the form C_S+ (constant) $(m_l/m_K)C_V$, where m_l is the mass of the emitted lepton, and similar relations hold for G'_S, F_S and F'_S. Therefore, as remarked by Treiman and Pais,[2] even in the absence of scalar interaction at the lepton vertex, the transition matrix will contain a scalar term proportional to m_l if any vector interaction is present. This term, however, is negligible for K_{e3} decay because of the smallness of the electron mass.

As first shown by Pauli and by Pursey, under the assumption of zero rest mass for the neutrino, there exist two commuting subgroups of transformations of the neutrino field which leave the free-particle equation and the commutation relations invariant and transform the interaction Hamilionian into a physically equivalent Hamiltonian.[4),5),6),7)] The transformations of the first subgroup are

$$\text{(I)}\quad \nu'=\alpha\nu+\beta\gamma_5 C^{-1}\bar{\nu},\quad \bar{\nu}'=\alpha^*\bar{\nu}+\beta^*\nu C\gamma_5$$

with α and β constants, satisfying

$$|\alpha|^2+|\beta|^2=1.$$

The transformations of the second subgroup are

$$\text{(II)}\quad \nu'=\exp\left[\frac{1}{2}i\tilde{\xi}+\frac{1}{2}i\eta\gamma_5\right]\nu,\quad \bar{\nu}'=\bar{\nu}\exp\left[-\frac{1}{2}i\tilde{\xi}+\frac{1}{2}i\eta\gamma_5\right]$$

with $\tilde{\xi}$ and η real. A transformation (I) is equivalent to a substitution on the scalar quantities G_j, G_j', F_j and F_j' of eq. (2) according to

$$G_j\to G_j\alpha^*+F_j'\beta^*,\quad G_j'\to G_j'\alpha^*+F_j\beta^*, \quad (3)$$

$$F_j\to F_j\alpha-G_j'\beta,\qquad F_j'\to F_j'\alpha-G_j\beta. \quad (3')$$

A transformation (II) is equivalent to

$$G_j+G_j'\to(G_j+G_j')\,e^{i(1/2)\eta+i(1/2)\tilde{\xi}},\quad G_j-G_j'\to(G_j-G_j')\,e^{-i(1/2)\eta+i(1/2)\tilde{\xi}} \quad (4)$$

$$F_j+F_j'\to(F_j+F_j')\,e^{i(1/2)\eta-i(1/2)\tilde{\xi}},\quad F_j-F_j'\to(F_j-F_j')\,e^{-i(1/2)\eta-i(1/2)\tilde{\xi}}. \quad (4')$$

Since we shall be concerned with the product transformation of (I) and (II), we can

choose $\xi=0$, because an arbitrary phase factor is already contained in (I). Following the method in references 4, 5 and 6, we construct the quantities

$$V_j^{(+)}=\begin{bmatrix} G_j^{(+)} \\ F_j^{(+)} \end{bmatrix} \qquad V_j^{(-)}=\begin{bmatrix} F_j^{(-)*} \\ G_j^{(-)*} \end{bmatrix}, \tag{5}$$

where $G_j^{(\pm)}=G_j\pm G_j'$, $F_j^{(\pm)}=F_j\pm F_j'$. Such quantities transform under the product of (I) and (II) according to

$$V_j^{(\pm)}\to MV_j^{(\pm)}; \tag{6}$$

the matrix M is given by

$$M=e^{i(1/2)\eta}\begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}. \tag{7}$$

Regarding $V_j^{(\pm)}$ as vectors in a two-dimensional space (with complex coordinates), one sees that they are transformed according to a unitary transformation. The invariants under the group of Transformations (7) are given by the scalar products

$$V_j^{(+)\dagger}V_l^{(+)}=G_j^{(+)*}G_l^{(+)}+F_j^{(+)*}F_l^{(+)}, \tag{8}$$

$$V_j^{(-)\dagger}V_l^{(-)}=G_j^{(-)}G_l^{(-)*}+F_j^{(-)}F_l^{(-)*}, \tag{8'}$$

$$V_j^{(+)\dagger}V_l^{(-)}=G_j^{(+)*}F_l^{(-)*}+F_j^{(+)*}G_l^{(-)*}, \tag{8''}$$

and their complex conjugates. Only invariants of the forms (8) and (8′) can appear in the description of the decay process. It will be convenient to substitute for them an equivalent set of invariants which have the property of remaining unchanged under one of the operations T, C, P or under all three such operations. To this purpose one has only to substitute into eqs. (8) and (8′) the expressions

$$\tfrac{1}{2}\boldsymbol{R}[V_j^{(+)\dagger}V_l^{(+)}\pm V_l^{(-)\dagger}V_j^{(-)}], \tag{9}$$

$$\tfrac{1}{2}\boldsymbol{I}[V_j^{(+)\dagger}V_l^{(+)}\pm V_l^{(-)\dagger}V_j^{(-)}]. \tag{9'}$$

One finds that eighteen different invariants can be constructed in this way. The list is as follows:

$$E_j=|G_j|^2+|G_j'|^2+|F_j|^2+|F_j'|^2, \tag{10}$$

$$D_j=\boldsymbol{R}[G_j^*G_j'+F_j^*F_j'], \tag{10'}$$

$$E_{jl}=\boldsymbol{R}[G_j^*G_l+G_j'^*G_l'+F_j^*F_l+F_j'^*F_l']=E_{lj}, \tag{10''}$$

$$D_{jl}=\boldsymbol{R}[G_j^*G_l'+G_j'^*G_l+F_j^*F_l'+F_j'^*F_l]=D_{lj}, \tag{10'''}$$

$$Q_{jl}=\boldsymbol{I}[G_j^*G_l+G_j'^*G_l'+F_j^*F_l+F_j'^*F_l']=-Q_{lj}, \tag{10''''}$$

$$N_{jl}=\boldsymbol{I}[G_j^*G_l'+G_j'^*G_l+F_j^*F_l'+F_j'^*G_l]=-N_{lj}. \tag{10'''''}$$

For each of the above six classes, there are three independent invariants. These invariants are not all independent, but have to satisfy relations and inequalities that follow from their algebraic structure.

In the K_{e3} decay modes it is a good approximation to neglect the electron mass in comparison with the large available energy. In this approximation, and for experiments that do not involve any observation of the possible *transverse* polarization of the electron, the transition probabilities can only depend on those invariants which remain unchanged under the group of transformations induced by the substitutions

$$\text{(III)} \quad e' = \exp\left[\tfrac{1}{2} i\epsilon + \tfrac{1}{2} i\phi\gamma_5\right] e, \quad \bar{e}' = \bar{e} \exp\left[-\tfrac{1}{2} i\epsilon + \tfrac{1}{2} i\phi\gamma_5\right]$$

on the electron-field operators. The group of transformations on the scalars G, G', F and F' induced by (III) is given by

$$G_j^{(\pm)} \to G_j^{(\pm)} e^{\pm i(\phi/2)-i(\epsilon/2)}, \quad F_j^{(\pm)} \to F_j^{(\pm)} e^{\pm i(\phi/2)-i(\epsilon/2)} \quad \text{for } j,=S,T, \tag{11}$$

$$G_j^{(\pm)} \to G_j^{(\pm)} e^{\mp i(\phi/2)-i(\epsilon/2)}, \quad F_j^{(\pm)} \to F_j^{(\pm)} e^{\mp i(\phi/2)-i(\epsilon/2)} \quad \text{for } j=V. \tag{11'}$$

Invariance under this group is therefore equivalent to requiring that there is no interference of S and T with V. Therefore only $S-T$ interference terms, and no $S-V$ or $T-V$ interference terms, can be measured in experiments in which no measurement is carried out of the transverse polarization of the emitted electron (i. e., only intensities or longitudinal polarizations are measured). The only invariants that can occur in the description of such experiments are E_j, D_j, E_{ST}, D_{ST}, Q_{ST} and N_{ST}.

The invariants D_j, D_{jl} and N_{jl} change sign under space reflection. In the expression for the transition probability they must be multiplied by pseudoscalars formed from the observed momenta and spins: they cannot appear in the description of experiments where no pseudoscalars are observed. Invariance under time reversal implies the relation

$$(\psi_f,\ T\psi_i) = (\tilde{\psi}_i,\ T\tilde{\psi}_f), \tag{12}$$

where $\tilde{\psi}$ is the time-reversed state of the state ψ. If one can show that the T matrix behaves for the particular transition as a Hermitian matrix, one then can substitute for eq. (12) the equation

$$(\psi_f,\ T\psi_i) = (\tilde{\psi}_f,\ T\tilde{\psi}_i)^* \tag{13}$$

which in our case for instance implies specific phase relations between the quantities G, G', F and F'. The transition matrix is Hermitian if calculated at first order in perturbation theory. Relation (13), however, also follows from time reversal, independent of the order of perturbation theory, for transitions such that contributions from intermediate states on the energy shell are absent or negligible (this condition is obviously satisfied in our case). If a weak interaction is essentially involved in the transition and if final-state interactions can be neglected, a general proof follows from the unitarity of the S matrix which is expressed by

$$T^\dagger - T = iTT^\dagger. \tag{14}$$

The largest contribution to a matrix element $(\psi_f TT^\dagger \psi_i)$ comes from terms, for instance, of the kind $(\psi_f T\psi_{f'})\,(\psi_{f'} T^\dagger \psi_i)$, where $(\psi_f T\psi_{f'})$ is due to the final-state interaction and $(\psi_{f'} T^\dagger \psi_i)$ to the weak interaction. Therefore T behaves as Hermitian as long as final-

state interactions can be neglected. Such a condition is verified in our case, for which the only final-state interaction is electromagnetic, and therefore the right-hand side in eq. (14) is, for the transition, smaller by a factor $\sim e^2$ than the left-hand side. To construct the time-reversed states we recall that time reversal corresponds to replacement of c numbers by their charge conjugates, and, for spinor fields, to the substitutions of

$$\epsilon U\psi \text{ for } \psi \text{ and } \epsilon^*\bar{\psi}U^{-1} \text{ for } \bar{\psi},$$

in which ϵ is a phase factor depending on the particular field and U is defined (apart from a phase factor) by[8]

$$U^{-1}\gamma_\mu^T U=\gamma_\mu, \quad UU^+=1. \tag{15}$$

It follows that $UU^*=-1$. A state of a fermion described by the spinor u can be represented by

$$\phi_u=\int \bar{\psi}\gamma_4 u\,dr\,\phi_0,$$

where ϕ_0 is the vacuum. Applying time reversal and using eq. (15), we find that the time-reversed state is

$$\tilde{\phi}_u=\epsilon^*\int \bar{\psi}\gamma_4 U^{-1}u^*\,dr\cdot\phi_0.$$

One finds in this way that under time reversal $u^*\to-\epsilon Uu$ and $\bar{u}^*=-\epsilon^*\bar{u}U^{-1}$. By a similar argument, and using the commutation relation $CU^*=UC^*$, one finds

$$u^{C^*}\to-\epsilon^* Uu^C.$$

We should also remark that under time reversal the four-momenta P_α and p_α of eq. (2) change sign (in fact for the time-reversed state)

$$P_\alpha\to(-P_i,\ P_4),$$

and by complex conjugation

$$P_\alpha\to(P_i,\ -P_4).$$

With such substitutions,[9] and making use of eq. (15), one finds that in order for eq. (13) to hold (invariance under time reversal) G_S, G_S', G_T and G_T' must have the same relative phases, and similarly G_T and G_T' must have the same relative phase, but their phase relative to G_S, G_S', G_V and G_V' is $\pi/2$. The same holds for the scalars F and F'. Making use of the TCP theorem, one then finds that measurements of the different invariants imply violations of P, T and C according to the following table (where we cross the symbol corresponding to that symmetry property which is violated) :

E_j : P, T, C

D_j : $\not{P}$, T, $\not{C}$

E_{jl} : P, T, C for $j=S$, $l=V$; P, $\not{T}$, $\not{C}$ for $j=S$, V, $l=T$

D_{jl} : $\not{P}$, T, $\not{C}$ for $j=S$, $l=V$; $\not{P}$, $\not{T}$, C for $j=S$, V, $l=T$

Q_{jl} : P, $\not{T}$, $\not{C}$ for $j=S$, $l=V$; P, T, C for $j=S$, V, $l=T$

N_{jl} : $\not{P}$, $\not{T}$, C for $j=S$, $l=V$; $\not{P}$, T, $\not{C}$ for $j=S$, V, $l=T$

In the decay probability, invariants that violate P can appear only as factors of pseudoscalars (number of momenta odd), invariants that violate T only as factors of pseudoscalars under time inversion (total number of momenta and of σ factors odd), and invariants that violate C as factors of pseudoscalars under charge conjugation (number of σ factors odd). For $K\to l+\pi+\nu$ the most general distribution will be of the form

$$W\sim a+b(\vec{p}_l\cdot\vec{\sigma}_l)+c(\vec{p}_\pi\cdot\vec{\sigma}_l)+d(\vec{\sigma}_l\times\vec{p}_l\cdot\vec{p}_\pi), \tag{16}$$

where a, b, c, d can depend only on $\vec{p}_l^{\,2}$, $\vec{p}_\pi^{\,2}$ and $(\vec{p}_l\cdot\vec{p}_\pi)$. Accordingly only E_j, E_{SV}, Q_{ST} and Q_{VT} can appear in the term a; D_j, D_{SV}, N_{ST} and N_{VT} in b and c; and E_{ST}, E_{VT} and Q_{SV} in d. If l is an electron, in the approximation in which the electron mass is neglected, only E_j and Q_{ST} can appear in a, and only D_j and N_{ST} in b, while no further general restrictions can be given for the other terms (involving transverse polarizations).

Observation of the last term in eq. (16) would be a proof of noninvariance under time reversal. From our discussion it follows that such a term will generally be expected to occur in the distribution, provided that in the leptonic vertex two of the invariants S, V and T are present. The experiment consists in detecting a possible polarization of the emitted lepton normal to the plane of decay. Detection of such polarization would imply noninvariance under time reversal. In $K^\pm\to\mu^\pm+\pi^0+\nu$ and in $K^0\to\mu^\pm+\pi^\mp+\nu$ the muon polarization can be inferred from the angular distributions of the electrons from the decay of the muon. In the cascade $K\to\mu+\pi+\nu$, $\mu\to e+\nu+\nu$ the electron distribution has the form

$$1+\alpha\langle\vec{\sigma}\rangle_\mu\cdot\vec{p}_e=1+A(\vec{p}_\pi\cdot\vec{p}_e)+B(\vec{p}_\mu\cdot\vec{p}_e)+C(\vec{p}_\pi\times\vec{p}_\mu)\cdot\vec{p}_e,$$

when A, B, C depend on $\vec{p}_e^{\,2}$ and on $\vec{p}_\pi^{\,2}$, $\vec{p}_\mu^{\,2}$, and $(\vec{p}_\pi\cdot\vec{p}_\mu)$. Therefore, any *forward-backward* asymmetry of the decay electrons with respect either to the pion direction or to the μ direction implies violation of P and of C; any *up-down* asymmetry with respect to the decay plane implies violation of T and of C.

We want finally to discuss the assumption of locality, which was explicitly introduced to derive expression (2) for T_{fi}. By such assumption we mean that the two momenta $P^{(l)}$ and $P^{(\nu)}$ of the two final leptons always appear in the form $P^{(l)}+P^{(\nu)}$ in the transition amplitude. If electromagnetic corrections in the final state are neglected such a requirement is satisfied if both leptons are emitted from the same local vertex. Lee and Yang have recently proposed to account for the deviation of ρ from 3/4 in μ decay with a nonlocal structure for the interaction between the four fermions.[10] We shall consider two different possibilities as to the origin of such nonlocality: (a) that an intermediate heavy boson is involved in the process; (b) that an intermediate pair of fermions, formed by a neutrino and a heavy fermion, is involved. This second possibility was suggested

by Lee[11] as a possibility for relating parity violation to the neutrino interactions only. In both cases the decay process would by a second-order process in the *semiweak* interactions of the intermediate bosons or of the intermediate fermion pair with the two spinor fields. Now, if in K decay the two emitted leptons interact with the intermediate boson, or fermion pair, at the same semiweak vertex, the locality hypothesis is of course satisfied. Therefore we have only to consider the possibility that they interact each at a different semiweak vertex. This is impossible however, in Case (a), because two other leptons would have to be emitted, one for each semiweak vertex, and their reabsorption would cause the process to be of higher order. In Case (b) a neutrino would have to be emitted at the μ vertex. Because of conservation of the nucleonic number, the baryon-antibaryon pair must be absorbed at the same vertex, and necessarily, at the ν vertex. Therefore, the ν emitted at the μ vertex must be reabsorbed at the μ vertex again. Diagrams of this kind are, however, usually assumed to give zero contribution because of the antisymmetrization of the interaction Hamiltonian with respect to all fermion operators. In conclusion: for both the above possibilities (a) and (b), it appears that no conceivable deviation from the locality asumption should occur, apart from electromagnetic interactions in the final state.

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Effect of Electron Exchange on the Dispersion Relation of Plasma Oscillations

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An investigation is made concerning the influence on the dispersion relation of plasma oscillations of the electron exchange which has been left as an open question since the work of Bohm and Pines. The dispersion relation is derived by adopting a canonical transformation which is a little different from that of Bohm and Pines. The result obtained shows that the electron exchange gives a small contribution to the Bohm-Pines' dispersion relation. We compare our result with that given recently by Ferrell using the Hartree-Fock self-consistent method, referring to the experimental data.

§ 1. Introduction

The dispersion relation of plasma oscillations in a dense electron gas derived by Bohm and Pines[1] is

$$\omega^2=\omega_p{}^2+(3\hbar^2k_0^2/5m)\,(k^2/m)+O(k^4), \tag{1·1}$$

where $\omega_p{}^2=4\pi ne^2/m$, n being the number of electrons per unit volume, and $\hbar k_0$ is the Fermi momentum. Wolff[2] has noted that electron exchange contributes a term proportional to k^2 to the dispersion relation. By the use of the Hartree-Fock self-consistent method, he obtained

$$\Delta\omega^2=(e^2\hbar^2/nm)\sum_{i,j}6\,(\boldsymbol{k}\cdot(\boldsymbol{p}_i-\boldsymbol{p}_j))\,(\boldsymbol{k}\cdot(2\boldsymbol{p}_i-\boldsymbol{p}_j))/(\boldsymbol{p}_i-\boldsymbol{p}_j)^4, \tag{1·2}$$

where $\boldsymbol{p}_i$ is the electron momentum. Ferrell[3] has recently calculated the effect of electron exchange.* His treatment is equivalent to that of Wolff, but his result is different from (1·2) and is given by

$$\Delta\omega^2=E_{ex}k^2/m, \tag{1·3}$$

where E_{ex} is the exchange energy per electron. Ferrell has replaced E_{ex} by the exchange energy $E_{ex}^{s.r.}$ for the screened Coulomb potential, although this replacement would not be justified within the framework of the Hartree-Fock self-consistent method. Then the dispersion becomes

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* We are indebted to Prof. Ferrell for sending his paper before publication.

$$\omega/\omega_p = 1 + (6\gamma^2/5)\,(1 + 5E_{ex}\,g(\beta)/6E_0)\,(k/k_0)^2, \tag{1·4}$$

where $\gamma = E_0/\hbar\omega_p$, E_0 being the Fermi energy and $g(\beta) = 1 - 4\beta/3 + \beta^2/2 - \beta^4/48$, β being k_c/k_0. Ferrell's result agrees with an estimate given by Pines,[4] except for sign. Pines' estimate gives

$$\Delta\omega^2 = -E_{ex}^{s.r.}\,k^2/m. \tag{1·5}$$

In the present note we show that a little modification of the treatment of Bohm and Pines leads to the dispersion relation under the influence of electron exchange and then we compare our result with Watanabe's[5] experimental one.

§ 2. Dispersion relation

The Hamiltonian for an aggregate of electrons embedded in a background of uniform positive charge may be written[1]

$$H = H_0 + H_1 + H'_{s.r.} + U + H'_{coll} - 2\pi ne^2 \sum_{k<k_c} 1/k^2, \tag{2·1}$$

where

$$H_0 = \sum_{p\sigma} \varepsilon(\boldsymbol{p})\, b^*_{p\sigma} b_{p\sigma} + \sum_{k<k_c} \hbar\omega\,(a_k^* a_k + \tfrac{1}{2}) - 2\pi e^2 \sum_{|p_1 - p_2| > k_c} \{1/|\boldsymbol{p}_1 - \boldsymbol{p}_2|^2\}\, n_{p_1\sigma} n_{p_2\sigma}, \tag{2·2a}$$

$$H_1 = (e\hbar/m) \sum_{\substack{k<k_c \\ p\sigma}} (2\pi\hbar/\omega)^{1/2} \{(\mathcal{E}_k \cdot \boldsymbol{p}) + k/2\}\,(a_k b^*_{p+k\sigma} b_{p\sigma} + a_k^* b^*_{p\sigma} b_{p+k\sigma}), \tag{2·2b}$$

$$H'_{s.r.} = 2\pi e^2 \sum'_{k>k_c} 1/k^2 \cdot b^*_{p_1+k\sigma} b^*_{p_2-k\sigma'} b_{p_2\sigma'} b_{p_1\sigma}, \tag{2·2c}$$

$$H'_{coll} = \sum_{k<k_c} (\hbar/4\omega)\,(\omega_p^2 - \omega^2)\,(a_k^* a_k + a_k a_k^* - a_k a_{-k} - a_k^* a^*_{-k}). \tag{2·2d}$$

Here $\varepsilon(\boldsymbol{p}) = \hbar^2 p^2/2m$, $n_{p\sigma} = b^*_{p\sigma} b_{p\sigma}$, $\mathcal{E}_k = \boldsymbol{k}/|\boldsymbol{k}|$, $b^*_{p\sigma}$ and $b_{p\sigma}$, a_k^* and a_k are respectively the creation and destruction operators of electron and plasmon, and σ denotes the electron spin. It is to be noted that the third term in H_0 represents the exchange energy. The prime in the summation in $H'_{s.r.}$ denotes a sum in which $\boldsymbol{p}_1 + \boldsymbol{k} = \boldsymbol{p}_2$ are excluded. U corresponds to the nonlinear plasma-electron interaction and may be neglected (random phase approximation).

We now consider a canonical transformation to eliminate the plasma-electron interaction H_1:

$$\begin{aligned} H_{new} &= e^{-iS} H e^{iS} \\ &= H_0 + H_1 + H'_{s.r.} + H'_{coll} + i[H_0, S] + i[H_1, S] + (i^2/2)[[H_0, S], S] + \cdots . \end{aligned} \tag{2·3}$$

Our generating function S is given by

$$S = (e\hbar/mi) \sum_{\substack{k<k_c \\ p\sigma}} (2\pi\hbar/\omega)^{1/2} \frac{(\mathcal{E}_k \cdot \boldsymbol{p}) + k/2}{\hbar\omega + E(\boldsymbol{p}) - E(\boldsymbol{p}+\boldsymbol{k})}\,(a_k b^*_{p+k\sigma} b_{p\sigma} - a_k^* b^*_{p\sigma} b_{p+k\sigma}), \tag{2·4}$$

where

$$E(\boldsymbol{p})=\varepsilon(\boldsymbol{p})-4\pi e^2\sum_{|\boldsymbol{p}'-\boldsymbol{p}|>k_c}\{1/|\boldsymbol{p}'-\boldsymbol{p}|^2\}n_{\boldsymbol{p}'\sigma}. \tag{2·5}$$

Noting that $[n_{\boldsymbol{p}\sigma},\ n_{\boldsymbol{p}'\sigma'}]=0$, we find

$$H_1+i[H_0,\ S]=0. \tag{2·6}$$

Hence

$$H_{\text{new}}=H_0+H'_{s.r.}+H'_{\text{coll}}+(i/2)[H_1,\ S]+\cdots. \tag{2·7}$$

We obtain

$$(i/2)[H_1,\ S]=H_{\text{self}}+H_{r.p.}+(i/2)[H_1,\ S]', \tag{2·8}$$

where

$$H_{\text{self}}=-(e^2\pi\hbar/m^2)\sum_{\substack{k\\ p\sigma}}(\hbar/\omega)\left[\frac{\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2\}^2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}+\frac{\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})-k/2\}^2}{\hbar\omega+E(\boldsymbol{p}-\boldsymbol{k})-E(\boldsymbol{p})}\right]n_{\boldsymbol{p}\sigma}, \tag{2·9}$$

$$\begin{aligned}H_{r.p.}=&(2\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ p\sigma}}(\hbar/\omega)\frac{\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2\}^2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}n_{\boldsymbol{p}\sigma}n_{\boldsymbol{p}+\boldsymbol{k}\sigma}\\ &-(\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ pp'\\ \sigma\sigma'}}(\hbar/\omega)\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2\}\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p}')+k/2\}(b^*_{\boldsymbol{p}'\sigma'}b_{\boldsymbol{p}'+\boldsymbol{k}\sigma'}b^*_{\boldsymbol{p}+\boldsymbol{k}\sigma}b_{\boldsymbol{p}\sigma}\\ &+b^*_{\boldsymbol{p}'+\boldsymbol{k}\sigma'}b_{\boldsymbol{p}'\sigma'}b^*_{\boldsymbol{p}\sigma}b_{\boldsymbol{p}+\boldsymbol{k}\sigma})\frac{1}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})},\end{aligned} \tag{2·10}$$

and

$$\begin{aligned}(i/2)[H_1,\ S]'=&(\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ p\sigma}}(\hbar/\omega)(a_k^*a_k+a_ka_k^*\\ &-a_ka_{-k}-a_k^*a_{-k}^*)\left[\frac{\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2\}^2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}-\frac{\{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})-k/2\}^2}{\hbar\omega+E(\boldsymbol{p}-\boldsymbol{k})-E(\boldsymbol{p})}\right]n_{\boldsymbol{p}\sigma}\\ &+(\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ pp'\sigma}}(\hbar/\omega)\{((\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2)((\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p}')\\ &+k/2)(a_k^*a_k[B,\ b_{\boldsymbol{p}'+\boldsymbol{k}\sigma}b_{\boldsymbol{p}'\sigma}]b^*_{\boldsymbol{p}\sigma}b_{\boldsymbol{p}+\boldsymbol{k}\sigma}-a_ka_k^*[B,\ b^*_{\boldsymbol{p}'\sigma}b_{\boldsymbol{p}'+\boldsymbol{k}\sigma}]b^*_{\boldsymbol{p}+\boldsymbol{k}\sigma}b_{\boldsymbol{p}\sigma})\\ &-((\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p})+k/2)((\boldsymbol{\varepsilon}_k\cdot\boldsymbol{p}')+k/2)(a_ka_{-k}[B,\ b^*_{\boldsymbol{p}'-\boldsymbol{k}\sigma}b_{\boldsymbol{p}'\sigma}]b^*_{\boldsymbol{p}+\boldsymbol{k}\sigma}b_{\boldsymbol{p}\sigma}\\ &-a_k^*a^*_{-k}[B,\ b^*_{\boldsymbol{p}'\sigma}b_{\boldsymbol{p}'-\boldsymbol{k}\sigma}]b^*_{\boldsymbol{p}\sigma}b_{\boldsymbol{p}+\boldsymbol{k}\sigma})\}.\end{aligned} \tag{2·11}$$

H_{self} is the electron self-energy and $H_{r.p.}$ is the residual weak electron-electron interaction. The first term in $H_{r.p.}$ represents the exchange energy of the interaction. In (2·11) B denotes $1/\{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})\}$. In obtaining (2·11) we have neglected a number of terms which are quadratic in the plasma field variables. These terms are of the same character as U. The second member in (2·11), each term of which is non-diagonal with respect to the electron variables, may also be neglected. In fact, for example, we get

$$H'=-(\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ pp'\sigma\sigma'}}(\hbar/\omega)\{(\mathcal{E}_k\cdot\boldsymbol{p})+k/2\}\{(\mathcal{E}_{-k}\cdot\boldsymbol{p}')$$
$$+k/2\}[B,\ b^*_{p'-k\sigma'}b_{p'\sigma'}]b^*_{p+k\sigma}b_{p\sigma}a_k a_{-k}$$
$$=(\pi e^2\hbar^2/m^2)\sum_{\substack{k<k_c\\ p\sigma}}(\hbar/\omega)\frac{(\mathcal{E}_k\cdot\boldsymbol{p})+k/2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}\left(\frac{4\pi e^2}{\hbar}\right)$$
$$\times\left\{\sum_{\substack{p'\sigma'\\ |\boldsymbol{p}-\boldsymbol{p}'|>k_c}}\frac{2}{|\boldsymbol{p}-\boldsymbol{p}'|^2}-\sum_{\substack{p'\sigma'\\ |\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k}|>k_c}}\frac{1}{|\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k}|^2}-\sum_{\substack{p'\sigma'\\ |\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k}|>k_c}}\frac{1}{|\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k}|^2}\right\}$$
$$\times\{(\mathcal{E}_k\cdot\boldsymbol{p})+k/2\}b^*_{p'\sigma'}b_{p'+k\sigma'}b^*_{p+k\sigma}b_{p\sigma}\frac{1}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}a_k a_{-k},$$

where we have used the relation $[1/A,\ B]=-1/A\cdot[A,\ B]/A$. An analogous expression is obtained for the term H'' which has the factor $a_k{}^*a_{-k}$. From second-order perturbation theory, we have

$$\Delta E=\sum_n\langle H''\rangle_{0n}\langle H'\rangle_{n0}/(E_0-E_n).$$

This is roughly of the order of $(\hbar^2k_0{}^2/m)\,(e^2k_0/\hbar\omega_p)\,(k_c/k_0)\,n'^2/n^3$ apart from a numerical factor, n' being $4\pi k_c{}^3/3\,(2\pi)^3$.

Then the equation $(i/2)[H_1,\ S]'+H'_{\rm coll}=0$ leads to the following dispersion relation

$$\omega^2-\omega_p{}^2=(4\pi e^2\hbar^2/m^2)\sum_{p\sigma}\left[\frac{\{(\mathcal{E}_k\cdot\boldsymbol{p})+k/2\}^2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}-\frac{\{(\mathcal{E}_k\cdot\boldsymbol{p})-k/2\}^2}{\hbar\omega+E(\boldsymbol{p}-\boldsymbol{k})-E(\boldsymbol{p})}\right]n_{p\sigma},\tag{2·12}$$

where $E(\boldsymbol{p})$ is given by (2·5).

The subsidiary conditions* of Bohm and Pines are given by

$$\Omega_k\Phi=0,\quad (k<k_c)\tag{2·13}$$

where

$$\Omega_k=a_k{}^*+a_{-k}-(8\pi e^2/\hbar\omega k^2)^{1/2}\sum_{p\sigma}b^*_{p+k\sigma}b_{p\sigma}.\tag{2·14}$$

Applying the canonical transformation, we obtain

$$(\Omega_k)_{\rm new}=e^{-iS}\Omega_k e^{iS}=\Omega_k+i[\Omega_k,\ S]+\cdots$$
$$=(\Omega_k)^{(1)}_{\rm new}+(\Omega_k)^{(2)}_{\rm new}+(\Omega_k)^{(3)}_{\rm new}+\cdots,\tag{2·15}$$

where

$$(\Omega_k)^{(1)}_{\rm new}=a_k{}^*\left\{1-\sum_{p\sigma}(4\pi e^2\hbar/m\omega k)\left(\frac{(\mathcal{E}_k\cdot\boldsymbol{p})+k/2)}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}+\boldsymbol{k})}-\frac{(\mathcal{E}_k\cdot\boldsymbol{p})-k/2}{\hbar\omega+E(\boldsymbol{p}-\boldsymbol{k})-E(\boldsymbol{p})}\right)n_{p\sigma}\right\}$$
$$+a_{-k}\left\{1-\sum_{p\sigma}(4\pi e^2\hbar/m\omega k)\left(\frac{(\mathcal{E}_k\cdot\boldsymbol{p})-k/2}{\hbar\omega+E(\boldsymbol{p}+\boldsymbol{k})-E(\boldsymbol{p})}-\frac{(\mathcal{E}_k\cdot\boldsymbol{p})-k/2}{\hbar\omega+E(\boldsymbol{p})-E(\boldsymbol{p}-\boldsymbol{k})}\right)n_{p\sigma}\right\},\tag{2·16}$$

* These subsidiary conditions are too stringent and accompanied by a mathematical difficulty.[6]

$$(\Omega_k)^{(2)}_{\text{new}} = -\sum_{p\sigma} (2\pi e^2\hbar^3/\omega m^2)^{1/2} \left\{ \frac{2m}{\hbar^2 k} + \frac{(\varepsilon_k \cdot \boldsymbol{p}) + k/2}{\hbar\omega + E(\boldsymbol{p}) - E(\boldsymbol{p}+\boldsymbol{k})} \right.$$

$$\left. - \frac{(\varepsilon_k \cdot \boldsymbol{p}) + k/2}{\hbar\omega + E(\boldsymbol{p}+\boldsymbol{k}) - E(\boldsymbol{p})} \right\} b^*_{p+k\sigma} b_{p\sigma}, \tag{2·17}$$

$$(\Omega_k)^{(3)}_{\text{new}} = -a_k^* \sum_{pp'\sigma} (4\pi e^2\hbar/m\omega k) \{(\varepsilon_k \cdot \boldsymbol{p}) + k/2\}$$

$$\times \left[\frac{1}{\hbar\omega + E(\boldsymbol{p}) - E(\boldsymbol{p}+\boldsymbol{k})},\ b^*_{p'+k\sigma} b_{p'\sigma} \right] b^*_{p\sigma} b_{p+k\sigma}$$

$$- a_{-k} \sum_{pp'\sigma} (4\pi e^2\hbar/m\omega k) \{(\varepsilon_k \cdot \boldsymbol{p}) - k/2\}$$

$$\times \left[\frac{1}{\hbar\omega + E(\boldsymbol{p}) - E(\boldsymbol{p}-\boldsymbol{k})},\ b^*_{p'+k\sigma} b_{p'\sigma} \right] b^*_{p-k\sigma} b_{p\sigma}. \tag{2·18}$$

In obtaining (2·15) we have made use of the random phase approximation. Expanding the coefficients of a_k^* and a_{-k} in (2·16) in powers of k and using the dispersion relation, we find that they are $\sim \Delta\omega^2/\omega_p^2$. Since this ratio is small as will be shown in the next section, we may possibly neglect $(\Omega_k)^{(1)}_{\text{new}}$ and also $(\Omega)^{(3)}_{\text{new}}$. Then the subsidiary conditions become

$$(\Omega_k)^{(2)}_{\text{new}} \Phi = 0, \quad (k < k_c).$$

§ 3. Comparison with experiment

Expanding the dispersion relation (2·12) in powers of k and assuming the Fermi distribution, we get

$$\omega^2 = \omega_p^2 + (2k^2/nm) \sum_{\substack{pp' \\ p-p' > k_c}} [3\hbar^2 (\varepsilon_k \cdot \boldsymbol{p})^2/m - 8\pi e^2 \{\hbar^2 f'_p/m$$

$$+ \hbar^4 (\varepsilon_k \cdot \boldsymbol{p}')^2 f''_{pp'}/m + 2(\varepsilon_k \cdot \boldsymbol{p})(\varepsilon_k \cdot \boldsymbol{p}') f'_{pp'}/m\}/|\boldsymbol{p}-\boldsymbol{p}'|^2] f_p + O(k^4), \tag{3·1}$$

where f_p is the Fermi distribution function and f'_p and f''_p are respectively $\partial f_p/\partial\mathcal{E}(p)$ and $\partial^2 f_p/\partial\mathcal{E}^2(p)$. Performing the summations over $\boldsymbol{p}$, we find

$$\omega^2 = \omega_p^2 + (3\hbar^2 k_0^2/5m)(k^2/m) + \Delta\omega^2 + O(k^4), \tag{3·2}$$

where $\Delta\omega^2$ is the contribution from electron exchange and is given by

$$\Delta\omega^2 = E_{ex}(k^2/m)\, g(\beta). \tag{3·3}$$

Here $E_{ex} = -(3e^2 k_0/4\pi)$ and $g(\beta) = -(4/5)\log(2/\beta) + 4/5 - 7\beta/6 + 113\beta^2/60 - \beta^3/9$, β being k_c/k_0. (3·3) may be rewritten as

$$\omega/\omega_p = 1 + 6\gamma^2/5\,(1 + 5E_{ex}/6E_0 \cdot g(\beta))\,(k/k_0)^2, \tag{3·4}$$

where $\gamma = E_0/\hbar\omega_p$. Thus we obtain for the correction factor due to exchange

$$1 + (5E_{ex}/6E_0)\, g(\beta) = 1 - 0.208 r_s g(\beta), \tag{3·5}$$

where $r_s = (3/4\pi a_0^3 n)^{1/3}$ is the radius of the unit sphere in Bohr radii. The result of our calculation is shown in the Table.

Dispersion coefficient

Eigenloss				Theory				
Metal	r_s	$\hbar\omega_p$	Exp.	No exch.	Exch.*	Exch.**	Exch.[3] (Ferrell)	Exp.[5]
Be	1.88	18.2	19.0	0.72	0.85	0.70	0.64	0.65±0.06
Al	2.07	15.7	15.0	0.65	0.77	0.62	0.58	0.74±0.07
Ge	2.08	15.6	16.5	0.65	0.77	0.62	0.58	1.22±0.22
Mg	2.65	10.9	10.5	0.51	0.58	0.42	0.44	0.81±0.05

* The values are calculated, using $\beta = 0.353\, r_s^{1/2}$ which was derived by Pines.

** The values are calculated, using $r_s = 6.02\,\beta^2\,[(2+\beta)\log(1+2/\beta) - 2]^{-1}$ which was derived by Ferrell.[3]

We see from Table that the exchange correction does not improve the agreement between theory and experiment. As has been pointed out by Ferrell,[3] the discrepancy may be due to departures, caused by the positive ion lattice, from free-electron behavior. Our approach may be extended to take the effect of the lattice into account.

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Note added in proof. It is to be noted that $i[H'_{s.r.}, S]$ and $(i^2/2)\,[[H'_{s.r.}, S], S]$ do not give diagonal terms. If the generating function S_{BP} of Bohm and Pines is used instead of our S, $(i^2/2)\,[[H_{s.r.}, S], S]$ (not $H'_{s.r.}$!) give diagonal tesms which contribute to the dispersion relation. However the dispersion thus obtained will not be such a closed form as (2.12).

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General Theory of Infrared Divergence

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The problems of infrared divergence in perturbation theory are generally discussed, including not only photons but neutrinos which interact with other particles in various ways. A completely general proof of the cancellation of infrared divergence in each order is given, as previous works on this problem seem insufficient.

§ 1. Introduction

As is well known, there appears infrared divergence in quantum electrodynamics owing to the vanishing mass of photon. But that is merely a defect of perturbation calculation, in which matrix elements are distinguished by the unobservable number of soft photons. In fact it was shown by Bloch and Nordsieck[1] and many others that this difficulty disappears if we calculate them by an appropriate method. Hence many people conjectured that the infrared divergence in perturbation theory would be cancelled out in each order of e^2 in the calculation of transition probability. This conjecture is easily verified in the lowest order. A general proof has been presented by Jauch and Rohrlich,[2] but it is not completely general since their proof does not include the divergence caused by many soft photons,* and also in their proof the natural reason why the cancellations take place is not clear. On this point Kinoshita's proof[3] is more instructive, but his method cannot be applied straightforwardly to general cases.** These details will be mentioned in the next section.

Now, the main purpose of this paper is to give a completely general proof of the cancellation of infrared divergence in perturbation theory for the most general case of the conventional field theory. This has the following significance.

1) Previous works on this problem have been restricted to quantum electrodynamics, in which the interaction Hamiltonian is very special and renormalizable. But there exist neutrinos with the vanishing mass which interact with other particles in various ways. Therefore it is necessary to throw light on the infrared problem of neutrino.

2) Since the proof of the cancellation remains insufficient as was pointed out above, it must be accomplished.

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* See §§ 2 and 3.

** Ito's proof,[4] which is its generalization, seems meaningless, because he considered the cancellation between physically distinguishable processes.

3) By clarifying the essential mechanism of the cancellation, we might be able to make a criticism on the attempts at modification of the present field theory, as was previously noticed by Bethe and Oppenheimer.

In § 2, we briefly review the infrared problem in perturbation theory. In § 3, we investigate the circumstances of the occurrence of infrared divergence, for which we obtain a power-counting theorem like that for ultraviolet divergence,[7] though there is a distinct difference between them. The poles which may contribute to infrared divergence are classified into two kinds—the first kind and second kind. Infrared divergence is generally caused by the addition of the poles of the second kind to those of the first kind. In § 4, we prove that *all the poles of the second kind always cancel out in any order of any process*. The disappearance of infrared divergence immediately follows from this fact, and we can well understand the mechanism of the cancellation.

§ 2. Review of infrared problem

Before entering into general considerations, we illustrate the situation of infrared problem and the outline of our proof of cancellation with some concrete examples. We always employ the Feynman-Dyson theory[6] hereafter.

The simplest example of infrared divergence is inelastic scattering of an electron, accompanied by a soft photon emission, by an external field. Its matrix elements corresponding to Fig. 1 (a) and (b) are given by

$$\left.\begin{aligned} M_a^{(1)} &= \frac{ie}{(2\pi)^2}\bar{u}(q')\frac{(\gamma\tilde{\xi})}{\sqrt{2k_0}}\cdot\frac{i\gamma(q'+k)-m}{(q'+k)^2+m^2-i\varepsilon}(\gamma a)\,u(q), \\ M_b^{(1)} &= \frac{ie}{(2\pi)^2}\bar{u}(q')(\gamma a)\frac{i\gamma(q-k)-m}{(q-k)^2+m^2-i\varepsilon}\cdot\frac{(\gamma\tilde{\xi})}{\sqrt{2k_0}}u(q). \end{aligned}\right\} \qquad (2\cdot1)$$

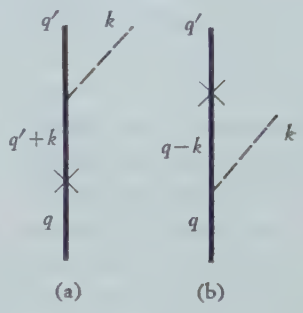

Fig. 1

Here q, q' and u respectively are the initial and final 4-momenta and the spinor amplitude of the free electron; k, k_0 and $\tilde{\xi}$ respectively are the 4-momentum, its energy component and polarization of the emitted photon; a is the 4-amplitude of the external field; and $\varepsilon\to+0$. Hence, when k_0 is small, we easily obtain

$$\int_{\text{low energy}}|M_a^{(1)}+M_b^{(1)}|^2d^3\boldsymbol{k}=\frac{e^2}{(2\pi)^4}|\bar{u}(q')(\gamma a)u(q)|^2\sum_{\text{pol.}}\int\frac{d^3\boldsymbol{k}}{2k_0}\left|\frac{(q\tilde{\xi})}{(qk)}-\frac{(q'\tilde{\xi})}{(q'k)}\right|^2 \qquad (2\cdot2)$$

as the (total) transition probability. This integral exhibits a logarithmic divergence at the low energy limit. As is well known, it originates from the artificial separation of real and virtual soft photons in the perturbation calculation. Since soft photons are unobservable, one cannot distinguish the above process from the scattering without photon emission. If we denote the matrix elements corresponding to Fig. 2 (a), (b), (c) and (d) by $M_a^{(2)}$, $M_b^{(2)}$, $M_c^{(2)}$ and $M_d^{(2)}$, respectively, we obtain

$$\int [M_a^{(2)*}(M_b^{(2)}+M_c^{(2)}+M_d^{(2)}) + (M_b^{(2)}+M_c^{(2)}+M_d^{(2)})^* M_a^{(2)}] d^3\boldsymbol{k} \quad (2\cdot3)$$

as the transition probability of the latter process at the second order. (The asterisk denotes the complex conjugate.) If (2·2) and (2·3) are explicitly evaluated and summed up, then one will find the exact cancellation of infrared divergence.

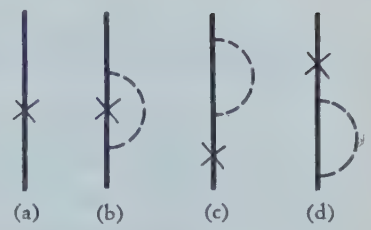

Fig. 2

It is of course not accidental, as was suggested by Bloch and Nordsieck's fundamental work. But since an approximation (i. e. second order perturbation) has been employed, the cancellation may not necessarily take place. So we want to explain clearly the reason why the cancellation always take place in perturbation. This has been attempted by Kinoshita.[3]

We first consider a partial sum

$$\int (M_b^{(1)*} M_a^{(1)} + M_b^{(2)*} M_a^{(2)}) d^3\boldsymbol{k}. \quad (2\cdot4)$$

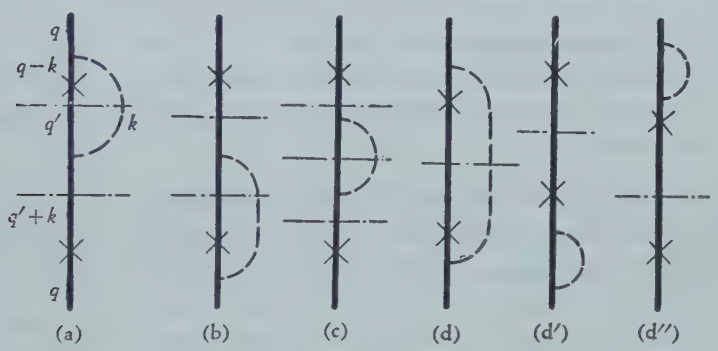

Fig. 3

If the Feynman diagrams corresponding to each term are connected at the final state, one obtains a common graph, Fig. 3 (a), for the both terms. He regarded it as a Feynman diagram, and then the matrix element is

$$\frac{e^2}{(2\pi)^6}\bar{u}(q)\gamma_\mu\int d^4k\frac{i\gamma(q-k)-m}{(q-k)^2+m^2-i\varepsilon}(\gamma a)\frac{1}{k^2-i\varepsilon}\cdot \cdot\frac{i\gamma q'-m}{q'^2+m^2-i\varepsilon}\gamma_\mu\frac{i\gamma(q'+k)-m}{(q'+k)^2+m^2-i\varepsilon}(\gamma a)u(q). \quad (2\cdot5)$$

Its real part is written as*

* Of course $i\gamma$'s should be regarded as real quantities.

$$\frac{e^2}{4(2\pi)^4}\bar{u}(q)\gamma_\mu\int d^4k\frac{i\gamma(q-k)-m}{(q-k)^2+m^2-i\varepsilon}(\gamma a)\,\delta(k^2)\,(i\gamma q'-m)\gamma_\mu\cdot$$

$$\cdot\left[\delta(q'^2+m^2)\frac{1}{(q'+k)^2+m^2-i\varepsilon}+\frac{1}{q'^2+m^2-i\varepsilon}\delta((q'+k)^2+m^2)\right]$$

$$\cdot\,(i\gamma(q'+k)-m)\,(\gamma a)\,u(q) \qquad (2\cdot6)$$

$+$some other terms without infrared divergence,

which is proportional to the corresponding term of the total transition probability because of the unitarity of S-matrix; that is to say, the first and second terms of (2·6) correspond to the first and second ones of (2·4), respectively. The part in the square bracket of (2·6) is rewritten as

$$\{\delta(q'^2+m^2)-\delta(q'^2+m^2+2q'k)\}/2q'k, \qquad (2\cdot7)$$

which tends to $-\delta'(q'^2+m^2)$ at the soft limit. Thus the degree of singularity diminishes and (2·6) exhibits no infrared divergence. He conjectured that the situation of every infrared divergence would be same as that of the above example. But the following difficulties appear for general cases.

1) The cancellation-counterparts, Fig. 3 (d), (d′) and (d″), cannot be drawn as a single graph. Actually he omitted the last two.

2) The treatment of self-energy part is ambiguous.

3) If the matrix element of basic process has an imaginary part, the use of the unitarity becomes inappropriate.

A more satisfactory proof was given by Jauch and Rohrlich. They correctly treated the self-energy part by taking account of mass and charge renormalizations. They made use of mathematical induction for the number of real soft photons. Hence their proof is essentially same as that for the lowest order. They asserted without any proof that the soft photons attaching to internal lines would not contribute to infrared divergence. But this assertion does not seem to be valid. For example, consider Fig. 4. A real soft photon k is emitted from an internal line. At the k-integration, however, infrared divergence does come out (see the next section). The reason is that when k' is also soft the electron line becomes essentially external. Such divergences as many soft photons concerning at the same time cannot be treated by their method.

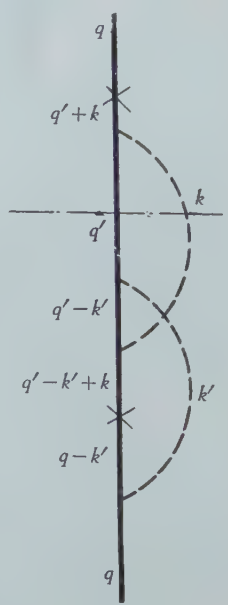

Fig. 4

Therefore a method applicable to the most general case is necessary to be proposed. Kinoshita's proof, though it is unsatisfactory, contains an essence of the mechanism of the cancellation. So we start from his idea. The aforementioned difficulties will be avoided in the following ways.

1) We connect the diagrams also at the initial state. This will be done by introducing K-diagram. But it should be noticed that real soft photons can always exist only at the final state.

2) The number of the couples of the corresponding Feynman diagrams is three for such a self-energy graph as Fig. 3 (c). *The key to the infrared problem is to extend the cancellation* (2·7) *to the case of such many cancellation-counterparts.* This will be given by two lemmas (4·9) and (4·11), which also solve the problem of the many-soft-photon divergence.

3) We should consider the cancellation directly without the aid of the unitarity. It is of importance to separate essentially external part intimately relating to infrared divergence from physical-process part irrelevant to it.

§ 3. Inspection of infrared divergence

We first investigate the mechanism of the occurrence of infrared divergence. The integrand of a perturbation term of S-matrix element is a product of such factors as

$$\frac{1}{(k^2+m^2-i\varepsilon)^n}=\frac{\mathrm{Pf}}{(k^2+m^2)^n}+i\pi\frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(k^2+m^2) \tag{3·1}$$

with some numerators. Here k is a 4-momentum, m a mass, and the symbol Pf denotes Hadamard's finite part.[7] The first term of the right-hand side of (3·1) has no pole by definition, unless it is multiplied by a discontinuous function. The second term can be written as

$$\delta(k^2+m^2)=\{\delta(k_0+\sqrt{\boldsymbol{k}^2+m^2})+\delta(k_0-\sqrt{\boldsymbol{k}^2+m^2})\}/2\sqrt{\boldsymbol{k}^2+m^2} \tag{3·2}$$

for $n=1$, where $\boldsymbol{k}$ and k_0 stand for momentum and energy parts of k_μ, respectively. For general cases, differentiating (3·2) by $\sqrt{\boldsymbol{k}^2+m^2}$, we have

$$\delta^{(n-1)}(k^2+m^2)$$
$$=\sum_{j=1}^{n}a_{nj}\{\delta^{(j-1)}(k_0+\sqrt{\boldsymbol{k}^2+m^2})+(-1)^{j-1}\delta^{(j-1)}(k_0-\sqrt{\boldsymbol{k}^2+m^2})\}/(\boldsymbol{k}^2+m^2)^{n-j/2}, \tag{3·3}$$

where a_{nj}'s are numerical coefficients. In (3·3) one of singularities is that of δ-function, but it merely diminishes the number of integration variables by one and does not give rise to any divergence. Thus if $m\neq 0$, infrared divergence never appears, as is well known. If $m=0$, a pole comes out when $k_0=|\boldsymbol{k}|=0$. *But infrared divergence is generally caused not by this kind of poles only, but by the cooperation with another kind.* The propagator of the massless fermion has no pole because the numerator is proportional to k_μ, but its presence leads to an essentially different situation from the case in which there are no propagators with the vanishing mass. The reason is that it is discontinuous at $k_\mu=0$, although it has the form $k_\mu/|\boldsymbol{k}|$. Therefore if the denominator of another Pf-factor multiplied by it vanishes at $k_\mu=0$, then a pole is induced and may contribute to infrared divergence.

In order to understand this fact more concretely, let us consider the case in which the integration 4-momentum is only one. By omitting the numerator for simplicity, the integral can be written as

$$\int \frac{1}{(k^2-i\varepsilon)^n} \prod_j \frac{1}{(k+q_j)^2+m_j^2-i\varepsilon} d^4k, \tag{3·4}$$

where q_j's are constant 4-momenta. If $n \geqq 1$, the propagators, in which equations

$$q_j^2+m_j^2=0 \tag{3·5}$$

hold, have the poles induced, since their denominators k^2+2q_jk vanish at $k_\mu=0$. If we denote the number of such propagators by l, $2n+l-4$ is the degree of infrared divergence. On the contrary, if $n=0$, infrared divergence never appears, however many propagators in which equations (3·5) hold are present, provided $m_j \neq 0$.

It is useful therefore to classify the poles, which may contribute to infrared divergence, into the following two kinds :—

The first kind : the pole which comes out from the imaginary part of the propagator with the vanishing mass.

The second kind : the pole which is induced by the pole of the first kind and comes out from the real part of the propagator in which (3·5) holds.

The first kind is, so to speak, essential and the second kind is additional.

Similarly, for the general case, if we neglect the ultraviolet part and assume

$$\lim_{k\to 0} Q(k) = \text{finite}, \tag{3·6}$$

where $Q(k)$ is the extra factor which is the integrated result of the part containing the other integration variables, then the above-mentioned consideration holds as it is. But (3·6) holds only when $Q(0)$, matrix element of "basic diagram", has no infrared divergence. Jauch and Rohrlich[2] considered only the case in which the infrared divergences are factorized and reduced to the above. But generally $Q(k)$ may contain an inverse power of k.

Thus the following power-counting theorem should be taken into account for the general case :—

theorem

If
$$D \equiv 2n+n'-d+l-4\nu < 0 \tag{3·7}$$

always holds in every subintegration by independent integration variables $k_j (j=1, 2, \cdots, \nu)$ which have the factors $1/(k^2-i\varepsilon)$, then infrared divergence does not occur.

Here

n : the number of boson propagators containing only k_j's,

n' : the number of fermion propagators containing only k_j's,

d : the number of derivative couplings appearing at the terminals of such propagators,

l : the number of the propagators which contain only k_j's as the integration variables and satisfy (3·5).

ν : the number of k_j's

To prove this theorem, we take out the part relating to infrared divergence and carry out the energy integration by means of (3·3). We then obtain usual 3ν-ple integral (the denominators have definite signs near the poles) and thus can prove the convergence by means of inequalities.

Applying this theorem we can investigate the degree of infrared divergence of a given integral. After some considerations we obtain an inequality $l \leqq 2\nu$ if there were not self-energy parts. From this fact only logarithmic divergence ($D=0$) can appear for renormalized quantum electrodynamics. More general consideration is also easy. But these are unimportant for our present purpose, and hence we omit the details.[8]

The considerations mentioned above are easily extended to the integral of transition probability, because the propagator of a real massless particle is $\delta(k^2)$ or $i\gamma\cdot k\delta(k^2)$. But we should exclude the case in which there are unintegrated δ-singularities, from which appears the so-called "infrared-catastrophe-like divergence[4)8)]" that essentially differs from infrared divergence.

§ 4. Proof of the cancellation

In this section a completely general proof of the cancellation of all the poles of the second kind is given for the unrenormalized case. The cancellation of all infrared divergences is its direct consequence.

Improving Kinoshita's idea we introduce "*K*-diagram" as follows. In a *K*-diagram we connect the two graphs not only at the final but also at the initial state. The connecting lines correspond to the propagators of real particles, which are

$$2\pi\theta(q_0)\,\delta(q^2+m^2) \tag{4·1}$$

as the precise form. In the following we omit the numerators that always appear as common factors. The propagators of real soft particles* are

$$2\pi\theta(k_0)\,\delta(k^2), \quad (k\approx 0) \tag{4·2}$$

and those of virtual soft particles are

$$\pi\delta(k^2), \quad (k\approx 0). \tag{4·3}$$

But the difference between (4·2) and (4·3) is

$$\pi\epsilon(k_0)\,\delta(k^2) = -\pi\{\delta(k_0+|\boldsymbol{k}|)-\delta(k_0-|\boldsymbol{k}|)\}/2|\boldsymbol{k}|, \tag{4·4}$$

which tends to

$$-\pi\delta'(k_0) \tag{4·5}$$

for $|\boldsymbol{k}|\to 0$. Thus the difference gives only a vanishing contribution at the soft limit. Therefore we may adopt only (4·3) as the propagators of soft particles.

In a *K*-diagram propagators are represented as follows:

soft particle: dotted line - - - -

other : solid line ——

---

* "Soft particle" means a massless particle with vanishingly small 4-momentum.

Let us call the part that is connected with the line of a real particle without vertices when all dotted lines vanish " essentially-external line ". This stands for the propagation of an observable particle. From the consideration in last section we know that the poles of the first kind appear from the propagators of dotted lines, and those of the second kind can appear only on the essentially-external lines but not on the other parts because of the absence of the propagators of real particles which represent the condition (3·5). So we denote these by hatching as G_1, G_2 in Fig. 5, which stand for the actual physical process. On the way of each essentially-external line there may be correction parts, which are self-energy parts or one-particle lines other than the real one when all dotted lines vanish. We also denote those by hatching and call S_1, S_2,⋯. The dotted lines whose both terminals are contained in hatched parts may be included in these.

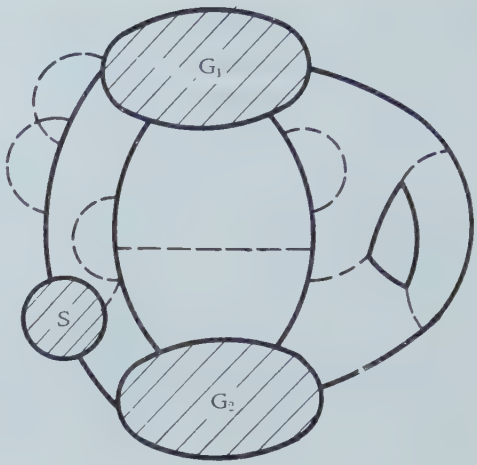

Fig. 5
(an example of K-diagram)

Now, inversely, we consider a K-diagram such as Fig. 5. When appropriate cuttings are given to it, one obtains various couples of Feynman diagrams representing the physical process in consideration. The cut portions correspond to real particles at the initial and final states. The cuttings are of course allowed only on the essentially-external lines and dotted ones, because one must not change observable quantities. We can indicate the cut portion of every essentially-external line *independently*. (If there are S-parts half-way, the whole except those should be regarded as an essentially-external line.) Then the dotted lines to be cut are generally determined automatically. But if there are the closed loops from the vertices of which only dotted lines start (we denote their number by α), the cuttings of the dotted lines are not determined uniquely. It is necessary to indicate further which side of every such closed loop the cutting line (i. e. the initial and final states) passes through.*

When we give all possible K-diagrams all possible cuttings indicated above corresponding to the one physical process, (including soft-particle corrections), their total sum manifestly represents the transition probability without omission and overlapping. Therefore we hereafter notice one of the K-diagrams and consider a set of the graphs which are obtained by giving all possible cuttings to it. Then they all have G_1, G_2, S_1, S_2, ⋯, and soft-particle propagators $\pi\delta(k_a^2)$, $\pi\delta(k_b^2)$, ⋯, as common factors. Their different part is a product of propagators belonging to essentially-external lines, namely

$$\prod_r F_r^{(s)}, \tag{4·6}$$

* The self-interactions of soft particles should be regarded as the special cases of the such closed loops.

where $F_r^{(s)}$ is a product of propagators belonging to an essentially-external line r which is cut at the s-th line:

$$F_r^{(s)} \equiv \left[\prod_{j=1}^{s-1} \frac{1}{(q_r+k_j)^2+m_r^2-i\varepsilon}\right] \cdot 2\pi\theta(q_{r0})\,\delta((q_r+k_s)^2+m_r^2)\left[\prod_{j=s+1}^{n_r} \frac{1}{(q_r+k_j)^2+m_r^2+i\varepsilon}\right]. \tag{4·7}$$

Here k_j's are linear combinations of soft momenta k_a, k_b, ···. The total sum over all possible cuttings of the K-diagram is

$$2^{\alpha} \cdot \prod_r F_r,$$

where

$$F_r \equiv \sum_{s=1}^{n_r} F_r^{(s)}. \tag{4·8}$$

lemma 1.

$$\sum_{s=1}^{n}\left[\left(\prod_{j=1}^{s-1} \frac{1}{A_j-i\varepsilon}\right)\pi\delta(A_s)\left(\prod_{j=s+1}^{n} \frac{1}{A_j+i\varepsilon}\right)\right] = \mathrm{Im}\prod_{s=1}^{n} \frac{1}{A_s-i\varepsilon}. \tag{4·9}$$

This identity is easily proved by mathematical induction.

lemma 2.

Generalized Feynman identity:

$$\prod_{s=1}^{n} \frac{1}{A_s-i\varepsilon} = (n-1)!\int \frac{dx}{(\sum_{s=1}^{n} x_s A_s-i\varepsilon)^n}, \tag{4·10}$$

where

$$dx = \prod_{s=1}^{n} dx_s \cdot \delta(\sum_{s=1}^{n} x_s-1), \quad (x_s \geqq 0).$$

The imaginary part of (4·10) is

$$\mathrm{Im}\prod_{s=1}^{n} \frac{1}{A_s-i\varepsilon} = \pi(-1)^{n-1}\int \delta^{(n-1)}(\sum_{s=1}^{n} x_s A_s)\,dx. \tag{4·11}$$

Applying (4·9) and (4·11) to (4·8) and (4·7), we obtain

$$F_r = 2\pi(-1)^{n_r-1}\theta(q_{r0})\int \delta^{(n_r-1)}(q_r^2+m_r^2+\sum_{s=1}^{n_r} x_s(2q_r k_s+k_s^2))\,dx. \tag{4·12}$$

Since $0 \leqq x_s \leqq 1$,

$$F_r \to 2\pi(-1)^{n_r-1}(1/(n_r-1)!)\,\theta(q_{r0})\,\delta^{(n_r-1)}(q_r^2+m_r^2) \tag{4·13}$$

uniformly at the limit of $k_s \to 0$. Thus F_r has no poles at the soft limit (cf. (3·3)), namely the total contribution from a K-diagram contains no poles of the second kind at the integrations of k_a, k_b, ···. The proof of the cancellation has thus been established completely. From this proof the remaining infrared divergence, if any, must be caused by poles of the first kind only. As is easily seen from (3·7) with $l=0$, the possible

cases are the self-mass parts of the massless particles, which should vanish due to some invariance properties such as Gauge invariance for photons and γ_5-invariance for two-component neutrinos,[9] and Coulomb forward scatterings, which are a kind of infrared-catastrophe-like divergences.

Finally, we briefly investigate the effect of renormalization, restricting ourselves to electrodynamics.

Mass renormalization δm: this does not contain any infrared divergence because $D \leqq -1$ for every subintegration (cf. (3·7)).*

Charge renormalization $Z_1^{-1} Z_2 Z_3^{1/2}$: this is equal to $Z_3^{1/2}$ because of Ward's identity[10] and obviously Z_3 does not contain any infrared divergence.

Thus all infrared divergences in renormalized quantum electrodynamics also cancel out.

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* $D \leqq -1$ can be proved straightforwardly for the irreducible self-energy graphs. For the reducible ones, one should employ mathematical induction for the number of irreducible parts, taking account of mass renormalizations.

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Theory of Direct Interaction in Nucleon Inelastic Scattering

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For the inelastic scattering of nucleons at low and intermediate energies an approximate method is proposed by which the direct interaction process is calculated. From the coupled equation for the radial wave functions in various channels a standing wave solution is obtained by perturbation approximation. For the asymptotic form of the wave function in outgoing channel we choose a proper form in such a way that the higher order contributions in perturbation expansion become small. The generalized K-matrix is calculated from the first order solution and later the S-matrix is transformed from the K-matrix. This method can give the exact solution in some simple cases and also is expected to give a good approximation in general cases.

By assuming the simple zero-range force between nucleon in the nucleus and the incident neutron the approximate method is applied to the inelastic scattering of neutrons by Mg^{24}. The effect of the Pauli principle is also estimated. The same process is calculated by assuming the collective excitation and both results are compared.

§ 1. Introduction

Direct interaction process was investigated by many workers to explain deviations from the statistical model in nuclear reactions. In the intermediate energy region, the incident particle interacts rather strongly with the nucleus and easily forms the compound nucleus once the impinging particle enters into the nucleus. If we consider the inelastic scattering of nucleon which leaves the residual nucleus at a low excited state described by the shell model, the direct interaction process takes place mainly at the diffused surface of the target nucleus. Austern, Butler and McManus[1] treated this process by the impact approximation whereas Butler[2] and Ui[3] used the distorted wave approximation.

At lower energy the interaction between the incident nucleon and the target nucleus becomes weak owing to the Pauli principle and the nucleon can travel rather freely inside the nucleus. The optical potential describes the interaction in good approximation and the single particle resonance is observed. In this energy region, the internal region of nucleus in addition to the boundary region seems to take part in the direct interaction process. Sasakawa, Watari and Kajikawa[4,5] and Lamarsh and Feshbach[6] evaluated this effect by applying the distorted wave approximation. However, in this energy region, the effect of the distortion of the incident particle by the optical potential is important and the distorted wave approximation fails near a single particle resonance where the amplitude of the incident wave becomes large. If the distorted wave approximation is applied near the single particle resonance one obtains a large contribution from the first order pertur-

bation term,* so the approximation in which the calculation is stopped at the first order becomes meaningless. Some authors[6),8),9)] noted this and carefully avoided the single particle resonance in their calculations.

When the initial and final states of the target nucleus are described by the collective model, another kind of mechanism of the excitation may be considered. The incident particle interacts with the surface of the nucleus, not with the single particles, and excites the surface rotation or oscillation mode. This type of the direct interaction process was studied by Hayakawa and Yoshida[10)] and Brink[11)] by using the distorted wave approximation, but it was found later[12)] that this process can be treated correctly without using the distorted wave approximation, provided the interaction is of the δ-function type. The latter method may be applied to the single particle resonance and its results are very different from those obtained from the distorted wave approximation. Unfortunately this method can only be applied to the δ-type surface interaction, hence it is desirable to find an approximation which is applicable to general cases.

The purpose of this paper is to propose the method which gives a good approximation for the general type of the interaction even at single particle resonance, and, in particular, in the case of the δ-type surface interaction, gives the same exact expression as given in reference 12. From the coupled equation for the radial wave functions in various channels a standing wave solution is obtained by perturbation approximation. For the asymptotic form of the wave function in outgoing channel we choose a proper form in such a way that the higher order contributions in perturbation expansion become small. The generalized K-matrix is calculated from the first order solution and later the S-matrix is tranformed from the K-matrix. By using this method the cross section of the inelastic scattering of neutron is calculated using a simplified zero range force between the incident neutron and nucleons in the nucleus. The effect of the Pauli principle, which is usually neglected, will be examined in this interaction.

The other purpose of this report is to give a comparison between the two different mechanisms of the direct interaction, i. e., the particle and collective excitation. The absolute cross section and angular distribution will be compared by using same distorting potential for the same target nucleus. Since the collective excitation is easy to calculate numerically the effect of the higher excited levels in the target nucleus is also taken into account.

§ 2. Generalized K-matrix

(I) *General theory*

We now consider the inelastic scattering of neutron by nuclei; the proton inelastic scattering and stripping reaction may be treated with a slight modification. For simplicity

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* If the incident energy is just the threshold energy for the neutron inelastic scattering and further the s- or p- wave resonance for the outgoing nucleon coincides with the threshold energy, the cross section for this reaction becomes infinite provided the real potential is used for the distorting potential.[7)] But this infinity is removed by introducing a small imaginary part in the distorting potential.

the Pauli principle will not be taken into account in this section. The Hamiltonian of the whole system is divided into three parts:

$$H=H_0(\vec{\xi})+T+U(r)+V(\vec{\xi}, r). \tag{2·1}$$

The first term represents the Hamiltonian of the target nucleus, the second is the kinetic energy of the incident neutron, $U(r)$ is the distorting potential* for the incident particle and $V(\vec{\xi}, r)$ is the interaction between the incident particle and the target nucleus.** The coordinate r is radial coordinate for the incident nucleon and $\vec{\xi}$ represents all other coordinates. The wave function for this system with a definite total angular momentum is expanded in terms of the channel wave function $\Phi_\alpha(\vec{\xi})$ which is a product of the target nucleus wave function and the spin-angle wave function of the incident nucleon. Here channel index α specifies the state of the target nucleus, angular momentum of the incident nucleon and channel spin.

The channel wave function $\Phi_\alpha(\vec{\xi})$ satisfies

$$[H_0(\vec{\xi})-\varepsilon_\alpha]\Phi_\alpha(\vec{\xi})=0, \tag{2·2}$$

where ε_α is the energy of the target nucleus. By using $\Phi_\alpha(\vec{\xi})$, the wave function is given by

$$\Psi(\vec{\xi}, r)=1/r\sum_{\alpha'}u_{\alpha'}(r)\,\Phi_{\alpha'}(\vec{\xi}). \tag{2·3}$$

The sum over α' includes only the neutron channel; other channels will be neglected. The expansion coefficient $u_{\alpha'}(r)$ is the radial wave function of the neutron for the channel α'. From the Schrödinger equation $(H-E)\Psi=0$, the following coupled equation is obtained,

$$\begin{aligned}&(T_\alpha+U_\alpha+\varepsilon_\alpha-E)\,u_\alpha(r)+\sum_{\alpha'}V_{\alpha\alpha'}(r)\,u_{\alpha'}(r)=0 \qquad (2·4)\\ &(T_{\alpha'}+U_{\alpha'}+\varepsilon_{\alpha'}-E)\,u_{\alpha'}(r)+\sum_{\alpha''}V_{\alpha'\alpha''}(r)\,u_{\alpha''}(r)=0\end{aligned}$$

where

$$V_{\alpha\alpha'}(r)=\int\Phi_\alpha^*(\vec{\xi})\,V(\vec{\xi}, r)\,\Phi_{\alpha'}(\vec{\xi})\,d\vec{\xi} \tag{2·5}$$

$$T_\alpha=-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+\frac{\hbar^2 l(l+1)}{2Mr^2}. \tag{2·6}$$

In the last equation M is the reduced mass of the neutron and l is the orbital angular momentum of the incident nucleon in unit of $\hbar$.

Our problem is to find an approximate solution of the coupled equation (2·4). If we assume the asymptotic forms of the $v_{\alpha'}(r)$ for the outgoing waves, then we can calculate

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* See reference 12.

** The actual form of the interaction, $V(\xi, r)$, depends on the mechanism of the excitation; if we assume the shell model for the target nucleus it is a sum of the interaction between the incident nucleon and the nucleons in the target nucleus, while it represents the interaction between the incident nucleon and the surface of the target nucleus in the case of the collective excitation.

the amplitude of the outgoing particle $u_{\alpha'}(r)$ by the first order perturbation approximation. This is the usual distorted wave approximation which fails near single particle resonance when its width is not large enough.

We shall now describe the generalized K-matrix formalism to obtain better approximation near single particle resonance. According to Teichmann and Wigner,[13] two independent wave functions outside the range of the distorting potential are defined;

$$\begin{aligned}\overline{F}_\alpha(r) &= \lambda_\alpha^{11} F_\alpha(r) + \lambda_\alpha^{12} G_\alpha(r) \\ \overline{G}(r) &= \lambda_\alpha^{21} F_\alpha(r) + \lambda_\alpha^{22} G_\alpha(r)\end{aligned} \tag{2·7}$$

where $F_\alpha(r)$ and $G_\alpha(r)$ are respectively usual regular and irregular solution of the Schrödinger equation in the external region. We choose the coefficients λ as real and further imply the following relation,

$$\lambda_\alpha^{11}\lambda_\alpha^{22} - \lambda_\alpha^{21}\lambda_\alpha^{12} = 1, \tag{2·8}$$

for the normalization of $\overline{F}_\alpha(r)$ and $\overline{G}_\alpha(r)$. Using these two functions the asymptotic form of the radial wave function in our formalism is expressed as

$$u_\alpha(r) \sim \overline{F}_\alpha(r) + K_{\alpha\alpha}\overline{G}_\alpha(r) \tag{2·9a}$$

for the entrance channel and

$$u_{\alpha'}(r) \sim k_{\alpha'}^{-1/2} K_{\alpha'\alpha} k_\alpha^{1/2} \overline{G}_{\alpha'}(r) \tag{2·9b}$$

for the other channels. In these equations K is the generalized K-matrix and k_α is the wave number of the incident or outgoing neutron in the channel α. If the K-matrix is obtained from the solution of (2·4) with asymptotic forms (2·9), then the S-matrix is calculated by the following transformation

$$S = -[\lambda^{12} + i\lambda^{11} + K(\lambda^{22} + i\lambda^{21})]^{-1}[\lambda^{12} - i\lambda^{11} + K(\lambda^{22} - i\lambda^{21})]. \tag{2·10}$$

Here the matrix notation is used in K and S, and λ is a diagonal matrix whose element is λ_α for the channel α.

The S-matrix obtained in (2·10) is unitary when the K-matrix is real and symmetric as shown easily. Further, the S-matrix should be independent of λ if the exact K is used in (2·10), while it depends on λ when an approximate K is inserted in (2·10). We shall make use of this fact to obtain the S-matrix in a better approximation.

We now proceed to calculate the K-matrix in perturbation approximation. For the zero-th order, all interactions $V_{\alpha'\alpha}(r)$ are neglected and (2·4) reduces to

$$(T_\alpha + U + \varepsilon_\alpha - E)\, u_\alpha(r) = 0, \tag{2·11}$$

where the solution is required to have the asymptotic form (2·9a). We assume that the distorting potential $U(r)$ is zero for $r > R$. Then from the logarithmic derivative of the wave function at $r = R$

$$f_\alpha = R\left[\frac{du_\alpha}{dr}/u_\alpha\right]_{r=R}, \tag{2·12}$$

we can obtain a relation between f_α and K-matrix in zero-th order by connecting the wave function for $r>R$ smoothly to the wave function for $r<R$;

$$f_\alpha = x_\alpha[\overline{F}_\alpha' + K_{\alpha\alpha}^{(0)}\overline{G}_\alpha']/[\overline{F}_\alpha + K_{\alpha\alpha}^{(0)}\overline{G}_\alpha]. \tag{2·13}$$

In the last equation, $\overline{F}_\alpha$ and $\overline{G}_\alpha$ without the argument r represent their value at $r=R$, the prime denotes the derivative with respect to x_α at $r=R$ where $x_\alpha = k_\alpha r$. By using Δ_α and s_α defined by[14]

$$\Delta_\alpha / x_\alpha = (G_\alpha G_\alpha' + F_\alpha F_\alpha')/(G_\alpha^2 + F_\alpha^2) \tag{2·14a}$$

$$s_\alpha / x_\alpha = (G_\alpha^2 + F_\alpha^2)^{-1}, \tag{2·14b}$$

the derivatives F_α' and G_α' are expressed in terms of F_α, G_α, Δ_α and s_α;

$$F_\alpha' = (s_\alpha G_\alpha + \Delta_\alpha F_\alpha)/x_\alpha \tag{2·15a}$$

$$G_\alpha' = (-s_\alpha F_\alpha + \Delta_\alpha G_\alpha)/x_\alpha \tag{2·15b}$$

where the Wronskian

$$G_\alpha F_\alpha' - F_\alpha G_\alpha' = 1 \tag{2·16}$$

is utilized.

If we introduce the two notations

$$l_\alpha = (f_\alpha - \Delta_\alpha)F_\alpha - s_\alpha G_\alpha \tag{2·17a}$$

$$m_\alpha = (f_\alpha - \Delta_\alpha)G_\alpha + s_\alpha F_\alpha \tag{2·17b}$$

the K-matrix in the zero-th order is obtained from (2·13) in a simple form

$$K_{\alpha\alpha}^{(0)} = -(\lambda_\alpha^{11} l_\alpha + \lambda_\alpha^{12} m_\alpha)/(\lambda_\alpha^{21} l_\alpha + \lambda_\alpha^{22} m_\alpha). \tag{2·18}$$

From the last equation the wave function is easily calculated;

$$u_\alpha^{(0)}(r) = u_\alpha^{(0)} \hat{u}_\alpha^{(0)}(r) \tag{2·19}$$

where $u_\alpha^{(0)}$ is the value of $u_\alpha^{(0)}(r)$ at $r=R$ which is given by

$$u_\alpha^{(0)} = x_\alpha/(\lambda_\alpha^{21} l_\alpha + \lambda_\alpha^{22} m_\alpha) \tag{2·20}$$

while $\hat{u}_\alpha^{(0)}(r)$ is the normalized wave function at $r=R$,

$$\hat{u}_\alpha^{(0)}(r) = [m_\alpha F_\alpha(r) - l_\alpha G_\alpha(r)]/x_\alpha. \tag{2·21}$$

The first order approximation $u_{\alpha'}^{(1)}(r)$ is obtained by solving the following equation,

$$(T_{\alpha'} + U_{\alpha'} + \varepsilon_{\alpha'} - E)\, u_{\alpha'}^{(1)}(r) + V_{\alpha\alpha'}(r)\, u_\alpha^{(0)}(r) = 0, \tag{2·22}$$

using $u_\alpha^{(0)}(r)$ obtained in (2·19). To solve this equation, it is necessary to introduce Green's function with the compatible boundary condition,

$$\begin{aligned} G_\alpha(r|r') &= (2M/\hbar^2 k_\alpha)\, u_\alpha^{(0)}(r)\overline{G}_\alpha(r') \quad \text{for } r<r' \\ &= (2M/\hbar^2 k_\alpha)\, u_\alpha^{(0)}(r')\overline{G}_\alpha(r) \quad \text{for } r>r'. \end{aligned} \tag{2·23}$$

Here $\overline{G}_\alpha(r)$, $F_\alpha(r)$ and $G_\alpha(r)$ for $r<R$ should be considered as the solution of (2·11) which are smoothly connected to their corresponding functions in the external region. Green's function given by (2·23), of course, depends on λ, so its dependence is explicitly shown by

$$G_\alpha(r|r')=[x_\alpha/(\lambda_\alpha^{21}l_\alpha+\lambda_\alpha^{22}m_\alpha)][\lambda_\alpha^{21}G_\alpha^{(1)}(r|r')+\lambda_\alpha^{22}G_\alpha^{(2)}(r|r')] \tag{2·24}$$

where $G_\alpha^{(1)}(r|r')$ and $G_\alpha^{(2)}(r|r')$ are independent of λ and obtained from (2·23) by replacing $u_\alpha^{(0)}(r)$ by $\hat{u}_\alpha^{(0)}(r)$, $\overline{G}_\alpha(r)$ by $F_\alpha(r)$ and $G_\alpha(r)$, respectively. Making use of (2·23), the solution of (2·22) is given by

$$\begin{aligned} u_{\alpha'}^{(1)}(r)&=\int_0^\infty G_{\alpha'}(r|r')\,V_{\alpha'\alpha}(r')\,u_\alpha^{(0)}(r')\,dr' \\ &\sim(2M/\hbar^2k_\alpha)\,\overline{G}_{\alpha'}(r)\int_0^\infty u_{\alpha'}^{(0)}(r)\,V_{\alpha'\alpha}(r)\,u_\alpha^{(0)}(r)\,dr\,. \end{aligned} \tag{2·25}$$

The last equation gives the first order K-matrix

$$K_{\alpha'\alpha}^{(1)}=-(2M/\hbar^2\sqrt{k_{\alpha'}k_\alpha})\int_0^\infty u_{\alpha'}^{(0)}(r)\,V_{\alpha'\alpha}(r)\,u_\alpha^{(0)}(r)\,dr\,. \tag{2·26}$$

The higher order term may be calculated in a similar way, but we shall give only the second order term;

$$K_{\alpha'\alpha}^{(2)}=(2M/\hbar^2\sqrt{k_{\alpha'}k_\alpha})\sum_{\alpha''}\iint u_{\alpha'}^{(0)}(r)\,V_{\alpha'\alpha''}(r)G_{\alpha''}(r|r')\,V_{\alpha''\alpha}(r')\,u_\alpha^{(0)}(r')\,dr\,dr'\,. \tag{2·27}$$

By collecting contributions in each order from (2·18), (2·26) and (2·27), the S-matrix may be calculated by inserting them in the equation (2·10),

$$S=-\left[-\frac{l-im}{x}+\hat{K}\frac{(1+i\lambda)x}{m+\lambda l}\right]^{-1}\left[-\frac{l+im}{x}+\hat{K}\,\frac{(1-i\lambda)x}{m+\lambda l}\right] \tag{2·28}$$

in which l, m and λ are diagonal matrices whose elements were defined by (2·17) and

$$\lambda_\alpha=\lambda_\alpha^{21}/\lambda_\alpha^{22}\,. \tag{2·29}$$

The matrix $\hat{K}$ is a sum of contributions other than the zero-th order when the normalized $\hat{u}_\alpha^{(0)}(r)$ are used instead of $u_\alpha^{(0)}(r)$, and expressed in a matrix form

$$\begin{aligned} \hat{K}=-(2M/\hbar^2)\,k^{-1/2}\Big[&\int_0^\infty \hat{u}^{(0)}(r)\,V(r)\,\hat{u}^{(0)}(r)\,dr-\int_0^\infty\int_0^\infty \hat{u}^{(0)}(r)\,V(r)\,G(r|r')\,V(r')\,\hat{u}^{(0)}(r')\,dr\,dr' \\ &+\cdots\cdots\Big]k^{-1/2}, \end{aligned} \tag{2·30}$$

where $\hat{u}^{(0)}(r)$ and $G(r|r')$ should naturally be taken as the diagonal matrix.

It is clear from (2·28) and (2·30) that the S-matrix depends only on the ratio λ

rather than each λ^{tj} and also the first term of $\hat{K}$ is independent of λ. The equation (2·28) may be deformed as

$$S=-\frac{1-i\lambda}{1+i\lambda}+2i\frac{m+\lambda l}{x(1+i\lambda)}\left[\frac{(-l+im)(m+\lambda l)}{x^2(1+i\lambda)}+\hat{K}\right]^{-1}\frac{m+\lambda l}{x(1+i\lambda)} \qquad (2\cdot 28a)$$

$$=-\frac{l+im}{l-im}-2i\frac{x}{l-im}\left[1+\hat{K}\frac{x^2(1+i\lambda)}{(-l+im)(m+\lambda l)}\right]^{-1}\hat{K}\frac{x}{l-im} \qquad (2\cdot 28b)$$

for the later use.

It will be worth while to note here that our method reduces to the usual distorted wave approximation when the energy is far from single particle resonances. In this case the logarithmic derivative f appearing in (2·28) through (2·17) is large and accordingly the second factor in the square bracket in the equation (2·28b) is small compared with the unity. Then it is a good approximation to neglect the second term in the square bracket in the equation (2·28b)

$$S=-\frac{l+im}{l-im}-2i\frac{x}{l-im}\hat{K}\frac{x}{l-im}.$$

The expression does not involve λ and it is easy to show that the expression coincides with the result of the usual distorted wave approximation.

(II) *Application*

In this section it will be shown that the method in the last section gives the exact solution in some special cases. First we treat the surface excitation which was already solved in reference 12. The interaction $V_{\alpha'\alpha}(r)$ is given in this case by

$$V_{\alpha'\alpha}(r)=(\hbar^2/2MR)\,\delta(r-R)W_{\alpha'\alpha}. \qquad (2\cdot 31)$$

Because of the δ-function in the interaction the K-matrix (2·30) is simply expressed as

$$\hat{K}=k^{-1/2}[-R^{-1}W+R^{-1}WG(R|R)\,(\hbar^2/2MR)W+\cdots\cdots]k^{-1/2} \qquad (2\cdot 32)$$

where W is a matrix whose element appeared in the equation (2·30). The contributions other than the first order include Green's function evaluated at $r=R$, so that those contributions may be reduced to zero by choosing the appropriate λ. From the equation (2·24) the λ which makes $G(R|R)$ zero is obtained by

$$G(R|R)=\{x/(m+\lambda l)\}[\lambda G^{(1)}(R|R)+G^{(2)}(R|R)]=0. \qquad (2\cdot 33)$$

from which we obtain

$$\lambda=-G^{(2)}(R|R)/G^{(1)}(R|R)=-G/F. \qquad (2\cdot 34)$$

Inserting (2·33) into (2·32), we get the exact K matrix

$$\hat{K}=-x^{-1/2}Wx^{-1/2}. \qquad (2\cdot 35)$$

The S-matrix is obtained by taking (2·34) and (2·35) into (2·28a)

$$S=u^{(-)}/u^{(+)}+2i(\sqrt{x}/u^{(+)})[f-\Delta-is+W]^{-1}(\sqrt{x}/u^{(+)}) \qquad (2\cdot 36)$$

where

$$u^{(\pm)}=G\pm iF. \tag{2·37}$$

This S-matrix is exactly same as given in reference 12.

This result suggests that a very good approximation can be obtained even if interaction $V(r)$ is not δ-function type but is concentrated at some surface $r=r_0$ in such a case as the surface excitation from the diffused nucleus treated by Chase and Wilets.[15] In this case the λ which is obtained from (2·33) by replacing R by r_0 should be used. Then the $\hat{K}$ matrix (2·30) becomes

$$\begin{aligned}\hat{K}\simeq&-(2M/\hbar^2\sqrt{k})\Big[\int_0^\infty \hat{u}^{(0)}(r)\,V(r)\,\hat{u}^{(0)}(r)\,dx-\hat{u}^{(0)}(r_0)\int_0^\infty V(r)\,dr\cdot G(r_0|r_0)\\&\times\int_0^\infty V(r)\,dr\cdot\hat{u}^{(0)}(r_0)+\cdots\Big]\\=&-(2M/\hbar^2)k^{-1/2}\int_0^\infty \hat{u}^{(0)}(r)\,V(r)\,\hat{u}^{(0)}(r)\,dr\cdot k^{-1/2}.\end{aligned} \tag{2·38}$$

When a computing machine is used in this calculation, this method facilitates the computation because the $\hat{K}$ is expressed in rapidly converging series.

Next we shall apply the method to the case of the separable potential;

$$V_{\alpha'\alpha}(r,r')=-(2M/\hbar^2)\,w_{\alpha'}(r)\,w_\alpha(r') \tag{2·39}$$

which is derived from the non-local but separable nucleon-nucleon potential used by Y. Yamaguchi.[16] The $\hat{K}$ matrix is

$$\hat{K}_{\alpha'\alpha}=k_{\alpha'}^{-1/2}\big[(\hat{u}_{\alpha'}w_{\alpha'})\,(w_\alpha\hat{u}_\alpha)+(\hat{u}_{\alpha'}\cdot w_{\alpha'})\sum_{\alpha''}(w_{\alpha''}G_{\alpha''}w_{\alpha''})\,(w_\alpha\hat{u}_\alpha)+\cdots\big]k_\alpha^{-1/2}, \tag{2·40}$$

where the abbreviation

$$(w_\alpha G_\alpha w_\alpha)=\int_0^\infty w_\alpha(r)\,G_\alpha(r|r')\,w_\alpha(r')\,dr\,dr' \tag{2·41a}$$

$$(\hat{u}_\alpha w_\alpha)=\int_0^\infty \hat{u}_\alpha(r)\,w_\alpha(r)\,dr \tag{2·41b}$$

is used. It is remarked that same expression $\sum_{\alpha''}(w_{\alpha''}G_{\alpha''}w_{\alpha''})$ appears always in all terms in (2·40) except the first. If this expression can be reduced to zero by choosing λ

$$\sum_{\alpha''}(w_{\alpha''}G_{\alpha''}w_{\alpha''})=\sum_{\alpha''}x_{\alpha''}/(\lambda_{\alpha''}l_{\alpha''}+m_{\alpha''})\big[\lambda_{\alpha''}(w_{\alpha''}G_{\alpha''}^{(1)}w_{\alpha''})+(w_{\alpha''}G_{\alpha''}^{(2)}w_{\alpha''})\big]=0 \tag{2·42}$$

the exact expression for $\hat{K}$ is obtained,

$$\hat{K}=k^{-1/2}(\hat{u}w)\times(w\hat{u})k^{-1/2}, \tag{2·43}$$

where $\times$ means a direct product of two vectors $(\hat{u}w)$ and $(w\hat{u})$ whose components are given in (2·41b). It is certainly possible to choose λ so as to satisfy (2·42) ;

$$\lambda = -(wG^{(2)}w)/(wG^{(1)}w). \tag{2·44}$$

If we put (2·43) into (2·28b), we obtain

$$S = -\frac{l+im}{l-im} + 2i\frac{1}{1+(p,q)}\frac{x}{-l+im}(p\times p)\frac{x}{-l+im} \tag{2·45}$$

where (p, q) is a scalar product and $(p\times p)$ is a direct product of the following vectors

$$p = k^{-1/2}(\hat{u}w),\ q = k^{-1/2}(w\hat{u})\,x^2(1+i\lambda)/(-l+im)(m+\lambda l). \tag{2·46}$$

In obtaining (2·45) a similar technique to that used by Wigner[17] was utilized. By using (2·42) it is possible to show that

$$(p,\ q) = (1/2i)\sum_{\alpha} k_{\alpha}^{-2}(w_{\alpha}G_{\alpha}^{(+)}w_{\alpha}),$$

where $G_{\alpha}^{(+)}$ is Green's function in the travelling wave which is obtained by replacing $\bar{G}(r)$ by $1/(2i)[G_{\alpha}(r)+iF(r)]$ in the equation (2·23). The result obtained in this example is naturally identified to that obtained by the straightforward method, by which more complex cases in which the interaction V is a sum of separable potentials can be solved exactly, while the method used above can be applicable only to the simple cases which are rather non-realistic.

We see that the method proposed in this section can give an exact solution in some special cases, but it gives only an approximate solution in the general case. None the less, we think the approximation is better than the usual distorted wave method particularly near the single particle resonance. We shall apply this method to the most familiar interaction, the usual two nucleon potential.* Before that we shall evaluate the potential $V(r)$ in this interaction taking into account the exchange effect between the incident nucleon and those of the target nucleus.

§ 3. Matrix elements for particle excitation

The target nucleus is assumed to be well described by the independent particle model. For the interaction between the incident nucleon and the nucleons in the nucleus, the simple spin-independent zero-range force is adopted ;

$$V(\boldsymbol{r}_0, \boldsymbol{r}_1) = V_0\,\delta(r_0-r_1)\,\delta(\cos\omega-1)/r_0r_1, \tag{3·1}$$

where $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are the position vectors of the incident nucleon and necleons in the nucleus, ω is the angle between the two vectors $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$, and V_0 is the strength of the potential. This choice of the potential is reasonable at least as the first approximation. The spin-dependent $\sigma_0\sigma_1$ force is considered as small, about 10% of the spin-independent force.[18] The effect of the finite range of the nuclear force was estimated to be small[5]

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* As for the choice of λ, see § 4.

and further the choice of the zero-range has an advantage to make the calculation much simpler.

We first consider the simplest case, in which the target nucleus has a single nucleon outside the core. The angular momenta of this nucleon in the initial and final state are denoted by j_1 and j_1' respectively, whereas those of the incident nucleon by j_0 and j_0'. The total angular momentum of this system is expressed by J, which is a vector sum of j_1 and j_0, or j_1' and j_0'. The normalized wave function for the initial state with the total angular momentum J is given by

$$|j_1 j_0 JM> = (1/r_0)\,\hat{u}_{l_0 j_0}(r)\,(1/r_1)\,\phi_{l_1 j_1}(r_1)\sum_m \chi^m_{l_1 j_1}\chi^{M-m}_{l_0 j_0}(j_1 j_0 mM-m|j_1 j_0 JM), \qquad (3\cdot 2)$$

where $\hat{u}_{l_0 j_0}(r_0)$ is the normalized zero-th order radial wave function of the incident neutron with orbital angular momentum l_0, $\phi_{l_1 j_1}(r_1)$ is that of the bound nucleon with orbital angular momentum l_1, χ^m_{lj} is the spin-angle wave function and $(j_1 j_0 mM-m|j_1 j_0 JM)$ is the Clebsch Gordan coefficient in which M is the Z-component of J. The normalization of $\hat{u}_{l_0 j_0}(r_0)$ is same as given in equations $(2\cdot 19)$ and $(2\cdot 21)$.

To calculate the matrix elements it is customary to decompose the two-body interaction into the spherical harmonics whose arguments are the angular coordinate of each nucleon. Then, $(3\cdot 1)$ is expressed as

$$V(\boldsymbol{r}_0\boldsymbol{r}_1) = V_0\sum_k v_k(r_0, r_1)\sum_\varkappa (-)^\varkappa C^{(k)}_\varkappa(0)\,C^{(k)}_{-\varkappa}(1), \qquad (3\cdot 3)$$

where

$$v_k(r_0,\ r_1) = (2k+1)\,\delta(r_0-r_1)/4r_0 r_1 \qquad (3\cdot 4a)$$

and

$$C^{(k)}_\varkappa(1) = \sqrt{\frac{4\pi}{2k+1}}\,Y_{k\varkappa}(\theta_1\varphi_1). \qquad (3\cdot 4b)$$

In the last equation $Y_{k\varkappa}(\theta_1\varphi_1)$ is the normalized spherical harmonics and $\theta_1\varphi_1$ is the angular coordinate of the particle 1. By using above equations the matrix element is calculated by the method of Racah[19]

$$\begin{aligned}\langle j_0' j_1' J|V|j_0 j_1 J\rangle = &\sum_k (2k+1)\,(-)^{j_0'+j_1-J} W(j_0' j_1' j_0 j_1;\, Jk)\\ &\times (l_0' j_0'\|C^{(k)}\|l_0 j_0)\,(l_1' j_1'\|C^{(k)}\|l_1 j_1)\,V_0\hat{I}(l_0' j_0' l_1' j_1';\, l_0 j_0 l_1 j_1), \qquad (3\cdot 5)\end{aligned}$$

where $W(j_0' j_1' j_0 j_1;\, Jk)$ is the Racah coefficient[19] and $(l' j'\|C^{(k)}\|l j)$ is the reduced matrix element. The last factor in the equation $(3\cdot 5)$ is the integral given by

$$\hat{I}(l_0' j_0' l_1' j_1',\ l_0 j_0 l_1 j_1) = (1/4)\int_0^\infty \hat{u}_{l_0' j_0'}(r)\,\phi_{l_1' j_1'}(r)\,\hat{u}_{l_0 j_0}(r)\,\phi_{l_1 j_1}(r)\,dr/r^2. \qquad (3\cdot 6)$$

By a similar procedure employed by de-Shalit[20] the summation over k in the equation $(3\cdot 5)$ may be carried out and the result is given by

$$\begin{aligned}\langle j_0' j_1' J|V|j_0 j_1 J\rangle = &[\{(2j_0'+1)\,(2j_1'+1)\,(2j_0+1)\,(2j_1+1)\}^{1/2}/(2J+1)]\cdot(j_0 j_1 \tfrac{1}{2}-\tfrac{1}{2}|j_0 j_1 J0)\\ &\times(j_0' j_1' \tfrac{1}{2}-\tfrac{1}{2}|j_0 j_1 J0)\,(\tfrac{1}{2})[1+(-)^{l_0+l_1+l_0'+l_1'}][(-)^{1+l_1-j_1+l_1'-j_1'}\end{aligned}$$

$$+1/J(J+1)\cdot\{(-)^{j_0+j_1-J}(j_0+\tfrac{1}{2})+(j_1+\tfrac{1}{2})\}\{(-)^{j_0'+j_1'-J}(j_0'+\tfrac{1}{2})+(j_1'+\tfrac{1}{2})\}]$$
$$\times(\tfrac{1}{2})V_0\hat{I}(l_0'j_0'l_1'j_1';\,l_0j_0l_1j_1). \qquad (3\cdot7)$$

From the matrix element obtained above and the equation (2·30) the normalized K-matrix of the first order is easily calculated

$$\langle\alpha' j_0' j_1' J|\hat{K}|\alpha j_0 j_1 J\rangle=-(2M/\hbar^2)k_{\alpha'}^{-1/2}\langle\alpha' j_0' j_1' J|V|\alpha j_0 j_1 J\rangle k_\alpha^{-1/2}, \qquad (3\cdot8)$$

where k_α and $k_{\alpha'}$ are the wave numbers of the nucleon in the initial and final state. The S-matrix is obtained from (3·8) by the transformation (2·28) or (2·28a), but the S-matrix thus obtained must be transformed further into the channel spin coupling scheme in order to calculate the cross section by applying Blatt and Biedenharn's formula,[21] because the S-matrix obtained directly from (3·8) is in the $j-j$ coupling scheme. The necessary formula for the second transformation was given in reference 12.

For even-even nucleus we assume that it has n_1 neutrons with angular momentum j_1 and n_2 protons with angular momentum j_2 outside the core. The resultant of n_1 angular momenta of neutrons is denoted by I_1 and that of protons by I_2 and the vector sum of I_1 and I_2, i. e., the spin of the nucleus, is expressed by I. For the ground state we assume $I_1=I_2=I=0$. The first excited state is assumed to be a mixture of two states, $I_1=2$, $I_2=0$ and $I_1=0$, $I_2=2$. If they are denoted by Φ_1 and Φ_2, the first excited state wave function is expressed by

$$\Psi=a\Phi_1+b\Phi_2 \qquad (3\cdot9)$$

$$\frac{a}{b}=\frac{-2H_{12}}{H_{11}-H_{22}-\sqrt{(H_{11}-H_{22})^2+4H_{12}^2}}, \qquad (3\cdot9a)$$

where H_{ij} is the matrix element of the energy between the state Φ_i and Φ_j in (3·9).

If the n_1 particles wave function is expressed in terms of those of n_1-1 particles by the coefficient of fractional parentage (c. f. p.),* the wave function for the target nucleus is expressed by

$$|j_1^{n_1}(I_1)j_2^{n_2}(I_2)I\rangle=\sum_{ef}\langle j_1^{n_1}I_1\{|j_1^{n_1-1}(e)j_1I_1\rangle\sqrt{(2I_1+1)(2f+1)}\,W(j_1I_1fI_2;eI)\,|eI_2(f)j_1I\rangle. \qquad (3\cdot10)$$

The angular momentum e is the resultant of (n_1-1) j_1's, and f is the vector sum of e and I_2. To calculate the two body matrix element it is convenient to express the wave function for the whole system in such a way that the n_1+n_2-1 nucleons are coupled to have the angular momentum f and the remaining nucleons with spin j_1, in the nucleus and the incident nucleon j_0 are coupled to make the resultant g. This transformation may be carried out by Racah's coefficient and one obtains

$$|j_1^{n_1}(I_1)j_2^{n_2}(I_2),\,I,\,j,\,J\rangle=\sum_{efg}\langle j_1^{n_1}I_1\{|j_1^{n_1-1}(e)j_1I\rangle\sqrt{(2I_1+1)}\,(2f+1)W(j_1I_1fI_2;eI)$$
$$\times\sqrt{(2I+1)(2g+1)}W(fj_1Jj_0;Ig)\,|f_1j_1j_0(g)J\rangle. \qquad (3\cdot11)$$

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* The value of c. f. p. needed in our calculation is tabulated in the paper by Edmonds and Flowers (Proc. Roy. Soc. **A214** (1952), 515).

The similar expression is obtained for the protons in the target nucleus. The wave function of the core is neglected in the above equation, since it has no effect on the non-diagonal matrix element.

The matrix element may be calculated by using the wave function (3·11)

$$\langle j_1^{n_1}(I_1')\, j_2^{n_2}(I_2')\, I'\, j_0';\, J|\sum V_{0t}|j_1^{n_1}(I_1)\, j_2^{n_2}(I_2)\, I j_0;\, J\rangle$$
$$=\sum_{efg}\langle j_1^{n_1} I_1'\{\, j_1^{n_1-1}(e)\, j_1 I_1'\rangle\langle j_1^{n_1} I_1\{|j_1^{n_1-1}(e)\, j_1 I_1\rangle[(2I_1+1)\,(2I_1'+1)\,(2I+1)\,(2I'+1)]^{1/2}$$
$$\times(2f+1)\,(2g+1)\,W(j_1 I_1' f I_2';\, eI')\,W(j_1 I_1 f I_2;\, eI)\,W(fj_1 J j_0';\, I'g)\,W(fj_1 J j_0;\, Ig)$$
$$\times n_1\delta_{I_2 I_2'}\langle j_1 j_0' g|V_{01}|j_1 j_0 g\rangle+\sum_{efg}\langle j_2^{n_2} I_2'\{|j_2^{n_2-1}(e)\, j_2 I_2'\rangle\langle j_2^{n_2} I_2\{|j_2^{n_2-1}(e)\, j_2 I_2\rangle$$
$$\times[(2I_2+1)\,(2I_2'+1)\,(2I+1)\,(2I'+1)]^{1/2}(2f+1)\,(2g+1)$$
$$\times W(I_1 e I j_2;\, fI_2)\,W(I_1' e I' j_2;\, fI_2')$$
$$\times W(fj_2 J j_0';\, I'g)\,W(fj_2 J j_0;\, Ig)\,n_2\delta_{I_1 I_1'}\langle j_2 j_0' g|V_{01}|j_2 j_0 g\rangle. \qquad (3\cdot12)$$

In our wave function at least one of I_1 and I_2 is zero, so we can assume $I_2=I_2'=0$. Then the above equation is simplified as

$$\langle j_1^{n_1}(I_1')\, j_2^{n_2}(0)\, I',\, j_0';\, J|\sum V_{0t}|j_1^{n_1}(I_1)\, j_2^{n_2}(0)\, I j_0;\, J\rangle$$
$$=\delta_{I_1' I'}\delta_{I_1 I}[\{(2I+1)\,(2I'+1)\}^{1/2} n_1\sum_{eg}\langle j_1^{n_1} I'\{|j_1^{n_1-1}(e)\, j_1 I'\rangle\langle j_1^{n_1} I\{|j_1^{n_1-1}(e)\, j_1 I\rangle$$
$$\times(2g+1)\,W(ej_1 J j_0';\, I'g)\,W(ej_1 J j_0;\, Ig)\langle j_1 j_0' g|V|j_1 j_0 g\rangle$$
$$+n_2(-)^{l_0-l_2}[(2j_2+1)\,(2j_0+1)]^{-1/2}\delta_{II'}\delta_{j_0 j_0'}\,2\langle j_2 j_2 0|V|j_0 j_0 0\rangle]. \qquad (3\cdot13)$$

In obtaining the last equation the following relation was used,

$$\sum_g(2g+1)\langle j_2 j_0 g|V|j_2 j_0 g\rangle=(-)^{l_0+l_2}[(2j_2+1)\,(2j_0+1)]^{1/2}\,2\langle j_2 j_2 0|V|j_0 j_0 0\rangle, \qquad (3\cdot14)$$

which may be obtained from (3·5) and (3·7).

We can further simplify the expression of the matrix element when $I_1=I_2=I_2'=0$. By using

$$\sum_g(2g+1)\,W(j_1 j_1 j_0 j_0';\, I'g)\,(-)^{j_1+j_0-g}[(2j_1+1)\,(2j_0+1)]^{-1/2}\langle j_1 j_0' g|V|j_1 j_0 g\rangle$$
$$=2[(2j_1+1)\,(2j_0+1)]^{-1/2}(-)^{l_1+l_0}\langle j_1 j_1 I'|V|j_0' j_0 I'\rangle \qquad (3\cdot15)$$

which is obtained in a similar way as (3·14), the right-hand side of the equation (3·13) is reduced to

$$\langle j_1^{n_1}(I_1')\, j_2^{n_2}(0)\, I',\, j_0';\, J|\sum V_{0t}|j_1^{n_1}(0)\, j_2^{n_2}(0),\, j_0;\, J\rangle$$
$$=\delta_{I_1' I'}[\sqrt{2I'+1}\,n_1\langle j_1^{n_1} I'\{|j_1^{n_1-1}(j_1)\, I'\rangle\{(2j_1+1)\,(2j_0+1)\}^{-1/2}(-)^{l_1+l_0}\,2\langle j_1 j_1 I'|V|j_0' j_0 I'\rangle$$
$$+n_2(-)^{l_0+l_2}\{(2j_1+1)\,(2j_0+1)\}^{-1/2}\delta_{I'0}\delta_{j_0 j_0'}\,2\langle j_2 j_2 0|V|j_0 j_0 0\rangle]. \qquad (3\cdot16)$$

We now consider the effect of the Pauli principle between the incident nucleon and the nucleons inside the nucleus. In this estimation we have to treat the exchange scattering and accordingly the equation to be solved becomes essentially an integro-differential

equation.[21] If the equation is solved in perturbation approximation along the line described in § 2, the zero-th order approximation has no change. In the first order approximation the equation (2·22) should be modified in two points. First, there appears an integral term like $\phi_{jl}(r)\int \phi_{jl}(r')\, u_{jl}(r')\, dr'$. Secondly, the exchange term should be added to the interaction term. We are concerned only about the second effect in this paper, because the first one is difficult to estimate without solving the integro-differential equation. Then the only change we have to make in calculating the first order K-matrix is to calculate the exchange integral beside the direct one just as in the calculation of energy levels of nuclei.

The antisymmetrical property of the wave function for two like nucleons requires a replacement of the matrix element (3·5) by the following one,

$$\langle j_1 j_0' g|V|j_1 j_0 g\rangle_A = \langle j_1 j_0' g|V|j_1 j_0 g\rangle - (-)^{j_1+j_0'-g}\langle j_0' j_1 g|V|j_1 j_0 g\rangle, \qquad (3\cdot 17)$$

where the matrix element without suffix A is same as given before, that is not yet antisymmetrized. One can estimate the effect of the Pauli principle by calculating $\hat{K}$-matrix from (3·17) for the case of odd neutron nuclei. For the even-even nucleus one should insert (3·17) into the equation (3·12) and take a sum over e, f and g. For the case $I=I_1=I_2=I_2'=0$, the following relation

$$\sum_g (-)^{j_0-j_0'-2j_1}(2g+1)\left[(2j_1+1)(2j_0+1)\right]^{-1/2} W(j_1 j_1 j_0 j_0'; I' g)\langle j_0' j_1 g|V|j_1 j_0 g\rangle$$
$$= (-)^{l_1+l_0}\left[(2j_1+1)(2j_0+1)\right]^{-1/2}\langle j_1 j_1 I'|V|j_0' j_0 I'\rangle \qquad (3\cdot 18)$$

simplifies (3·14) with the exchange term (3·17), and leads to the result

$$\langle j_1^{n_1}(I_1') j_2^{n_2}(0) I', j_0'; J|\sum V_{0t}|\, j_1^{n_1}(0) j_2^{n_2}(0) 0, j_0; J\rangle,$$
$$= \delta_{I_1' I'}\left[\sqrt{2I'+1}\, n_1\langle j_1^{n_1} I'\{|j_1^{n_1-1}(j_1) I'\rangle (-)^{l_1+l_0}\left[(2j_1+1)(2j_0+1)\right]^{-1/2}\langle j_1 j_1 I'|V|j_0' j_0 I'\rangle\right.$$
$$\left. + n_2(-)^{l_0+l_2}\left[(2j_1+1)(2j_0+1)\right]^{-1/2}\delta_{j_0 j_0'}\delta_{j_0 j_0'}\, 2\langle j_2 j_2 0|V|j_0 j_0 0\rangle\right]. \qquad (3\cdot 19)$$

In obtaining the last equation the fact was utilized that the exchange integral is identical to the direct integral in the case of the zero-range interaction. The first term of the right-side of (3·19) is just a half of the corresponding term of (3·16), thus the Pauli principle makes the effective strength of the nuclear force between like nucleons half as large as that of the nuclear force when the Pauli principle is neglected. On the contrary, the second term of (3·19) is same as that of (3·16) because the Pauli principle has no effect on the neutron-proton interaction. If neither I nor I' is zero, the resultant expression is not so simple that one cannot estimate the effect of the Pauli principle without detailed calculations. However, the actual calculation (see § 4) shows that the non-diagonal matrix element for $I=I'=2$ are small compared with those for $I=0$ and $I'=2$, so we can infer the effect of the Pauli principle only from the equation (3·19).

We found that the Pauli principle makes the effective strength of the nuclear force weak in the case of even-even nuclei, but we should consider it qualitatively, because we made over-simplifications in our calculation. The zero-range force makes us over-estimate the effect of the Pauli principle, because the exchange integral is same as the direct one

in the zero-range force. On the contrary, if the long-range approximation is used the exchange term vanishes. The effect of the term involving integrals were not taken into account in our calculation.

If we use a potential without spin-dependence for the incident and outgoing nucleon, then we have no need to take account of the spin of the incident neutron at all. This simplifies the calculation considerably. In this case it is convenient to use the following formula

$$\langle Il_0JM|\sum V_{0i}|I'l_0'JM\rangle = V_0\sum_k (2k+1)\,(-)^{I+l_0'-J}W(Il_0I'l_0';\,Jk)\sqrt{(2I+1)\,(2I'+1)}$$
$$\times(l_0\|C^{(k)}\|l_0')\,[(-)^{I_1'-I_2-k+I}(I_1\|\sum C^{(k)}(i)\|I_1')\,W(I_1I\,I_1'I';\,I_2k)\,\delta_{I_2I_2'}\hat{I}(l_1j_1l_0,\,l_1j_1l_0')$$
$$+(-)^{I_1-I_2+k-I'}(I_2\|\sum C^{(k)}(i)\|I_2')\,W(I_2I\,I_2'I_1;\,I_1k)\,\delta_{I_1I_1'}\hat{I}(l_2j_2l_0,\,l_2j_2l_0')\,],$$

where

$$(l_0\|C^{(k)}\|l_0') = \sqrt{2l_0'+1}\,(l_0'k00\,|\,l_0'kl_00)$$

and

$$(I_1\|\sum C^{(k)}\|I_1') = n_1\sum_e \langle j_1^{n_1}I_1\{|j_1^{n_1-1}(e)j_1I_1\rangle\langle j_1^{n_1-1}(e)j_1I_1'|\}j_1^{n_1}I_1'\rangle$$
$$\times(-)^{e-j_1+k-I'}\sqrt{(2I_1+1)\,(2I_1'+1)}\,W(j_1I_1j_1I_1';\,ek)\,(-)^{j_1'-1/2}(2j_1+1)/\sqrt{2k+1}$$
$$\times(j_1j_1\tfrac{1}{2}-\tfrac{1}{2}\,|j_1j_1k0)\,\{1+(-)^{l_1+l_1'+k}\}/2.$$

These expressions were obtained by applying Racah's method[19] directly. In this case, of course, the effect of the Pauli principle is not taken into account, because we should consider the incident neutron without spin as different from the neutron in the nucleus.

§ 4. Results and discussions

Now the neutron scattering by Mg^{24} will be investigated in this section as an example. First the particle excitation is considered; for the target nucleus Mg^{24} we assume that four neutrons and protons whose orbit is assigned as $d_{5/2}$ are moving in a square well potential. The depth of potential is assumed as 30.5 Mev and its radius $R=1.45\times10^{-13}$ cm$\times(24)^{1/3}$, which give a right binding energy of the last neutron of Mg^{24}, 4.64 Mev.

The ground state has spin 0, so both resultant spins of neutrons and protons are assumed to be 0; in the notation of § 3 $I_1=I_2=I=0$. The first excited state is assumed as a mixture of two states $I_1=0, I_2=2$, $I=2$ and $I_1=2$, $I_2=0$ and $I=2$ whose mixing ratio is taken as $a=b=1/\sqrt{2}$. This ratio is derived from (3·9a) if $H_{11}=H_{22}$ and $H_{12}>0$. The higher excited states other than the first are all neglected in the calculation of particle excitation.

For distorting potential an energy independent, real square well potential is assumed

$$\begin{aligned} V(r) &= -42 \text{ Mev} && r\leq R \\ &= 0 && r>R \end{aligned} \qquad (4\cdot1)$$

where $R=1.45\times10^{-13}$ cm$\times(24)^{1/3}$. This choice of the potential is same as used by

Feshbach, Porter and Weisskopf[22] except the imaginary part. The omission of the imaginary part is for the convenience of the numerical calculation. The presence of the direct interaction changes the elastic scattering considerably, so we have to use the different distorting potential from the usual optical potential which gives a right elastic scattering as shown in our previous paper.[12] However, as we have interest mainly in the application of our method and comparison of two different mechanisms of the excitation, and not so much in the comparison with experiment, our choice of the distorting potential seems to be sufficient for our purposes.

The inter-nucleon potential is assumed as zero-range, spin independent as discussed in § 3. Its strength V_0 is fixed from the pairing energy of the last neutron in Mg^{26}, which is expressed as[23]

$$E_{\text{pair}}=(2j+1)I(j)V_0/2 \tag{4·2}$$

where j is the angular momentum of the last neutron, i.e., $j=5/2$, V_0 is the strength of the nuclear force appeared in (3·1) and $I(j)$ is the usual integral analogous to (3·6);

$$I(j)=(1/4)\int_0^\infty [\phi_{jl}(r)]^4 dr/r^2, \tag{4·3}$$

where $\phi_{jl}(r)$ is the radial wave function of a bound nucleon with angular momentum j and orbital angular momentum l. The experimental pairing energy of the last neutron in Mg^{26} is taken from the table of Talmi and Thieberger[24] which gives

$$E_{\text{pair}}=1.30 \text{ Mev}. \tag{4·4}$$

Using a wave function solved for the square well potential described above, we obtain from (4·2), (4·3) and (4·4)

$$V_0=-182\times 10^{-39} \text{ cm}^3 \text{ Mev}. \tag{4·5}$$

We calculated the cross section for incident energies $E=2$, 4 and 6 Mev neglecting the Pauli principle. Omission of the exchange effect between the incident nucleon and nucleons inside the nucleus is partly for simplification of the numerical calculation and partly because of the fact that one of the effects of the Pauli principle is the reduction of the effective strength of the nuclear force in the even-even nucleus and also the determination of the strength has some uncertainty. For the non-diagonal matrix elements of K, the first order contribution is calculated and for the diagonal matrix element only the zero-th order contribution is calculated. The matrix elements between the first excited state are also included in the calculation.

From these $\hat{K}$ matrix elements the S-matrix is transformed by (2·28). The parameter λ appearing in (2·28) is fixed so that $\bar{G}_\alpha(r)$ may have zero value at a point where the interaction $V_{\alpha'\alpha}(r)$ has maximum value. By choosing λ in this way it is true that the higher order contributions become much smaller than those of the usual perturbation approximation on the basis discussed in § 2, II. However it is not clear those

contributions are small compared with the first order contribution. The numerical solution of our original equation (2·4) might reveal the validity of our approximation near a single particle resonance. For even-parity wave the usual distorted wave approximation is applied because these waves are far from the single particle resonances. The calculation was carried out also for odd parity waves in perturbation approximation for comparison.

Before discussing the results of calculation, the parameters used in the calculation of the surface excitation will be described here. The target nucleus is assumed to be described by the collective model and further the strong coupling approximation is valid. The parameter of the deformation of the nucleus ε, which is defined in the previous paper,[12] is fixed as 0.3, that is same as used before.[12]

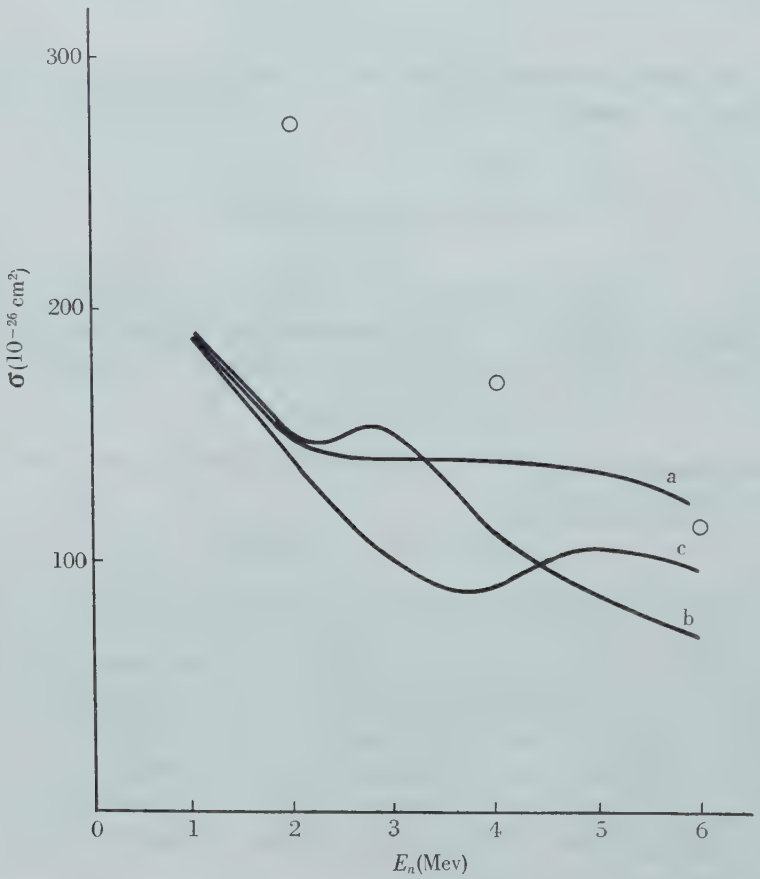

Fig. 1. Elastic total cross sections for Mg^{24} (n, n) Mg^{24}
Curve a: The scattering by the square well potential (4·1).
Curve b: The surface excitation by the exact calculation taking account of the first two states of the target nucleus.
Curve c: The surface excitation by the exact calculation taking account of the first three states of the target nucleus.
Circle o: The particle excitation by the standing wave method.

The ground state, the first (1.368 Mev, spin 2) and the second (4.122 Mev, spin 4) excited states are identified as the first three levels in the rotational spectra. For the distorting potential the same potential (4·1) is used also in this case. The calculation was carried out every one Mev up to 6 Mev firstly including the lowest three levels of the target nucleus, next including only the ground and the first excited states and lastly in perturbation approximation.

The elastic total cross section for four cases are shown in Fig. 1. In contrast to the simple curve of the scattering by square well potential, the curve calculated taking

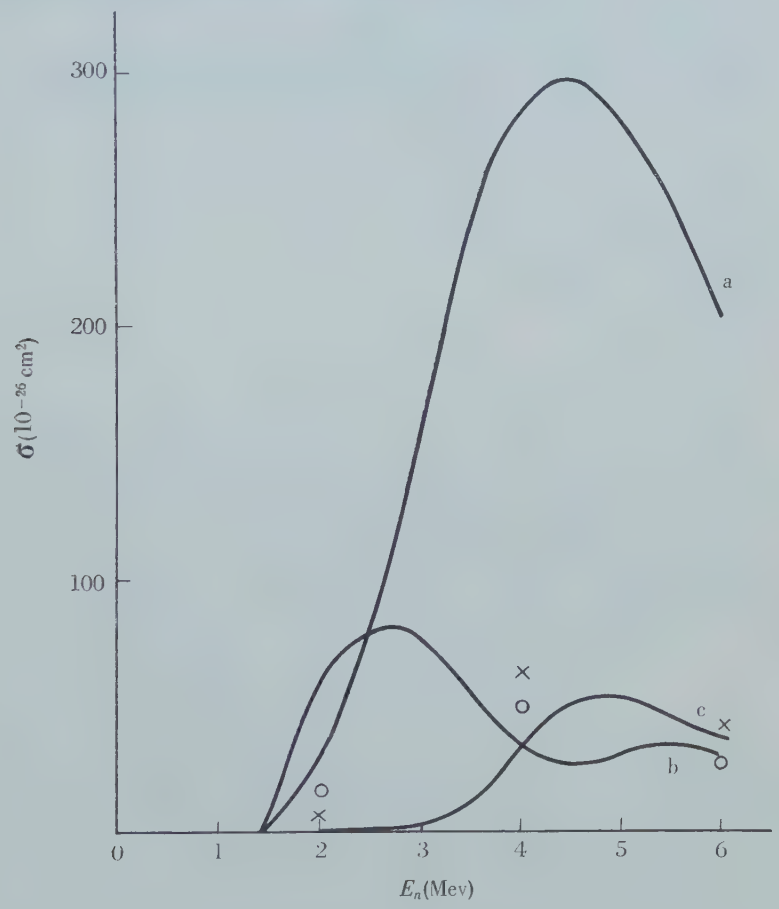

Fig. 2. Total inelastic scattering for Mg^{24} (n, n') Mg^{*24} leaving the residual nucleus at the first excited state.

Curve a: The surface excitation by the perturbation approximation.

Curve b: The surface excitation by the exact calculation taking account of the first two states of the target nucleus.

Curve c: The surface excitation by the exact calculation taking account of the first three states of the target nucleus.

Circles o: The particle excitation by the standing wave method.

Crosses x: The particle excitation by the perturbation approximation.

into account the first two and three levels of the target nucleus in the surface excitation shows a gradual peak. The cross section of the particle excitation is much higher than those of the surface excitation at lower energies.

In Fig. 2 the total inelastic scattering leaving the residual nucleus in the first excited state is shown in five cases. The surface excitation cross section calculated in the perturbation approximation has a very large value at 4.5 Mev. The two cases, the surface excitation including the first two states and first three states have very different excitation curves. The former has a peak near 2.5 Mev and a gradual peak at 5.5 Mev while the latter cross section is very small up to 3 Mev. The two cases of the particle excitation are not so different in the absolute value at each energy.

The angular distributions for elastic scattering are shown in Fig. 3 in the absolute scale, and those of inelastic scattering are shown in Fig. 4, where the cross section is normalized

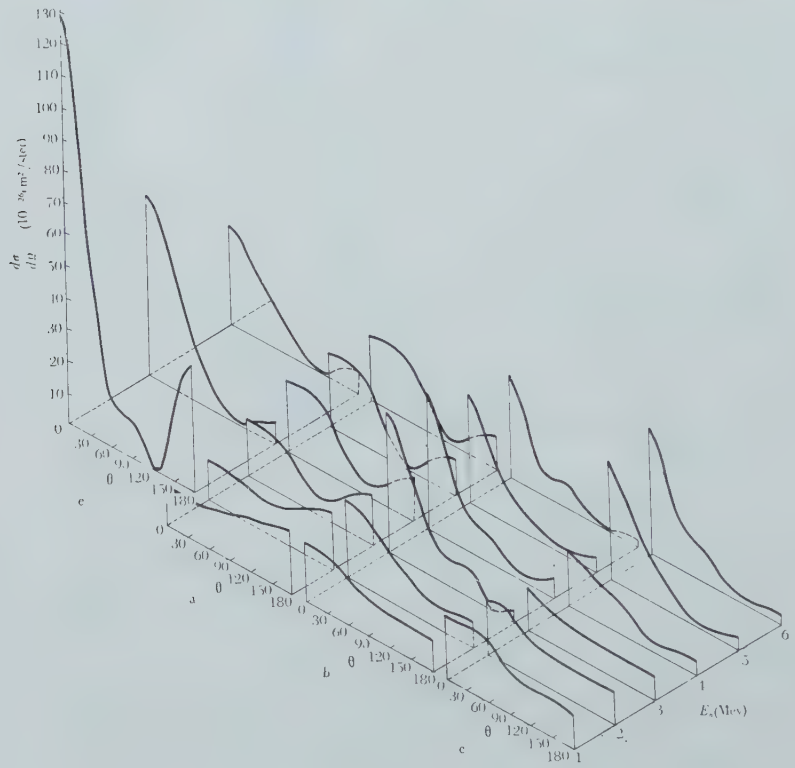

Fig. 3. Angular distribution of the elastic scattering Mg^{24} (n, n) Mg^{24}
θ: Scattering angle in c. m. system (degree), E_n: incident energy of the neutron in lab. system (Mev), $d\sigma/d\Omega$: differential cross section.
a: The scattering by the square well potential (4·1).
b: The surface excitation by the exact calculation taking account of the first two states of the target nucleus.
c: The surface excitation by the exact calculation taking account of the first three states of the target nucleus.
e: The particle excitation by the standing wave method.

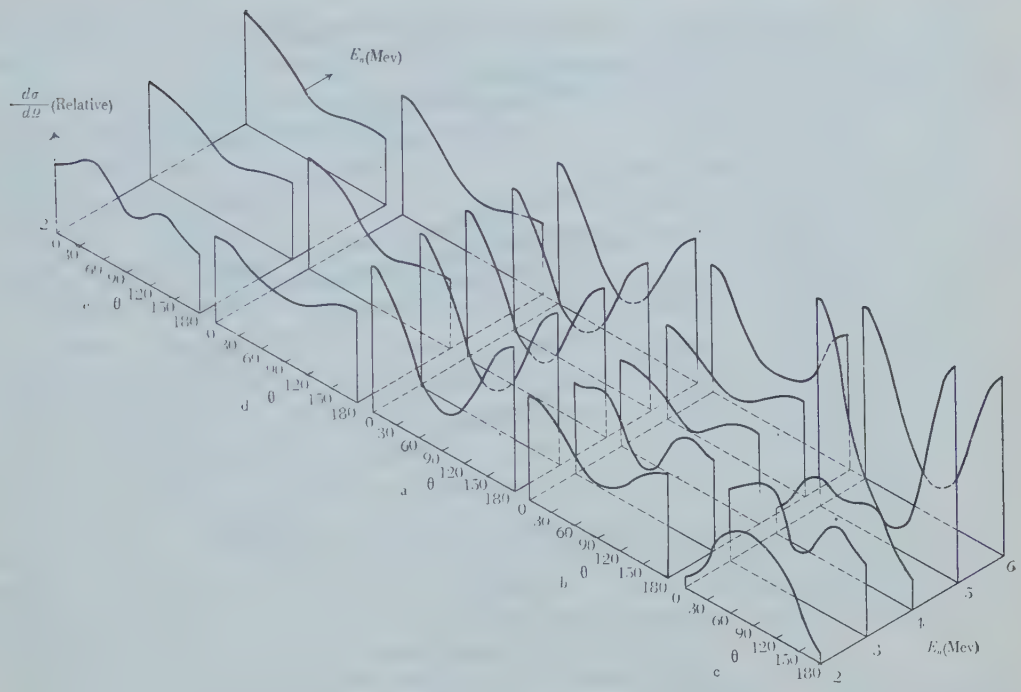

Fig. 4. Angular distribution of the inelastic scattering Mg^{24} (n, n') Mg^{24} leaving the residual nucleus at the first excited state.
θ: Scattering angle in c. m. system (degree), E: incident energy of the neutron in lab. system (Mev), $d\sigma/d\Omega$: differential cross section (in arbitrary scale).
a: The surface excitation by the perturbation approximation
b: The surface excitation by the exact calculation taking account of the first two states of the target nucleus.
c: The surface excitation by the exact calculation taking account of the first three states of the target nucleus.
d: The particle excitation by the perturbation approximation.
e: The particle excitation by the standing wave method.

so as to give a same total cross section for all. In the elastic scattering the particle excitation and the surface excitation cross section deviate from the case of the square well potential. The energy dependence of the angular distribution is strong for the standing wave calculation compared with the perturbation approximation calculation. Especially the case of the surface excitation taking into account the first and the second excited states varies very much with energy.

Before discussing differences between two mechanisms, it is important to know how the partial waves are distorted by the potential (4·1). In Fig. 5 the quantity, the logarithmic derivative for the solution of the square well potential f_l minus the real part of the logarithmic derivative of the outgoing wave at the same energy J_l, is plotted against the incident energy. This quantity corresponding to the outgoing particle leaving the target nucleus at the excited state is also plotted up to 6 Mev incident energy although

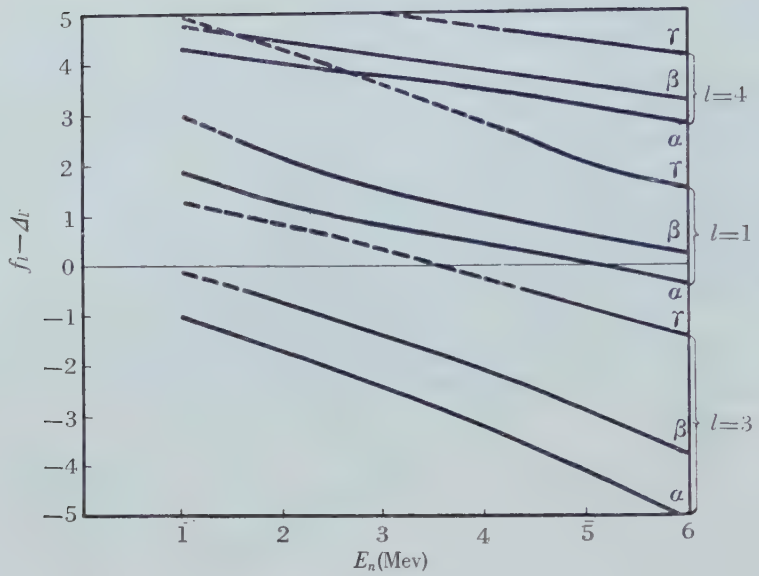

Fig. 5. Energy dependence of $f_l - \Delta_l$
l indicates the orbital angular momentum.
α: curves corresponding to the ground state.
β: curves corresponding to the first excited state.
γ: curves corresponding to the second excited state.

the dotted curve is used for the corresponding quantity for the bound state. For s and d waves the curves are not drawn because they are outside the frame of the figure. The point on which the curve crosses the line $f_l - \Delta_l = 0$ corresponds to the resonance energy; the p wave for the ground state has a resonance near 5 Mev. The dotted f wave curve for the second excited state crosses the line $f_l - \Delta_l = 0$ near 3.5 Mev; at this energy the zero-th order wave function inside the nuclear surface is connected smoothly to the exponentially decaying wave function outside. This corresponds to the usual resonance phenomena at positive energy. In the case of the surface excitation, however, it should be noted that the direct interaction gives rise to a shift of this resonance energy. This resonance effect will be discussed shortly.

When the differences between two excitation mechanism are discussed, it is convenient to institute a comparison by the K matrix. For the particle excitation $\hat{K}$ matrix is written by (3·8) as

$$\langle \alpha I l J | \hat{K}_p | \alpha' I' l' J \rangle = -(2MV_0/\hbar^2 R) x_\alpha^{-1/2} \langle I l J | M_p | I' l' J \rangle \langle \alpha l | \hat{I} | \alpha' l' \rangle x_{\alpha'}^{-1/2}, \qquad (4 \cdot 6a)$$

while for the surface excitation

$$\langle \alpha I l J | \hat{K}_s | \alpha' I' l' J \rangle = -(2MR^2/\hbar^2) U_0 (2\varepsilon/15) x_\alpha^{-1/2} \langle I l J | M_s | I' l' J \rangle x_{\alpha'}^{-1/2}. \qquad (4 \cdot 6b)$$

M_p and M_s are the spin angle part of the interaction matrix element in particle and surface excitation respectively. In our calculation they are of the same order of magnitude

and have the same sign for the diagonal element and the opposite sign for the non-diagonal element. The quantities V_0 and U_0 are the strength of the zero-range two nucleons interaction for the particle excitation and the depth of the distorting potential for the surface excitation respectively. Both of them have a negative sign. ε is the deformation parameter of the target nucleus in the surface excitation and its sign depends on the nucleus; for the case of Mg^{24} it is positive. $\hat{I}$ is the radial integral of the single particle wave functions normalized at the nuclear surface, and has usually a positive value. The wave function has a large value at the nuclear boundary at its resonance energy if the wave is normalized at infinity, but the wave function has a large amplitude for an off resonance energy since it is normalized at the nuclear boundary. Thus $\hat{I}$ has a small value for the resonance energy and a large value at the off resonance unless the integrand of $\hat{I}$ oscillates with r and the integral is partly cancelled. However, the contribution to the K-matrix is, of course, large at resonance and small at the off resonance because the K-matrix is obtained from $\hat{K}$ by multiplying the values of the wave function at the nuclear surface. Nevertheless one of differences between two excitation mechanism rests on this point; in the surface interaction the effect of the resonance is amplified compared with the particle resonance. This is the reason why the angular distribution for inelastic scattering has the larger 90° asymmetry than the case of the surface excitation (c. f. Fig. 4).

The second point to be discussed is the sign of the $\hat{K}$ matrix. In the preceding paragraph it was shown that the sign of the non-diagonal matrix element is opposite in the case of Mg^{24}. The cross section depends on the sign of the matrix element unless the perturbation approximation is used. Then it is not surprising that the angular distribution and the absolute value of the cross section in two excitation mechanisms differ very much. In the case of the particle excitation, the large elastic cross section may be due to the approximation in our calculation. But it is difficult to discuss it without an exact solution of the original coupled equation (2·4).

The third point is the magnitude of the $\hat{K}$ matrix. The ratio of these two $\hat{K}$ matrices is given by

$$\hat{K}_p/\hat{K}_s = (V_0/R^3U_0)\,(15/2\varepsilon)\,\hat{I}\cdot(M_p/M_s) \qquad (4\cdot7)$$

from (4·6a) and (4·6b). The first factor (V_0/R^3U_0) is about 1/2 as estimated from (4·1) and (4·5), and the second factor is 25. $\hat{I}$ changes from 10^2 to 10^{-1} according to the partial waves. The ratio $\hat{K}_p/\hat{K}_s$ is from about 1 to 1/70 for the odd parity wave transitions. It is also interesting to study the dependence of $\hat{K}$ on the number of nucleon outside the core. In the case of the surface excitation ε has a large value (about 0.4) when the neutron and proton are half filled in the shells. On the other hand, in the case of the particle excitation the non-diagonal matrix element between the spin zero state and spin I state is proportional to

$$n(j^{n-1}(j)\,I|\}\,j_1{}^n\,I) = [2n(2j+1-n)/(2j-1)]^{1/2} \quad \text{for } I=2,\,4,\cdots\cdots(2j-1) \qquad (4\cdot8)$$

where the excited state of the target nucleus with spin I is assumed to have n neutrons

(or protons) with total angular momentum j. For $j=5/2$ the expression (4·8) gives 2 for $n=2$, 4. For large j it has a maximum when the shell is half filled, but the maximum value is not so different from the case for $n=2$ in such a simple model. Thus we see that the surface excitation gives large matrix element compared with the particle excitation for nuclei whose shells are half filled. Again returning to the case of Mg^{24}, the diagonal elements of K matrix have the same order as the non-diagonal matrix element for the surface excitation. In such a situation the perturbation approximation fails and the cross section for inelastic scattering becomes extremely large compared with the exact calculation as seen from Fig. 2. For the particle excitation both perturbation and standing wave approximations are not so different in the total cross section owing to the small non-diagonal matrix element of $\hat{K}$.

We now discuss the behaviour of the cross section when the quantity $f_l-\Delta_l-W_{ll}$ for the closed channel approaches to zero, the resonance energy. For surface excitation of the even-even nucleus we take into account only three channels, 1, 2, 3, which correspond to the spin 0, 2, 4 states respectively. The S-matrix for this process is described by the equation (2·36) which contains the inverse matrix of $f-\Delta-is+W\equiv H$. If we write its matrix elements by

$$H=\begin{pmatrix} G_1 & W_{12} & W_{13} \\ W_{21} & G_2 & W_{23} \\ W_{31} & W_{32} & G_3 \end{pmatrix} \tag{4·9}$$

$G_3=f_3-\Delta_3+W_{33}$ approaches to zero because $s=0$ for the closed channel. Further $W_{13}=0$ owing to the assumed interaction which contains only the spherical harmonic of order 2. Then the matrix element of $Q\equiv H^{-1}$ is calculated as

$$Q_{11}=1/G_1, \quad Q_{21}\to 0, \quad Q_{31}=-W_{21}/(W_{22}G_1). \tag{4·10}$$

From the last equations it is seen that the inelastic scattering leaving the target nucleus to the first excited state becomes zero. This three channels calculation is of course over-simplification, but we can expect a similar situation occurs in general cases. In our numerical calculation $f_l-\Delta_l-W_{ll}$ becomes zero for the channel $J=3$ $l=3$ near 3 Mev and $J=1$ $l=3$ near the threshold. It may be explained by this fact that the cross section is very depressed up to 4 Mev when the first three excited states of the target nucleus are taken into account in contrast to the calculation neglecting the second excited state.

Finally we should like to point out that the inclusion of the many excited states in the calculation of the direct interaction makes the energy dependence of the cross section complex. The angular distribution of inelastic scattering calculated by including the first three states has a strong dependence on the energy as seen from Fig. 4. The usual optical model can explain the gross structure of the energy dependence only and the single particle resonances appear. However if a few low lying states have strong transition probability from the ground state, the single particle resonance state splits into the same number of resonances as the excited states taken into account. There will appear smaller resonances but not so small as the fine structure.

Our calculation were carried out in order to apply the method proposed here and compare two excitation mechanisms. So we used some assumptions which are not so realistic to avoid complexity. However if this method is used to calculate the direct interaction effect with reasonable assumptions in the case where the experimental data are available we can have some hope to resolve the difficulties encountered in explaining the experimental data by taking account of the effects discussed here.

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Multiple Meson Production in Antinucleon-Nucleon Annihilation

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The multiple meson production in the antinucleon-nucleon annihilation is studied. Assuming that the whole system establishes a statistical equilibrium, we adopt the temperature of the system as the only parameter, and deciding the critical temperature in the similar way as Landau's, we obtain the numerical values which agree with experimental data and clear up the question why one must take a large magnitude of the interaction volume as a parameter in the Fermi statistical model.

§ 1. Introduction

As regards the multiple production of mesons Fermi[1] proposed the statistical model which has been useful to analyse the experimental data but gave lower multiplicities of π-mesons than the experimental ones in antinucleon-nucleon annihilations as well as in nucleon-nucleon collisions. The experimental multiplicity forces us to take fifteen times of the usual volume in the Fermi statistical model whose radius is π-meson Compton wave length.[2]

Recently Gotō[3] treated the meson production in the annihilation introducing the $\pi-\pi$ interaction of resonance type with certain resonance energy which is based on the experimental momentum distribution of the nucleon in $\pi^-+P\rightarrow nucleon+2\pi$ with 1 Gev. His modification is in favour of high multiplicities but he also must take the large interaction volume to agree with the experimental multiplicity. Itō and Minami[4] pointed out that the experimental multiplicity could be explained by introducing an appropriate $\pi-\pi$ interaction. However it is necessary to closely examine the nature of $\pi-\pi$ resonance interaction which they took. Takeda[5] shows the large multiplicity by considering mesons which are virtually surrounding nucleonic cores and are really emitted for losses of these cores after the annihilation.

Pomerančuk[6] proposed a modification of the statistical model which includes the nature of strong $\pi-\pi$ interactions but does not depend on its detail form. He considered the system should be enlarged so that the emitted particles might fly away freely. Assuming the $\pi-\pi$ interaction range to be the Compton wave length of π-meson, the volume of the final state V may be expressed as

$$V\sim N\mu^{-3}\ *\qquad(1)$$

in the production of N π-mesons, where μ is the rest mass of a π-meson. Taking eq.

* We assume $\hbar=c=1$.

(1) into account, Sudarshan[7] and the Soviet group[8] calculated the multiplicity of the π-meson in the annihilation process but these results show the low multiplicity too.

In this paper, instead of eq. (1), we shall introduce in Sec. 2 the critical temperature of the system at which mesons fly away as free particles. In Sec. 3 the physical meaning of this temperature is studied and the numerical values of various quantities determined by this temperature are evaluated and compared with the experimental values. The agreement between both values is good.

§ 2. Fundamental idea

2·1 *General aspects*

We have no precise method to treat the whole process of annihilation but a statistical treatment may be admitted because of the strong interaction in meson fields surrounding a nucleon. The statement that we expect some sort of statistical equilibrium should be justified as follows.

When an antinucleon annihilates with a nucleon the rest energies of these particles will be suddenly released in a small spherical volume whose radius is about the Compton wave length of a π-meson. At this moment the state of the messic cloud is similar to its state before the annihilation but is different from the state described by the Planck distribution because of the sudden losses of the nucleonic and antinucleonic cores. Just after the annihilation, therefore, the system is, of course, not in the equilibrium state. However, since the various particles (π-mesons, heavy mesons and other particles) are virtually created and absorbed in this small volume, the concept "the number of particles" is inadequate to describe this state. Indeed the number of particles regains its physical meaning only after the main process of interaction has been over and all particles have become fairly free. Thus, the energy spectrum of particles, which we observe, is different from that just after the annihilation.

From such a point of view the condition that the number of particles should have a physical meaning is given as follows: the mean free path of particles, λ, in the system is comparable in magnitude to the linear dimension of the total system, L, as argued by Landau and Belen'kij[9),10)];

$$\lambda \sim L. \tag{2}$$

If λ is determined mainly by the $\pi-\pi$ interaction and if the total system is in a thermal equilibrium state we may obtain the temperature of system T from eq. (2), that is,

$$T \sim \mu,^{*} \tag{2'}$$

where we assume the $\pi-\pi$ interaction cross section is geometrical one, $\pi\mu^{-2}$. We can easily understand that under the condition (2) $L > \mu^{-1}$ since the λ is inversely proportional to the matter density, $\lambda \sim (L^3/N)\mu^2$, so that the system must be enlarged.

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* We assume that Boltzmann constant $k=1$.

In the stage where the volume of the system is large, the system might attain to a statistical equilibrium as a whole due to strong interactions and when $\lambda \gtrsim L$ all particles might be submitted to the Planck distribution.

One can easily find that eq. (2) is quite different from eq. (1) used by Pomerančuk. In the case of $N\pi$ production, the volume of final system $V(\sim L^3)$ is approximately $N^{5/2}\mu^{-3}$ from eq. (2). The N-dependence of this volume is different from the one given by eq. (1), and we may expect that the multiplicity of particles is larger than in the case of Pomerančuk.

The difference between Landau's hydrodynamical treatment[9] and ours is that no hydrodynamical consideration taken into account on the system after the annihilation because this system has no special translational motion but only a thermal motion (we neglect the kinetic energy of a nucleon and an antinucleon). In order to justify the hydrodynamical treatment each local system must establish an equilibrium and must include many particles. In our system, however, the number of particles is not enough to justify such a treatment. In view of the increase of entropy in the expansion process, our aspect looks rather like Heisenberg's[11] than Landau's.

From the above discussion, we may consider that the final system is in a thermal equilibrium state whose temperature is determined by the condition (2). As far as the average values of the physical quantities is concerned, one can thermodynamically treat the annihilation process introducing a temperature by no reference to the tedeous phase space volume treatment.

2·2 *The examination of the Planck distribution*

In order to examine whether the whole system is in the equilibrium or not we compare the Planck distribution with the experimental spectrum of kinetic energies of π-mesons. The reasoning of the application of the Planck distribution to emitted π-mesons is not so clear but may be understood from the fact that in the flying away stage the $\pi-\pi$ interaction mean free path is of the same order as that of the linear dimension of the total system

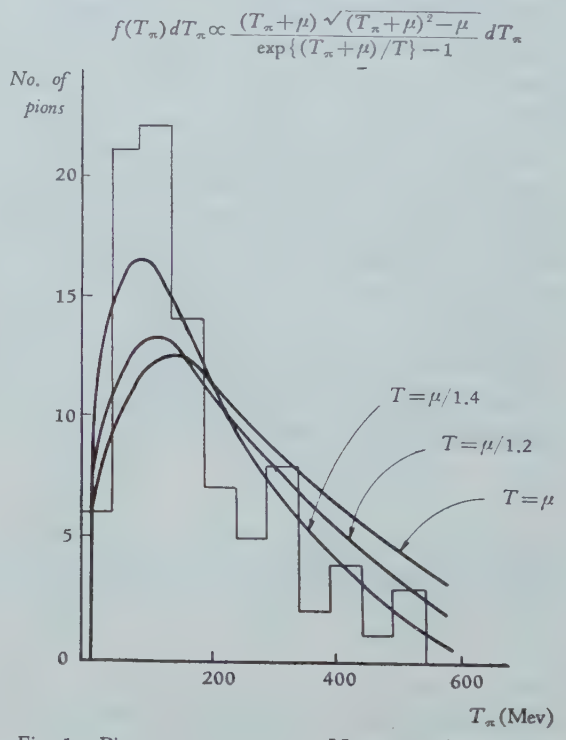

Fig. 1. Pion energy spectrum. Histogram shows experimentally found charged pion spectrum. Solid curves are computed from the Planck distribution for $T=\mu$, $\mu/1.2$ and $\mu/1.4$.

(see Sec. (3·3)).

The normalized energy spectra of π-mesons are plotted in Fig. 1, where the Planck distribution with $T=\mu$, $\mu/1.2$ and $\mu/1.4$ is assumed. It is pointed out that approximately 5% of the experimentally observed π-mesons are expected to have lost the energy by inelastic scattering.[2] Therefore the plotted curves should be slightly depressed at high energies and raised at low energies in order to compare them directly with the experimental spectrum. It seems that the Planck distribution with $T=\mu/1.4$ or $\mu/1.2$ agrees as well with the experimental spectrum as the Fermi model do, provided that the multiplicity of π-meson is put to be 5.

Since the total energy is nearly 2 Gev and T is about one-tenth of the total energy, the concept of the equilibrium seems to be unreasonable because of the large fluctuation. Nevertheless, this fluctuation does not play an important role in our calculation as far as we consider only the average quantities.

§ 3. Quantitative treatment

According to the discussion in Sec. (2·2), we proceed on by using the distribution function of ideal gas under the condition (2) and give the detailed calculation for the average physical quantities.

3·1 *The density of π-mesons*

We use the dimensionless parameter x instead of the temperature T:

$$x=\mu/T. \tag{3}$$

The particle density n_π and the energy density $\mathcal{E}_\pi$ of the π-meson are given by the Planck distribution function and we have[10),12)]

$$n_\pi=(3/2\pi^2)\,\mu^3 x^{-3}F(x), \tag{4a}$$

$$\mathcal{E}_\pi/\mu=(3/2\pi^2)\,\mu^3 x^{-4}G(x), \tag{4b}$$

in which

$$F(x)=x^2\sum_{\nu=0}^{\infty}K_2((1+\nu)x)/(1+\nu),$$

$$G(x)=x^2\sum_{\nu=0}^{\infty}(1+\nu)^{-1}\{3K_2((1+\nu)x)/(1+\nu)+xK_1((1+\nu)x)\},$$

where the statistical weight of the pion is 3 and $K_m(x)$ is the modified Bessel function of the m-th order.

The total number of the pion N_π is expressed by

$$N_\pi=(2/\pi)\,ax^{-3}F(x), \tag{5}$$

where a is the dimensionless measure of the volume defined by

$$a=V/(\tfrac{4}{3}\pi\mu^{-3}). \tag{6}$$

The value a is determined by the following equation,

$$(\mathcal{E}_\pi+\mathcal{E}_K)V=2M, \tag{7}$$

where $\mathcal{E}_K$ is the K-meson energy density and M is the rest energy of a nucleon. In eq. (7) the kinetic energies of a nucleon and an antinucleon are neglected. We must know $\mathcal{E}_K$ to obtain the value a.

3·2 *The density of K-meson*

It is necessary that the conservation law for the strangeness is considered in K-meson productions. For introduction of the strangeness two ways are possible : the one takes the strangeness in a form of canonical distributions and another in a form of micro-canonical distributions. The difference between the two formalisms lies mainly in the treatment of the fluctuation, any kind of which are not essentially important in the system including many particles. On the system consisting of several particles, however, the fluctuation can not be neglected and one should necessarily regard the system as a microcanonical ensemble.

On high energy phenomena Koba[12] introduced the strangeness in the canonical formalism in which the conservation for the strangeness does not affect the distribution function because of the initial strangeness being just zero, and the particle density n_K and the energy density $\mathcal{E}_K$ of K-meson is given by replacing $3.5x$ instead of x in the functions F and G in eq. (4) respectively (the value 3.5 represents the ratio of the K to the π-meson mass). This formalism for the strangeness, however, is not appropriate in the annihilation process and we shall use the partition function Z for the K-meson in the microcanonical formalism (for the strangeness) shown as follows ;

$$Z=\sum_{n_s,\,\bar{n}_s}\Big(\prod_s e^{-(n_s+\bar{n}_s)\epsilon_s/T}\,\delta_{\sum_s n_s,\,\sum_s \bar{n}_s}\Big), \tag{8}$$

where n_s and $\bar{n}_s$ are the number of K-meson and $\bar{K}$-meson in the state s respectively and δ is Kronecker's symbol.

Magalinskij and Terletzkij[13] derived in the extreme relativistic case the particle number from the similar equation. It is easy to generalize their calculation to the case where particles have rest masses and the particle density and the energy density of K and $\bar{K}$-meson are given as follows;

$$n_{\bar{K}}=\frac{\mu^3}{2\pi^2}x^{-3}\sum_{\nu=0}^{\infty}F_\nu(3.5x)\int_0^{2\pi}\cos(1+\nu)\varphi\, e^{\Phi(\varphi)}d\varphi\Big/\int_0^{2\pi}e^{\Phi(\varphi)}d\varphi, \tag{9a}$$

$$\mathcal{E}_K/\mu=\frac{\mu^3}{2\pi^2}x^{-4}\sum_{\nu=0}^{\infty}G_\nu(3.5x)\int_0^{2\pi}\cos(1+\nu)\varphi\, e^{\Phi(\varphi)}d\varphi\Big/\int_0^{2\pi}e^{\Phi(\varphi)}d\varphi, \tag{9b}$$

where

$$F_\nu(y)=y^2K_2((1+\nu)y)/(1+\nu),$$

$$G_\nu(y)=y^2(1+\nu)^{-1}\{3K_2((1+\nu)y)/(1+\nu)+yK_1((1+\nu)y)\},$$

$$\Phi(\varphi)=(8/3\pi)ax^{-3}\sum_{\nu=0}^{\infty}\cos(1+\nu)\varphi\,F_\nu(3.5x)/(1+\nu).$$

Here the statistical weight of K and $\bar{K}$-meson is assumed to be equal to 2. In the series of $\Psi(\varphi)$ we calculate only the first term because the function $F_\nu(3.5x)$ rapidly decreases with the increase of ν. By this approximation we can integrate eq. (9) over φ and obtain

$$n_K = (2/\pi^2)\,\mu^3 x^{-3} \sum_{\nu=0}^{\infty} F_\nu(3.5x)\, f_\nu(\rho), \tag{10a}$$

$$\varepsilon_K/\mu = (2/\pi^2)\,\mu^3 x^{-4} \sum_{\nu=0}^{\infty} G_\nu(3.5x)\, f_\nu(\rho), \tag{10b}$$

with

$$f_\nu(\rho) = I_{\nu+1}(\rho)/I_0(\rho),$$

$$\rho = (8/3\pi)\, a x^{-3} F_0(3.5x),$$

where I_ν is the imaginary Bessel function of the ν-th order.

The factor $f_\nu(\rho)$ is shown always to be less than unity for any value of ρ by the character of the Bessel function. Therefore one finds that the particle number of K-meson is restricted by the strangeness conservation.

As $f_\nu(\rho) \to 1$ for $\rho \to \infty$, n_K and ε_K are written down by the same expression as obtained by Koba in high energy phenomena. This fact can be seen as follows: In high energy phenomena one considers that the open local system is in the statistical equilibrium but the total system is not (see Sec. 4) and thus one must use the distribution function in which the fluctuations in the strangeness as well as in the energy have been taken into account, namely the canonical distribution function for these quantities. Therefore, the expression obtained by Koba, who used the canonical distribution, agrees with eq. (10) when $\rho \to \infty$ (the large ρ corresponds to the system whose volume and energy are large, and in such a large system the microcanonical distribution does not differ essentially from the canonical distribution except their fluctuations). In this case the probability for the state having the equal numbers of K and $\bar{K}$-mesons is not essentially restricted by the conservation of the strangeness in the system which has zero initial strangeness.

On the other hand, as $f_\nu(\rho) \to 0$ for $\rho \to 0$, $f_\nu(\rho)$ largely affects the values of n_K and ε_K in the small system and this is just the case considered in this paper.

3·3 *Determination of critical temperature*

We can obtain the value of x corresponding to the temperature at which π-mesons begin to fly away without interacting with each other. As pointed out in Sec. (2·1) the stage of flying away as a free particle is decided under the condition that the mean free path of particles λ is of the same order as that of the linear dimension of the total system L or larger than it. We may take $L = (3/4\pi \cdot V)^{1/3}$ and $\lambda = 1/n_\pi \sigma_{\pi\pi}$ if the interaction for $\pi - \pi$ pairs is strong compared with the others, where $\sigma_{\pi\pi}$ is the total cross section for the $\pi - \pi$ interaction. Using eq. (7), we obtain the expression

$$\lambda/L = \frac{1}{n_\pi \sigma_{\pi\pi}} \left(\frac{4\pi}{3} \frac{\varepsilon_\pi + \varepsilon_K}{2M} \right)^{1/3}.$$

Assuming $\sigma_{\pi\pi}$ to be the geometrical cross section $\pi\mu^{-2}$, we estimate this quantity for each

value of x as shown in Table I. Since the estimation depends on the value of $\sigma_{\pi\pi}$, it is correct only in the order of magnitude consideration. From this table we can roughly decide the possible range of x-values as follows:

Table I

x	λ/L
0.5	0.2
1.0	0.8
1.2	1.2
1.4	1.6
2.0	3.8
3.0	12.5

In the case of $x<1$ which corresponds to $\lambda/L<1$, since the particles strongly interact with each other, their number changes through the virtual process and has no any physical meaning even if it is calculated thermodynamically.

While, in the case of $x>2$ the L is too small compared with the λ for the system to maintain itself* and thus a thermodynamical treatment should not be employed. In order to treat this stage we must consider such a $\pi-\pi$ correlation in the form of the final interaction. This final interaction, however, seems to give small contribution to the number of emitted particles (see Sec. 4).

From the above discussion the x-value, at which it has a physical meaning to define the particle number thermodynamically, is circumscribed within $1 \lesssim x \lesssim 2$, but one would not be able to decide theoretically which of these x-values is more suitable.

3·4 *Numerical results*

Along the preceding discussions we shall calculate the various quantities thermodynamically, that are the multiplicity of π-meson N_π, the average kinetic energy of π-meson T_π, the ratio of the number of K-meson to that of π-meson N_K/N_π, the total energy attributed to all K and $\bar{K}$-mesons $E_{K\bar{K}}$ and the measure of the interaction volume a. These numerical results are compared with the experimental values[2] in Table II.

Table II. To evaluate the values in this table we must first of all know the value a from eqs. (7), (4b) and (10b). If the value a is known all quantities corresponding to each x are obtained.

x	N_π	N_K/N_π (%)	T_π (Mev)	$E_{K\bar{K}}$ (Mev)	a
1.0	3.4	14	310	351	3.0
1.2	4.1	8	260	225	7.0
1.4	5.1	4	210	133	14.5
Experimental value	5.3±0.6	*several*	206±20	150±120	—

One finds, as shown in Table II, that all the listed values in the case of $x=1.4$ agree with the observed values.** These agreements are equally well as those in the calculation of the usual statistical model.[2] ***

* The dependence of λ/L on x is $x^{5/3}$ for small x and $x\,e^{2x/3}$ for large x.

** If we take the case of $x=2$ we have the results, $N_\pi \sim 6.8$ and $N_K/N_\pi \sim 3\%$. Generally the multiplicity of π-meson increases slowly for the large x as x increases and approaches to the definite constant value $2M/\mu$, as a matter of course, in the limit of $x\to\infty$.

*** As regards N_K/N_π, according to Table XIV of reference (2) the probability of producing a K-meson pair is 12.3% for the case of $a=15$ and this percentage gives the ratio $N_K/N_\pi = 2\times 12.3/5 \sim 5\%$.

We have taken into account the conservation of strangeness in the form of micro-canonical formalism. If we dare to assume the canonical distribution in the annihilation process the choice of $x=1.4$ gives $N_\pi=4.3$ and $N_K/N_\pi=13\%$ which are inconsistent with the experimental values.

§ 4. Concluding remarks

In preceding sections we have consistently studied the antinucleon-nucleon annihilation problem by assuming that the total system reaches rapidly to thermal equilibrium through a variety of virtual process and so thermodynamical treatment may be ensured. Here we shall closely examine this assumption.

In the case of $\lambda\sim L$, as the interactions between particles fairly remain it seems that one would not be able to consider the system to be in the state of ideal gas and should take account of the final $\pi-\pi$ interactions. The effects of the final interaction, however, may not be so important by two reasons as shown below. The one is the point that the kinetic energy distribution of π-meson resembles to the Planck distribution with $x=1.4$ (Fig. 1) and there is not any correlation between the emitted particles. Another reason is the fact that the final interaction gives little change to the number of emitted particles. According to the calculation given by Gotō,[1] if the π-mesons interact strongly with each other in the final state we obtain $N_\pi=5.2$ for $a=10$. Since the other estimate without the final interaction[2] shows $N_\pi=4.5$ in the similar case, $a=10$, the contribution of the final interaction to the whole process is of the magnitude of $0.7/5.2\sim 0.13$ and falls almost in the range of experimental error. Therefore we may say that the contribution of the final interaction does not play so important role in the meson production by the nucleon-antinucleon annihilation and the main character of the whole process is decided in the stage of $\lambda\ll L$.

In the case of $\lambda\ll L$, we cannot study the whole process by the terms of each elementary process, for example $\pi-\pi$, $\pi-K$ or a more complicated type of interaction, because these elementary interactions are not yet well-known and even if they are known the calculation of them is complicated. At present we have only the statistical treatment as a useful method.

As shown in Sec. (3·3) we have seen that the interaction volume is enlarged because of the $\pi-\pi$ interactions. It is in the region of $1\lesssim x\lesssim 2$ (or $\lambda/L\sim 1$) that the thermodynamical treatment is justified and the particle number decided by this treatment has a physical meaning. Since the condition $\lambda\sim L$ corresponds to the enlarged volume (see Table I) one can understand qualitatively the reason why one should take the large interaction volume to agree the multiplicity expected by the Fermi statistical model with the experimental one. Thus we conclude that the character of the meson production in the antinucleon-nucleon annihilation is not so curious as we can not understand by means of the usual concept.

In this paper we have used the temperature T as a parameter instead of the interaction volume which has been employed in the usual statistical model. The reason for using the temperature is found in the simpleness of calculating average quantities. The

most fitted value of this new parameter for analysis of the experiments is not inconsistent with the one at which π-mesons may be emitted as free particles (shown in Sec. (3·3)). The thermodynamical calculation should not differ from the usual statistical one as far as we consider the average quantities.

Finally it is interesting to remark the difference between the annihilation process and the high energy phenomena. The definition of the critical temperature in this paper is essentially not different from the one taken by Landau and Belen'kij.[10] Regarding the expansion process, however, our case differs from their case. In ours constant temperature over the closed system as a whole has a definite meaning because the system has no special translational motion but only thermal motion. While in high energy phenomena the system is characterized by the direction of incident particles and in the first step expands one-dimensionally along this direction. Then the energy of the translational motion is so large that the system is no longer in equilibrium as a whole and the individual local part of the system has a different temperature.

According to above reasons, in high energy phenomena, mesons are produced as soon as the local temperature becomes about μ in each individual system. Therefore the volume of the total system, which has been used in determination of the particle number in our calculation (eq. (5)), is physically meaningless and the particle number should be defined by another conservative quantity, such as entropy. Furthermore, when we obtain the energy distribution of emitted particles we require a theory taking into account the fact that each local part of the system moves having its appropriate Lorentz factor and retaining, the local equilibrium in itself, for example, the relativistic hydrodynamics used by Landau.[9] For annihilation events no hydrodynamical treatment is necessary because the system expands uniformly and is characterized only by a temperature. This is supported by the resemblance of the experimental energy spectra to the Planck distribution as has already been shown in Fig. 1.

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Remarks on Spin-Wave Theory for the Ferromagnetic Exchange Problem

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A theory which is essentially equivalent to Bloch-Bethe-Van Kranendonk's formalism is developed of the properties of an ideal ferromagnet with isotropic exchange coupling between nearest neighbour spins 1/2. Concerning the thermodynamic properties of the system at low temperatures far below the Curie point a complete treatment is obtained. In this temperature region the excitation of the system is naturally described in terms of spin waves which obey Fermi-Dirac statistics. Interactions between spin waves are discussed in the scheme of the second quantization, and their effects to the spontaneous magnetization are also discussed.

§ 1. Introduction

In this paper, the system of localized spins coupled ferromagnetically by the exchange interaction between nearest neighbours is assumed to be a model for the ferromagnet. The discussions are mainly concerned with the case of spin 1/2.

As early as in 1930, in his treatment of the properties of the ferromagnet at low temperatures far below the Curie point, Bloch[1] proposed the concept of spin wave which takes place as an excitation in the system in which the almost every spin is parallel to each other. After Bloch's proposal the same problem was re-formulated within the scope of *free* spin waves by a few authors[2),3)], and his famous $T^{3/2}$-law for the spontaneous magnetization was confirmed and the agreement with experiment was tested successfully as far as the low temperatures are concerned. However, with the raise in temperature the disagreement between theory and experimental data is increased. This may be explained by the inclusion of the interaction between spin waves, because as the temperature is raised the number of spin waves excited thermally becomes so many that the scattering between spin waves may offer a tangible effect.

Bethe[4] discussed the spin wave interaction by the use of his rigorous solution of Bloch's secular equation*, although he used a rather hypothetical one-dimensional array of spins. Recently this problem has been treated and it gives the term proportional to $T^{7/4}$ [5] and T^4 [6),7)] to the spontaneous magnetization and so the matter is not conclusive at the present stage. Now we shall present here a theory of spin wave with interaction in the second quantized formalism following Fock[8]. But the present formalism is equivalent to Van Kranendonk's one and in contrast to his $T^{7/4}$ a term like T^3 is obtained

* Eq. (5) of the reference 1).

as a correction to the spontaneous magnetization. The present calculation is much more straightforward than his treatment and T^3-correction is much more trusted. Indeed his calculation contains many ambiguities especially as to interaction potentials (his V_a and V_r). Its disagreement with Dyson's T^4 correction will be discussed in the future in a separate paper. In the next section the symmetry property and the statistics of spin wave are discussed and the calculation is given in the following two sections. § 5 is devoted to some remarks on the conventional theory of spin wave.

§ 2. Statistics of spin wave

For the sake of definiteness, we briefly reproduce some elements of Bethe's rigorous treatment[4),9),10)]. Our consideration is limited to the system of one dimensional equidistant array of N localized spins $1/2$ with exchange coupling of the Heisenberg type between nearest neighbour spins. Periodic boundary condition is adopted. Then in the case of two reversed spins which are not located at the nearest neighbour sites the secular equation takes the form

$$E_2 a_{n_1 n_2} = J(4a_{n_1 n_2} - a_{n_1-1\, n_2} - a_{n_1+1\, n_2} - a_{n_1\, n_2-1} - a_{n_1\, n_2+1}), \tag{1}$$

where J is the exchange integral between nearest neighbour sites, $a_{n_1 n_2}$ stands for the amplitude of the state in which two reversed spins occupy the n_1- and n_2-th sites and E_2 is the energy of two reversed spin states. On the other hand the state of two reversed spins at nearest neighbour sites (say the n- and $n+1$-th ones) gives

$$E_2 a_{nn+1} = J(2a_{nn+1} - a_{n-1\, n+1} - a_{nn+2}). \tag{2}$$

Bethe introduced the foreign variables a_{nn} with the subsidiary conditions

$$a_{nn} + a_{n+1\, n+1} - 2a_{nn+1} = 0 \tag{3}$$

so as to unite eqs. (1) and (2). Now the solution $a_{n_1 n_2}$ is assumed to take the form

$$a_{n_1 n_2} = c_1 e^{i(k_1 n_1 + k_2 n_2)} + c_2 e^{i(k_2 n_1 + k_1 n_2)}. \tag{4}$$

Further one may assign without loss of generality $e^{i\varphi}$ and $e^{-i\varphi}$ to c_1 and c_2, respectively, then eqs. (3) and (4) give

$$2\cot\varphi(k_1 k_2) = \cot(k_1 a/2) - \cot(k_2 a/2) \tag{5}$$

where a is the lattice spacing. Eq. (5) tells us that

$$\begin{aligned} &\cot\varphi(k_1 k_2) + \cot\varphi(k_2 k_1) = 0, \quad \cot\varphi(kk) = 0, \\ &\varphi(k_1 k_2) + \varphi(k_2 k_1) = \pi, \quad \varphi(kk) = \pi/2, \quad 0 \leqq \varphi < \pi. \end{aligned} \tag{6}$$

And one obtains finally with abbreviation $(12) = \varphi(k_1 k_2)$

$$\begin{aligned} a_{n_1 n_2}(k_1 k_2) &= e^{i(k_1 n_1 + k_2 n_2 + (12))} + e^{i(k_2 n_1 + k_1 n_2 - (12))} \\ &= e^{i(k_1 n_1 + k_2 n_2 + (12))} - e^{i(k_2 n_1 + k_1 n_2 + (21))}, \\ a_{n_1 n_2}(k_1 k_2) &= -a_{n_1 n_2}(k_2 k_1), \quad a_{n_1 n_2}(kk) = 0. \end{aligned} \tag{7}$$

The antisymmetric property of $a_{n_1n_2}(k_1k_2)$ as to the interchange of k_1 and k_2 ensures the same property of $\Psi(k_1k_2)$ which is the desired solution for the above secular problem and represents the state of two spin waves running with the wave numbers k_1 and k_2:

$$\Psi(k_1k_2)=\sum_{n_1<n_2} a_{n_1n_2}(k_1k_2)\,\Phi_{n_1n_2}\,,$$
$$\Phi_{n_1n_2}=\det|\psi_1(1)\alpha(1)\cdots\psi_{n_1}(n_1)\beta(n_1)\cdots\psi_{n_2}(n_2)\beta(n_2)\cdots\psi_N(N)\alpha(N)| \tag{8}$$

where the notations have usual meanings.

Further the cases of $n(>2)$ reversed spins have been discussed by D. Frank[10] and the above antisymmetricity is also obtained. So one can regard the state of n reversed spins as the system consisting of n spin waves (*magnons*) which obey Fermi-Dirac statistics.[9,10,11]

§ 3. Hamiltonian of system of spins $S=1/2$

The above mentioned ideal ferromagnets are assumed to have the following simple forms of crystal structure: (I) linear chain, (II) square lattice and (III) simple cubic lattice. In each case the lattice constant is a and the exchange interaction is assumed to be isotropic and exists between nearest neighbour pairs. The periodic boundary condition is adopted for the 'Grundgebiet' of the lattices with N spins——(I) a chain of length Ga, (II) a square of area $(Ga)^2$ and (III) a cube of volume $(Ga)^3$, and $N=G$, G^2 and G^3, respectively. Then the Hamiltonian of the spin system in the uniform external magnetic field H is given in terms of spin variables

$$\mathcal{H}=-J\sum_{n\,\delta}\boldsymbol{S}_n\cdot\boldsymbol{S}_{n+\delta}+\tfrac{1}{4}NzJ-2\mu_BH\sum_n S_n^z=E-2\mu_BH\sum_n S_n^z \tag{9}$$

where the sum $\sum_{n,\delta}$ runs twice over all pairs of nearest neighbours, μ_B is the Bohr magneton, z the number of nearest neighbours and δ the lattice vector between them.

The z-component of total spin operator $S^z=\sum_n S_n^z$ commutes with $\mathcal{H}$ and has the allowed values $m=N/2,\ N/2-1,\ \cdots -N/2$. We introduce the spin deviation $n=N/2-m$ which is also the quantum number of the system described by eq. (9) and takes the values 0, 1,$\cdots$. Therefore, one can treat the states with $n=0,\ 1,\cdots$ separately. From (9)

$$\mathcal{H}_n=E_n+2\mu_BHn-\mu_BHN. \tag{10}$$

Case $n=0$. There is only one state (ground state):

$$E_0=0. \tag{11}$$

Case $n=1$. In this subspace there are N states. Bloch's equation[1] takes the form

$$E_1a_n=J\textstyle\sum_\delta(a_n-a_{n+\delta})\equiv Ta_n \tag{12}$$

the solutions of which are the Bloch free spin waves

$$a_n=\frac{1}{\sqrt{N}}\,e^{ik\cdot n} \tag{13}$$

with eigenvalues of the kinetic energy operator T

$$E_1(\boldsymbol{k}) = J\sum_{\delta}(1-e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}) \equiv Jz(1-\gamma_k). \tag{14}$$

These may be pictured as free 'particles' with kinetic energy $E_1(\boldsymbol{k})$ and $\boldsymbol{k}$ itself is determined by the boundary condition:

$$\begin{aligned} &k_i = 2\pi/Ga\cdot p_i\,;\ p_i = -G/2,\ -G/2+1,\cdots,\ G/2-1, \\ &\text{(I)}\ \ i=1,\quad \text{(II)}\ \ i=1,\ 2\quad \text{and}\quad \text{(III)}\ \ i=1,\ 2,\ 3, \end{aligned} \tag{15}$$

and there are just N modes in each case.

Case $n=2$. The sets of secular equations are given by (1) and (2) in the case of (I) and are easily written down in the cases of (II) and (III). Following Van Kranendonk[5] we take the point of view that the difference between eqs. (1) and (2) (similar equations in the cases of (II) and (III)) represents the existence of the attractive potential U between two reversed spins and U itself takes the form

$$\begin{aligned} U(\boldsymbol{n},\ \boldsymbol{m}) &= -2J,\ \text{for}\ \ \boldsymbol{n}-\boldsymbol{m}=\hat{0}, \\ &= 0,\ \text{otherwise}. \end{aligned} \tag{16}$$

On the other hand, Van Kranendonk's repulsive potentials corresponding to the so-called kinematical interactions do not enter into the present formulation, because these interactions have been effectively taken into account by the statistics of spin waves.

Instead of Bloch-Van Kranendonk's formulation discussed till now in the *configuration* space, we will develop the second quantized scheme for the description of the state of the spin-deviation. To this purpose it is most convenient to use the representation given by Fock[8]. Then the Hamiltonian operator describing the system of spin wave is

$$\mathcal{H} = \sum_{k}(\boldsymbol{k}|T|\boldsymbol{k})\,a_k^* a_k + \tfrac{1}{2}\sum_{k_1k_2k_3k_4}(\boldsymbol{k}_1\boldsymbol{k}_2|U|\boldsymbol{k}_3\boldsymbol{k}_4)\,a_{k_1}^* a_{k_2}^* a_{k_4} a_{k_3}\,, \tag{17}$$

where a_k^* and a_k are the creation and annihilation operators for the spin wave running freely through the lattice with wave number vector $\boldsymbol{k}$, and satisfy the following relations

$$[a_k,\ a_{k'}^*]_+ = \delta_{kk'},\ [a_k,\ a_{k'}]_+ = [a_k^*,\ a_{k'}^*]_+ = 0, \tag{18}$$

where $[a,\ b]_+ = ab+ba$. In the case of (I) the discussions of § 2 ensure for us to use these commutation relations. In the cases of (II) and (III) there have not been any confirmations as to the use of these representations, but these are assumed to be correct in these cases too. Concerning to this point a further comment is given in Appendix.

Eqs. (12), (13) and (16) give

$$\begin{aligned} &(\boldsymbol{k}|T|\boldsymbol{k}) = Jz(1-\gamma_k) \equiv \varepsilon_k^0 \\ &(\boldsymbol{k}_1\ \boldsymbol{k}_2|U|\boldsymbol{k}_3\ \boldsymbol{k}_4) = -(2Jz/N)\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4-\boldsymbol{K})\,\gamma_{k_1-k_3}\,, \end{aligned} \tag{19}$$

where δ is a Kronecker's delta symbol, that is, $\delta(\boldsymbol{k}) = 1$ for $\boldsymbol{k}=0$ and otherwise $\delta(\boldsymbol{k}) = 0$, and $\boldsymbol{K}$ is a primitive vector in the reciprocal space. The second term of eq. (17) represents the scattering of two spin waves by the attractive potential U and the δ-factor ensures the momentum conservation with mod. $\boldsymbol{K}$. However we are only interested in

the thermodynamic properties of the system at low temperatures and so the thermally excited spin waves have mainly very long wave lengths, therefore the processes $\boldsymbol{K}\neq 0$ (Umklappprozesse) give too small contribution to the thermodynamic properties to be reserved and the process $\boldsymbol{K}=0$ is only retained.

As to the field dependent energy of the system, eq. (10) gives

$$-\mu_B HN+2\mu_B H\sum_k a_k^* a_k. \tag{20}$$

Finally the Hamiltonian of the system takes the form

$$\begin{aligned}
&\mathcal{H} = \mathcal{H}_0+\mathcal{H}_2+\mathcal{H}_4,\\
&\mathcal{H}_0 = -\mu_B HN\\
&\mathcal{H}_2 = \sum_k \varepsilon_k a_k^* a_k, \quad \varepsilon_k=\varepsilon_k^0+2\mu_B H\\
&\mathcal{H}_4 = \frac{Jz}{N}\sum_{k_1k_2k_3k_4}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\gamma_{k_1-k_3}a_{k_1}^* a_{k_2}^* a_{k_3}a_{k_4}.
\end{aligned} \tag{17a}$$

This interaction Hamiltonian $\mathcal{H}_4$ is easily verified to be a simple Fourier transform of Frank's interaction term, but does not appear in the symmetrical form. Physically speaking, however, we are dealing with the system of identical particles (magnons), and in the collision process

$$\boldsymbol{k}_3+\boldsymbol{k}_4 \rightarrow \boldsymbol{k}_1+\boldsymbol{k}_2, \tag{21}$$

the interchanges $(\boldsymbol{k}_3\longleftrightarrow\boldsymbol{k}_4)$, $(\boldsymbol{k}_1\longleftrightarrow\boldsymbol{k}_2)$ and $(\boldsymbol{k}_3\longleftrightarrow\boldsymbol{k}_4\,;\,\boldsymbol{k}_1\longleftrightarrow\boldsymbol{k}_2)$ do give nothing but the process (21), so $\mathcal{H}_4$ must consist of the terms invariant with respect to the above interchanges. Thus with the use of eq. (18) it is reasonable to write down the interaction Hamiltonian in the following symmetrical form

$$\mathcal{H}_4=\frac{Jz}{2N}\sum_{k_1k_2k_3k_4}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,(\gamma_{k_1-k_3}-\gamma_{k_1-k_4})\,a_{k_1}^* a_{k_2}^* a_{k_3}a_{k_4}. \tag{22}$$

In the case of discussing the each elementary process which corresponds to each summand of the interaction Hamiltonian one has to start with eq. (22). However, in the present course of calculation we are interested in thermodynamic behaviours of the system in which the interaction Hamiltonian appears as a whole (see the next section) and we may safely start with eq. (17a) within the limitation of "two body interaction"*.

§ 4. Spontaneous magnetization

In this section the thermodynamic properties of the system described by the Hamiltonian operator (17a) are given by the use of the compact formalism introduced by Ichimura[12]. The thermodynamic properties can be derived from the free energy

$$F=-\beta^{-1}\ln Z, \qquad \beta^{-1}=kT \tag{23}$$

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* By Van Kranendonk it has been proved that the many body interactions have little effect on the thermodynamic properties.

where $Z=\mathrm{Tr}\exp(-\beta\mathcal{H})$ is the canonical partition function of our system. Then the spontaneous magnetization M is given by

$$M=-1/V\cdot\partial F/\partial H \tag{24}$$

which is reduced to

$$M/M_0=1-2/N\cdot\mathrm{Tr}\left(\sum_k a_k^* a_k\exp(-\beta\mathcal{H})\right)/\mathrm{Tr}\left(\exp(-\beta\mathcal{H})\right) \tag{25}$$

where $M_0=\mu_B NV^{-1}$ is the saturation magnetization. Now we define the grand partition function Ξ of the system described by the Hamiltonian $\tilde{\mathcal{H}}=\sum_k\varepsilon_k^0 a_k^* a_k+\mathcal{H}_4$:

$$\Xi=\mathrm{Tr}\left(\exp\left[-\alpha\sum_k a_k^* a_k-\beta\tilde{\mathcal{H}}\right]\right). \tag{26}$$

Comparing Z with Ξ we see that apart from the factor $\exp(-\beta\mathcal{H}_0)$ the canonical partition function Z of the system described by eq. (17a) is equal to the grand partition function Ξ of the system with Hamiltonian $\tilde{\mathcal{H}}$, when the identification

$$\alpha=2\beta\mu_B H \tag{27}$$

has been made. Then the spontaneous magnetization is written in terms of Ξ;

$$\begin{aligned} &M/M_0=1-2/N\cdot\bar{n}\,,\\ &\bar{n}=-\partial\ln\Xi/\partial\alpha. \end{aligned} \tag{28}$$

Now let us calculate Ξ by expanding it in the power series of $\beta\mathcal{H}_4$[12]. For a moment, we treat Ξ of the form

$$\Xi=\mathrm{Tr}\left(\exp\left[-\sum_k(\alpha+\beta\varepsilon_k^0)a_k^* a_k-\kappa\beta\mathcal{H}_4\right]\right) \tag{29}$$

where κ is assumed to be the perturbing parameter, and Ξ is expanded in the power series of κ and we restrict ourselves up to the linear term of κ. In the final result we shall put $\kappa=1$. Ξ is expanded in terms of κ as

$$\begin{aligned} \Xi=&\mathrm{Tr}\left(\exp\left(-\sum_k(\alpha+\beta\varepsilon_k^0)a_k^* a_k\right)\right)\\ &-\kappa\beta\,\mathrm{Tr}\left(\mathcal{H}_4\exp\left[-\sum_k(\alpha+\beta\varepsilon_k^0)a_k^* a_k\right]\right)+\cdots. \end{aligned} \tag{30}$$

The first term of (30) is reduced to

$$\begin{aligned} \Xi_0=&\sum_{n_{k_1}=0}\cdots\sum_{n_{k_N}=0}\prod_k\exp\left(-(\alpha+\beta\varepsilon_k^0)n_k\right)\\ =&\prod_k\left\{1+\exp(-\alpha-\beta\varepsilon^0{}_k)\right\}, \end{aligned} \tag{31}$$

and in the second the operator $\exp\left(-\sum_k(\alpha+\beta\varepsilon_k^0)a_k^* a_k\right)$ is diagonal and the following two cases

$$\begin{aligned} &\boldsymbol{k}_1=\boldsymbol{k}_3\equiv\boldsymbol{k}\quad\text{and}\quad\boldsymbol{k}_2=\boldsymbol{k}_4\equiv\boldsymbol{k}',\\ &\boldsymbol{k}_1=\boldsymbol{k}_4\equiv\boldsymbol{k}\quad\text{and}\quad\boldsymbol{k}_2=\boldsymbol{k}_3\equiv\boldsymbol{k}' \end{aligned}$$

contribute to Ξ and one can arrive at

$$\Xi=\Xi_0\left(1+(\kappa\beta\,Jz/N)\sum_{kk'}(1-\gamma_{k-k'})f_k f_{k'}+\cdots\right) \tag{32}$$

where

$$f_k=(\exp(\alpha+\beta\varepsilon_k^0)+1)^{-1}, \tag{33}$$

and finally

$$\ln \Xi=\ln \Xi_0+(\kappa\beta Jz/N)\sum_{kk'}(1-\gamma_{k-k'})f_k f_{k'}+\cdots. \tag{34}$$

Eqs. (34) and (28) tell us that at vanishing H

$$\begin{aligned} M/M_0=1&-(2/N)\sum_k f_k^0 \\ &-2Jz\beta/N^2\cdot\sum_{kk'}(1-\gamma_{k-k'}) \\ &\times(f_k^0(1-f_k^0)f_{k'}^0+f_k^0(1-f_{k'}^0)f_{k'}^0) \end{aligned} \tag{35}$$

where $f_k^0=(\exp(\beta\varepsilon_k^0)+1)^{-1}$. The first two terms of the right-hand side of (35) correspond to free spin waves and the last term is the desired correction which stems from the scattering between spin waves.

In the low temperature region far below the Curie point the k-sums in eq. (35) can be replaced by the integrations as usual and in the scope of the so-called effective mass approximation for ε_k^0 the results are

$$\begin{aligned} &\text{(I)} \quad M/M_0=1-(\sqrt{2}/\pi)z(-1/2)\theta^{1/2}-(2/\pi^2)\theta[y(1/2)z(-1/2) \\ &\qquad\qquad +z(1/2)y(-1/2)], \\ &\text{(II)} \quad M/M_0=1-(1/\pi)z(0)\theta-(1/\pi^2)\theta^2[y(1)z(0)+z(1)y(0)], \\ &\text{(III)} \quad M/M_0=1-(\sqrt{2}/\pi^2)z(1/2)\theta^{3/2}-(2/\pi^4)\theta^3[y(3/2)z(1/2) \\ &\qquad\qquad +z(3/2)y(1/2)], \end{aligned} \tag{36}$$

where $\theta=kT/2J$ and

$$z(n)=\int_0^\infty x^n dx/(e^x+1), \quad y(n)=\int_0^\infty x^n dx/(e^x+1)(1+e^{-x}). \tag{37}$$

It is easily shown that

$$\begin{aligned} &z(n)=(1-2^{-n})\Gamma(n+1)\zeta(n+1), \quad \text{for} \quad n>0, \\ &z(-1/2)=1.072, \qquad z(0)=\ln 2, \\ &y(n)=n\ z(n-1), \quad \text{for} \quad n>0 \quad \text{and} \quad n\neq 1, \\ &y(-1/2)=0.6749, \quad y(0)=1/2, \quad y(1)=\ln 2, \end{aligned} \tag{38}$$

where ζ is Riemann's ζ-function and Γ the Γ-function. In the case of (III) the coefficient of $\theta^{3/2}$ is smaller than the usual one[1] by a factor $1-1/\sqrt{2}$. In the cases of (II) and (III) the deviations of ε_k^0 from the effective mass approximation are also introduced in the calculation of $\sum_k f_k^0$. For example in the case of (III) if ε_k^0 is averaged over angles in k-space,

$$\varepsilon_k^0=J(k^2a^2-(1/20)k^4a^4+\cdots). \tag{39}$$

And the canonical average gives

$$\beta J/20\cdot\langle k^4\rangle_{AV}a^4=\theta/10\cdot z(5/2)/z(1/2)=0.4547\theta \tag{40}$$

so in the low temperature region one can expand f_k^0 in terms of $\beta\varepsilon_k^0-\beta Jk^2a^2$ and the elementary calculations give

$$\text{(III)}\quad M/M_0=1-(\sqrt{2}/\pi^2)\theta^{3/2}[z(1/2)+\theta/10\cdot y(5/2)]$$
$$-(2/\pi^4)\theta^3[y(3/2)z(1/2)+z(3/2)y(1/2)] \tag{36a}$$

and similarly

$$\text{(II)}\quad M/M_0=1-(1/\pi)\theta[z(0)+\theta/8\cdot y(2)]$$
$$-(1/\pi^2)\theta^2[y(1)z(0)+z(1)y(0)]. \tag{36b}$$

Finally we obtain that

$$\text{(I)}\quad M/M_0=1-0.4826\,\theta^{1/2}-0.2092\,\theta,$$
$$\text{(II)}\quad M/M_0=1-0.2206\,\theta-0.1558\,\theta^2, \tag{36c}$$
$$\text{(III)}\quad M/M_0=1-0.09715\,\theta^{3/2}-0.04128\,\theta^{5/2}-0.02684\,\theta^3.$$

Eqs. (36)$\sim$(36c) show that the one and two dimensional lattices as well as the three dimensional one reveal the ferromagnetic nature at low temperatures far below the Curie point. Indeed the experiment[13] of Ni thin film has reported that the ferromagnetic ordering sets in when the sample cooled and the 'Curie point' decreases with decreasing thickness of the sample, and the agreements with the theory of Klein-Smith[14] are rather satisfactory.

In the sample of thin film the spin waves running perpendicularly to the surface layer have so large energies that in low temperature region the thermally excited spin waves are mainly progressing parallel to the layers and the temperature dependence of the spontaneous magnetization is expected to be the same as the case (II). Using the free spin wave approximation we have also calculated the spontaneous magnetization M of the film of thickness pa* and

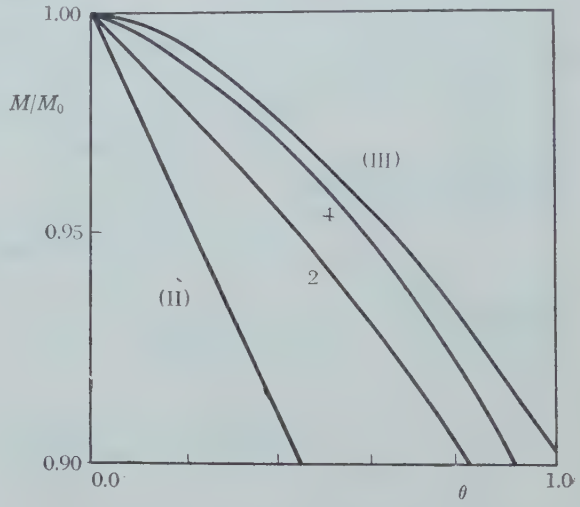

Fig. 1. Magnetization *versus* temperature curves of films of thickness $pa=2a$ and $4a$, and of the cases of (II) and (III) which correspond to the effective mass approximation without interaction.

* The calculation is performed in the same manner as Klein-Smith's one[14]. Though the mode $\boldsymbol{k}=0$ has to be excluded in the calculation of the thermodynamic properties, in the present calculation the inclusion of $\boldsymbol{k}=0$ does not change the result in practice and the mode $\boldsymbol{k}=0$ is included.

the result is

$$M/M_0=1-(\theta/\pi)\phi_p(\theta),$$
$$\phi_p(\theta)=(1/p)\sum_{n=1}^{p}\ln(1+\exp[(\cos 2\pi n/p-1)/\theta]). \tag{41}$$

The summation in $\phi_p(\theta)$ is performed directly in the cases of $p=2$, 4, 8, 36, and the curves M/M_0 versus θ are shown in Fig. 1. In the case of $p=1$, $\phi=\ln 2$ and eq. (41) is equal to (II) of (36) except the interaction term. In the cases of small p Fig. 1 shows that the curves are represented approximately by the equation

$$M/M_0=1-\alpha\theta \tag{41a}$$

in the low temperature region, where α is a constant.

§ 5. Remarks on the conventional theory of spin wave

In this section S is not always equal to $1/2$. In usual treatment of the ferromagnetic exchange problem the operator system introduced by Holstein-Primakoff[3] is commonly adopted. One starts with eq. (9) and introduces spin deviation operators by the following equations

$$S_n^+=S_n^x+iS_n^y=\sqrt{2S}f_n a_n,\quad S_n^-=S_n^x-iS_n^y=\sqrt{2S}\,a_n^* f_n,$$
$$S_n^z=S-a_n^* a_n,\quad f_n=(1-a_n^* a_n/2S)^{1/2}. \tag{42}$$

a_n and a_n^* satisfy the relations

$$[a_n,\ a_{n'}^*]=\delta_{nn'},\quad [a_n,\ a_{n'}]=[a_n^*,\ a_{n'}^*]=0, \tag{43}$$

and the eigenvalues of the operator $n_n=a_n^* a_n$ are equal to 0, 1, 2,$\cdots$. In the subspace $n_n=0$, 1,$\cdots$, $2S$ eq. (42) is exact. However, a_n and a_n^* are expressed as infinite matrices and eq. (42) introduces the fictitious states foreign to the system described by the Hamiltonian (9). Setting aside for a moment the discussion of this difficulty, we proceed further. The Fourier transforms are now defined by

$$a_n=(1/\sqrt{N})\sum_k e^{-ikn}a_k,\quad a_n^*=(1/\sqrt{N})\sum_k e^{ikn}a_k^* \tag{44}$$

where $\boldsymbol{k}$ runs over all the vectors defined by eq. (15). Then

$$[a_k,\ a_{k'}^*]=\delta_{kk'},\quad [a_k,\ a_{k'}]=[a_k^*,\ a_{k'}^*]=0 \tag{45}$$

and

$$\sum_n a_n^* a_n=\sum_k a_k^* a_k, \tag{46}$$

and the Hamiltonian becomes

$$\mathcal{H}=-2\mu_B HSN+\sum_k(2J_z S(1-\gamma_k)+2\mu_B H)a_k^* a_k, \tag{47}$$

where the interaction between spin waves has been omitted and the notations used here have the usual meanings.

The canonical partition function Z of the system described by (47) is

$$Z=\mathrm{Tr}\exp(-\beta\mathcal{H}), \tag{48}$$

from which the spontaneous magnetization M becomes at $H=0$

$$\begin{aligned} &M/M_0=1-(1/SN)\textstyle\sum_k \bar{n}_k \\ &\bar{n}_k=\mathrm{Tr}\,(n_k e^{-\beta\sum_k \varepsilon_k{}^0 n_k})/\mathrm{Tr}\,(e^{-\beta\sum_k \varepsilon_k{}^0 n_k}) \end{aligned} \tag{49}$$

where $\varepsilon_k{}^0=2JzS(1-\gamma_k)$. From eq. (45) $n_k=0,\ 1,\cdots$, but $n_n=0,\ 1,\cdots,\ 2S$ so eq. (46) gives $0\leq\sum_k n_k\leq 2SN$. Then the operation Tr becomes

$$\textstyle\sum_{n=0}^{2SN}\sum_{n_{k_1}\cdots n_{k_N}}(n_k\geq 0,\ \sum_k n_k=n),$$

which is too complicated to be performed exactly. Now the rather drastic approximation

$$\mathrm{Tr}\longrightarrow\textstyle\sum_{n_{k_1}=0}^{P}\cdots\sum_{n_{k_N}=0}^{P},$$

is introduced where P is an integer which satisfies $P^N\simeq\Omega_N$, Ω_N standing for the volume of the N-dimensional domain

$$0\leq\textstyle\sum_k n_k\leq 2SN \quad\text{and}\quad n_k\geq 0.$$

Thus $P\simeq 2Se$. With this approximation

$$\bar{n}_k\simeq(e^x-1)^{-1}-(P+1)\,(e^{(P+1)x}-1)^{-1}\equiv f_P(x), \tag{50}$$

where $x=\beta\varepsilon_k{}^0$, and we obtain that

$$\begin{aligned} &\text{(I)}\quad M/M_0=1-(\theta^{1/2}/\sqrt{2}\,\pi S)\textstyle\int_0^\infty f_P(x)/\sqrt{x}\cdot dx, \\ &\text{(II)}\quad M/M_0=1-(\theta/2\pi S)\textstyle\int_0^\infty f_P(x)\,dx \\ &\qquad\qquad =1-(\theta/2\pi S)\ln(P+1), \\ &\text{(III)}\quad M/M_0=1-(\theta^{3/2}/\sqrt{2}\,\pi^2 S)\textstyle\int_0^\infty f_P(x)\sqrt{x}\,dx \\ &\qquad\qquad =1-(\theta^{3/2}/\sqrt{2}\,\pi^2 S)\,(1-1/\sqrt{P+1})\,\Gamma(3/2)\zeta(3/2). \end{aligned} \tag{51}$$

It is easily observed from eq. (51) that

(i) in each case the spontaneous magnetization M is finite,

(ii) if we consider the *hypothetical* case $S=1/2$ and $P=1$, we see that eq. (51) is exactly equal to eq. (36) within the scope of the free spin waves.

In the derivation of eq. (51) we have used the following assumptions :

(a) free spin wave approximation,

(b) approximate evaluation of $\bar{n}_k$.

The approximation (a) is justified as far as the low temperature region is concerned, but (b) is rather serious. However in the low temperature region the factor $e^{-\beta\varepsilon_k{}^0 n_k}$ in (49) decreases rapidly with increasing n_k values and (b) may be justified, too.

In the usual treatment[1] one simply puts $P\to\infty$, and the divergences of M

which one encounters in the cases of (I) and (II) are due entirely to this careless procedure. However the present calculation is only approximate one and the more exact evaluation of Tr is much desired.

§ 6. Discussions

First of all we begin with the thermodynamical considerations. We consider the system with magnetization M in the external magnetic field H. If the Nernst heat theorem[15] is applied to the process

$$(T, H) \to (T, H+\Delta H), \tag{52}$$

it gives

$$\lim_{T\to 0} (\partial M/\partial T)_H = 0. \tag{53}$$

Namely the magnetization versus temperature curves must have horizontal tangent at $T=0$[16]. Now we return to our problem of the spin system. The case (III) satisfies eq. (53) but the cases (I) and (II) do not[16]. However this conjecture is a superficial one because at such low temperature that the Nernst theorem holds kT may be smaller than ε_k^0 and the replacement

$$\sum_k \longrightarrow \int dk$$

is not justified. Hence the k-sums must be performed directly and in this temperature region the free spin waves are naturally a good approximation. So we have

$$\partial (M/M_0)/\partial T = -(2k/N)\sum_k \exp(\beta\varepsilon_k^0)/(\exp(\beta\varepsilon_k^0)+1)^2 \cdot (\beta\varepsilon_k^0)^2/\varepsilon_k^0$$

which becomes exponentially null as $T\to 0$.

Secondly we discuss the corrections given by spin wave interactions. As has been pointed out, the present formalism is essentially equivalent to the description given by Van Kranendonk and at first sight it seems rather curious that the correction terms to the spontaneous magnetization are T^3 (present result) and $T^{7/4}$ (Van Kranendonk's one). However his calculation contains some ambiguous points. He defined the correct form of the attractive potentials between spin waves by his careful observations of the secular equation in the case $n=2$ where two spin deviations are present and was able to show that the so-called *Spinkomplexe* does not exist in the case (III). But he neglected completely the existence of the attractive potential in the actual calculation (II of the reference 5)) and included only the effect of the repulsive potentials. And in II of the reference 5) he used also the approximations such as the uniform distribution of E_n. But it must be noted that the present calculation is limited to the very low temperature region because of the expansion (32) and (34).

Finally we remark on the correction T^4 given by Dyson[6]. We have also obtained T^4-correction[7] by the use of Holstein-Primakoff's description. But in the case of small

S it seems that the corrections like T^2 and T^3 are also present in the case of (III) when the deformed counting given in § 5 is adopted, and are in accordance with the present result (III) of eq. (36) and the prediction given in Appendix.

The author wishes to express his sincere gratitude to Profs. K. Tomita, T. Matsubara and T. Yamamoto for helpful discussions. Thanks are also due to Mr. F. Terasaki for stimulating discussions.

Appendix

In the present model of ferromagnets the Hamiltonian of the system of spins 1/2 is

$$\mathcal{H}_{ex}=-2\sum_{l<m}J_{lm}(\boldsymbol{S}_l\cdot\boldsymbol{S}_m), \tag{A1}$$

where S_l^z and $S_l^{\pm}=S_l^x\pm iS_l^y$ satisfy the usual relations

$$\begin{aligned}&[S_l^+,\ S_m^-]=\delta_{lm}\cdot 2S_l^z,\quad [S_l^{\pm},\ S_m^z]=\delta_{lm}\cdot\mp S_l^{\pm},\\ &[S_l^+,\ S_m^+]=[S_l^-,\ S_m^-]=0,\end{aligned} \tag{A2}$$

and l's are lattice site labels. We define the operators α_l^* and α_l[8),17)] by the following equations

$$\begin{aligned}&\alpha_l^*g(n_1,\cdots,n_l,\cdots)=(1-n_l)\,g(n_1,\cdots,n_l+1,\cdots),\\ &\alpha_l g(n_1,\cdots,n_l,\cdots)=n_l\,g(n_1,\cdots,n_l-1,\cdots),\end{aligned} \tag{A3}$$

where g is an arbitrary function of $n_1,\ n_2,\cdots$ and n's take the values 0,1. From this definition it is easily verified that

$$\begin{aligned}&\alpha_l^*\alpha_l=n_l,\quad \alpha_l\alpha_l^*=1-n_l,\\ &[\alpha_l,\ \alpha_m^*]=[\alpha_l,\ \alpha_m]=[\alpha_l^*\alpha_m^*]=0,\quad l\neq m\end{aligned} \tag{A4}$$

and that the operator system $(\alpha_l,\ \alpha_l^*)$ is equivalent to $(S_l^{\pm},\ S_l^z)$ when the substitutions

$$S^+\to\alpha,\quad S^-\to\alpha^*,\quad S^z\to\tfrac{1}{2}-\alpha^*\alpha \tag{A5}$$

have been made. Then in terms of α and α^* the Hamiltonian is written as

$$\mathcal{H}_{ex}=\sum_{l<m}J_{lm}(-\tfrac{1}{2}+2\alpha_l^*\alpha_l-\alpha_l\alpha_m^*-\alpha_l^*\alpha_m-2\alpha_l^*\alpha_l\alpha_m^*\alpha_m). \tag{A6}$$

In order to secure the anticommutation relations for the operators referred to different lattice sites too we further introduce the operator system $(a,\ a^*)$ by the following equations[8)]

$$\begin{aligned}&a_l=\alpha_l\nu_l,\qquad a_l^*=\nu_l\alpha_l^*,\\ &\nu_l=(-1)^{n_1+\cdots+n_l}=\prod_{p=1}^{l}(1-2n_p),\end{aligned} \tag{A7}$$

and obtain finally that

$$\begin{aligned}&\mathcal{H}_{ex}=\sum_{l<m}J_{lm}\left\{-\tfrac{1}{2}+2a_l^*a_l-(-)^{n_{l+1}+\cdots+n_{m-1}}(a_l^*a_m+a_m^*a_l)-2a_l^*a_la_m^*a_m\right\},\\ &[a_l,\ a_m^*]_+=\delta_{lm},\quad [a_l,\ a_m]_+=[a_l^*,\ a_m^*]_+=0.\end{aligned} \tag{A8}$$

Now we can start with either (A4) and (A6) or (A8). In the former case the Hamiltonian is simple but the commutation relation (A4) is rather complicated, and the latter is in somewhat opposite situation to the former. We choose the latter point of view. In the case (I) with nearest neighbour interaction

$$F \equiv \prod_{p=l+1}^{m-1} (1-2n_p) = 1,$$

and we can easily verify eqs. (17)$\sim$(19). In the cases of (II) and (III), however, we meet with the difficulty that F is not always equal to unity even in the approximation of nearest neighbour interaction. If we expand F in n's, we obtain

$$F = \prod_{p=l+1}^{m-1} (1-2n_p) = 1-2\sum_{p=l+1}^{m-1} n_p + 0(n^2). \tag{A9}$$

In the first approximation ($F=1$), (A8) is equivalent to eqs. (17)$\sim$(19). And the second term in the right-hand side of (A9) gives the energy of order of $\mathcal{H}_4$. Though it is difficult to estimate this effect to the spontaneous magnetization, it seems that this correction is of the type $T^n (n \geq 2)$.

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Letters to the Editor

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The Exciton-Lattice Interaction and the Line Shape of the Exciton Absorption Band

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November 28, 1957

The exciton in insulators has recently been investigated from various sides. Theoretically, the most of the attention is directed to the electronic structure[1] of the exciton state; one can compare the results of the energy calculations with the peak position of the exciton absorption band. Some of recent papers concern the exciton-phonon interaction; however, to correlate stationary[2] (the so-called excitaron) and the dynamical[3] (scattering process) problems with the experimental information seems to be difficult, at least in the present stage. On the other hand, essentially no progress has been made in the theory of the line shape of the exciton absorption band since the pioneer work by Peierls.[4] It seems to the present author that thorough investigation of the line shape is the most effective way of obtaining information, such as the energy band structure of the exciton and the strength of exciton-phonon interaction, from experimental data available at present. In this note, we briefly report the result of theoretical investigation of the line shape, together with the comparison with observations, and further application of the information obtained thereby to various speculations on the dynamical processes of the exciton.

First, we derive the exciton-phonon interaction Hamiltonian with the use of Wannier's exciton wave function. The derived Hamiltonian consists of the contributions from the electron in the conduction band and the hole in the valence band, each multiplied with effective charge which depends on the phonon wave number as well as on the internal states (bound or ionized) of the exciton before and after scattering. Due to the electrical neutrality of an exciton, the optical mode of vibration is not important, especially in the scattering process of thermal excitons to which only the long wave phonons can contribute.

Next we investigate the line shape of the exciton absorption band with the method of generating function.[5] The mathematical treatment is considerably simplified due to the optical selection rule characteristic of the exciton case, that is, $\boldsymbol{K} \approx 0$ where $\boldsymbol{K}$ is the wave number of the created exciton. The generating function is expanded into infinite power series in the exciton-phonon intercation Hamiltonian, but in the following two limiting cases the series reduces to simple forms as a whole, leading to typical line shapes.

(i) When the exciton-phonon interaction and the exciton effective mass is not large and the temperature is not too high, the exciton absorption has a Lorentzian shape (motional narrowing), provided that $\boldsymbol{K}=0$ is neither at the top nor at the bottom of the exciton energy band. The breadth is given by $\hbar/\tau$, where τ is the lifetime of $\boldsymbol{K}=0$ exciton due to the phonon scattering.

(ii) When the interaction or the effective mass is very large, or when the temperature is very high, the absorption band is of a Gaussian shape. In this case the localized exciton model is sufficiently good, and the situation is similar to the case of a trapped electron.[5]

The ultra-violet exciton absorption bands observed in alkali-halides seem to belong to case (i). The line shapes are Lorentzian in the main,[6),7)] and the breadths are proportional to absolute temperature[7)] (except at low temperatures where zero point energy of lattice vibration is not negligible) as is expected from the theory. When the crystal becomes very imperfect, which has the same effect as the increase in temperature, the line shape is expected to approach the Gaussian type. Fischer's data[6)] support this conclusion. On the other hand, the X-ray exciton should belong to case (ii) at any temperature, due to its large effective mass. Parratt and Jossem's[8)] experiment on KCl crystal is in favor of this view, though the observed breadth is too large to be explained with the exciton-phonon interaction only.

Taking into account the interaction between *different* exciton energy bands through phonons, which has been neglected in the above discussion, one can explain the repulsion (as temperature rises) of adjacent absorption bands observed in CsI[9)], as well as the asymmetric shape of one of them.[7)] The detailed feature of Fischer's data is also qualitatively understood in the same way.

The information obtained above by comparing theory and experiment is utilized to calculations of the probabilities of various processes which the exciton possibly suffers after it is created, such as direct and indirect annihilations, thermal dissociation and so on. The fact that simultaneous luminescence is not observed when the excitons are produced optically in alkali-halide crystals forces us to conclude that $\boldsymbol{K}=0$ is not the bottom of the exciton energy band in these crystals. For, if the bottom were at $\boldsymbol{K}=0$, the direct annihilation of the exciton (spontaneous emission) would occur in as short a time as 10^{-12} sec. When the energy of the optically produced $\boldsymbol{K}=0$ exciton is higher than the bottom ($\boldsymbol{K}\neq 0$) of the ionization continuum by 0.1 eV or more, the exciton will, after a number of collisions with phonons, certainly dissociate into a free pair of an electron and a hole before it is thermalized. In this case we expect the existence of photoconductivity even if the exciton absorption band may be sufficiently sharp. However, when the contribution of this dissociation process to the line breadth $\hbar/\tau$ is larger than the binding energy of the exciton, the exciton absorption band may not be discriminated from the continuum spectra. This is a possible explanation for the absence of exciton absorption bands in Si and Ge.

The detail of this work will shortly be published in this journal.

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On Puppi's Determination of the Pion-Nucleon Coupling Constant

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December 2, 1957

Recently, Puppi and Stanghellini investigated the coupling constant in the interactions between nucleon and charged pions using the dispersion relation for pion-proton scattering[1], and found that

$$g^2_{\pi^+-p}/4\pi = 0.095 \pm 0.005,$$
$$g^2_{\pi^--p}/4\pi \sim 0.04. \qquad (1)$$

It is very curious that there is a so large difference between the coupling constants as cannot be explained through the violation of charge symmetry which comes from electromagnetic effects[2]. If Puppi's result were correct, not only the charge symmetry but also the locality of the interaction Hamiltonian* would be violated.

We should like to make a comment on Puppi's result since it seems to lead to serious conclusions as mentioned above.

The pion-nucleon coupling constant can also be determined in problems of nuclear forces[4]. At large inter-nucleon distances ($r \gtrsim 1.5/\mu$, where μ denotes the pion mass), the nuclear force due to the one-pion-exchange process is dominant. The one-pion-exchange potential does not depend on the detailed form of the pion-nucleon interaction nor the approximation method for the derivation and is proportional to the square of the pion-nucleon coupling constant. Thus the determination of the coupling constant can be done by analysing low energy experimental data in the two-nucleon system. Accordingly, the data of the two-nucleon system should be used to check Puppi's curious result.

When the coupling constants between nucleon and π^+, π^- and π^0 are denoted as g_+, g_- and g_0 respectively, the one-pion-exchange potential is given by

---

* For example, the $ps(ps)$ pion-nucleon interaction Hamiltonian density which has different coupling constants for positive and negative pions should have the form:

$$H_{\text{int}} = i\bar{\psi}_p\gamma_5\psi_n(G_+\varphi^+{}_\pi + G_-\varphi^-{}_\pi) + i\bar{\psi}_n\gamma_5\psi_p(G_+\varphi^-{}_\pi + G_-\varphi^+{}_\pi)$$

where suffices + and − in φ denote the positive and negative frequency parts of the pion field respectively[3]. However, this type of non-locality affects larger interaction region than the Compton wave length of pion.

$$V^{(1)}(x)=\frac{1}{3}\frac{1}{4\pi}\Big[(g_+^2+g_-^2)\frac{1}{2}\{(\tau_1\cdot\tau_2)-\tau_{z1}\cdot\tau_{z2}\}+g_0^2\tau_{z1}\cdot\tau_{z2}\Big]$$
$$\times\Big[(\sigma_1\cdot\sigma_2)+S_{12}\Big(1+\frac{3}{x}+\frac{3}{x^2}\Big)\Big]\mu\frac{e^{-x}}{x} \quad (2)$$

where $x=\mu r$.

The τ-dependent factor, which is to be investigated, is :

$-(g_+^2+g_-^2+g_0^2)$ for the triplet even state (deuteron),

$(g_+^2+g_-^2-g_0^2)$ for the singlet even state in the $n-p$ system,

g_0^2 for the singlet even state in the $p-p$ system.

The determination of the coupling constant was done in the following way[4] :

(i) by analysing the deuteron parameters

$$(g_+^2+g_-^2+g_0^2)\,(1/3)\,(1/4\pi)\lesssim 0.09. \quad (3)$$

(ii) by analysing the low energy parameters in the singlet even state

$$0.07\lesssim(g_+^2+g_-^2-g_0^2)\,(1/4\pi) \text{ and } g_0^2/4\pi. \quad (4)$$

Eqs. (3) and (4) result in an inequlity

$$0.07\lesssim(1/2)\,(g_+^2+g_-^2)\,(1/4\pi)\lesssim 0.10. \quad (5)$$

The strength of the one-pion-exchange potential (2) contains the charged pion coupling constant only in the combined form $(1/2)\,(g_+^2+g_-^2)/4\pi$. According to Puppi's result, we have

$$0.06\lesssim(1/2)\,(g_+^2+g_-^2)\,(1/4\pi)\lesssim 0.07. \quad (6)$$

Comparing eq. (6) with eq. (5), we cannot conclude that his result is inconsistent with the experimental data on nuclear forces when the experimental error is taken into account. However, we could safely conclude that his result, especially $g_-^2/4\pi \sim 0.04$, is not probable at all.

More accurate experiments on $\pi-p$ scattering and cautious analyses of them are highly desirable.

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On the Applicability of the Pion-theoretical Nuclear Potential

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As is well known, the pion theory can properly account for phenomena of both nuclear forces and pion reactions when the available energy of the system is low. However, it is still an open question at what energies and in what manner the low

energy pion theory must be modified and then replaced by a new theory. In phenomena of nuclear forces, the agreement between the experiment and the pion theory is very satisfactory at energies below at least 20 MeV[1]. At higher energies, however, the situation has not been well clarified, though qualitative agreement has already been obtained up to about 100 MeV.

It may be said that there are two different types of approach in the research on nuclear forces at high energies. One type of approach is the phase shift analysis.[2] Unfortunately, the works on the phase shift analysis hitherto seem to have been performed on purely phenomenological and mathematical programs, without properly taking into account the fact that the characteristic features of the low energy pion theory can still remain unchanged to some extent even at high energies. Moreover, solutions were not always unique. The other type of approach is the formal application of the static pion theoretical potential even to scattering at 300 MeV[3]. From this, however, one can get rather little information on what are indispensable modifications of the low energy pion theory. There have also been some attempts in which the spin-orbit coupling potential plays an important role: That of Gammel and Thaler[4] based on Stapp's[5] phase shift analysis and that of Signell and Marshak[6] based on Gartenhaus' potential[7]. Although the assumption of the $\boldsymbol{L}\cdot\boldsymbol{S}$ potential is interesting, the potentials introduced so far seem to be unreasonably strong. In fact, they are comparable to the one-pion-exchange potential even in the outer region $r\sim\hbar/\mu c$, because of their large value of potential depth. Such strong $\boldsymbol{L}\cdot\boldsymbol{S}$ potentials are hardly expected from the present pion theory. There would be many other types of velocity dependence than the spin-orbit coupling force and they would be effective only at quite small distances and be very complicated.

In order to find out what features of the pion theory play the essential role at high energies arround 100 MeV, we applied the static pion-theoretical potential to the unpolarized and polarized nucleon-nucleon scattering, taking notice of the fact that some parameters should be regarded as phenomenological from the pion-theoretical point of view. One of the characteristic features of the pion-theoretical potential is its strong tensor force which is due to the one-pion-exchange process. Therefore it cannot be justified at all to neglect the mixing parameters ϵ_J between $L=J-1$ wave and $L=J+1$ wave, as is often done in the phase shift analysis. We found that both polarized and unpolarized angular distributions are very sensitive to the mixing parameters. Since the mixing parameters are dependent upon the inner interactions as well as the outer ones, we varied them to some extent. For higher waves ($L\geq 2$), we assumed phase shifts calculated with the static pion-theoretical potential. For P-wave phase shifts, the sign and the order of magnitude were retained. The S-wave phase shifts were varied extensively. The result was that if $\epsilon_1\sim 0$ and $\epsilon_2\sim -\pi/6$, the static pion-theoretical potential without any serious modification could reproduce experimental data up to 100 MeV very well.

Thus, it seems to us that the scattering phenomena around 100 MeV may lie within the applicability of the low energy pion theory. The velocity-dependent interactions, whose existence we have no reason to deny,

are expected to be confined within a small region (probably $r < 0.7\hbar/\mu c$).

More detailed results as well as an analysis of higher energy phenomena will be published soon in this journal.

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High Energy Gamma-Rays from the Crab Nebula

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December 28, 1957

In a recent paper Burbidge[1] has suggested a model of the Crab nebula in which high energy protons are primary particles which produce secondary electrons by nuclear collisions, the electrons being responsible for the synchrotron radiation. The proton flux is so intensive that it could produce high energy photons observable in the upper atmosphere of the earth. In this note we wish to point out that the observation of such photons will facilitate to examine whether Burbidge's model of the Crab will be acceptable or not.

According to Burbidge's model, the intensity of the photons under consideration is estimated to be about 3×10^{-5} cm^{-2} sec^{-1} on the top of the earth atmosphere. Since the average energy of the photons is of the order of 10^{11} ev, this intensity is compared with the intensity of nuclear particles of the same energy, about 10^{-2} cm^{-2} sec^{-1}, or with that giving photons of the same energy, 10^{-4} cm^{-2} sec^{-1}. This comparison indicates that the observation of the photons from the Crab will be possible with nuclear emulsion; the photon may appear as a cascade shower not accompanied with charged pions. It will be of much interest to investigate if the cascade showers of such appearance[2)3)] are identified with the photons from the Crab. In the model proposed by Oort and Walraven[4], the intensity of such photons is too low to be observable.

The author is grateful to Dr. G. R. Burbidge who communicated his idea to him prior to publication.

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Quantum Electrodynamics with the Indefinite Metric

——Non-Lorentz-Invariance of the Gupta Formalism——

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Gupta's quantum electrodynamics was investigated in the several kind of representations, and it was shown that the Gupta metric operator for the free field can be used even in the case of the interacting fields. It was also shown that the Gupta theory is not invariant under the Lorentz transformation.

Introduction

The Gupta theory of the quantum electrodynamics[1)2)] based on the indefinite metric is believed to be the most satisfactory formulation concerning the treatment of the longitudinal and scalar photons.

Recently, Källen,[3)] Gell-Mann and Low[4)] and others[5)] investigated the general properties of the quantum electrodynamics subjected to no approximation. In their treatment, the photon Green-function for the interacting fields is defined in the form

$$\Psi_0^* \eta \boldsymbol{A}_\mu(x) \boldsymbol{A}_\nu(x') \Psi_0, \tag{1}$$

and the norm of the Heisenberg state Ψ_H is given by

$$\Psi_H^* \eta \Psi_H = (-1)^{n_0}, \tag{2}$$

where $\boldsymbol{A}_\mu(x)$ and Ψ_0 are the Heisenberg operator and the true vacuum state, respectively. η is the metric operator which was defined by Gupta only in the case of the free field, and is expressed as $\eta = (-1)^{N_0}$ by using the number operator N_0 of the bare scalar photons. At first sight, it seems to be inadequate to use such a metric operator η in the definitions of the propagator (1) and of the norm (2) for the interacting fields, because it does not hold that

$$\eta \Psi_H = (-1)^{n_0} \Psi_H.$$

The first purpose of this paper is to clarify the reason why we can adopt the above definitions as the correct ones. For this purpose, it is very convenient to use the "in" field in the Yang-Feldman formalism.[6)7)8)] When we use the interaction representation, the treatment of the Lorentz condition becomes cumbersome on account of occurrence of the charge-dependent term. On the other hand, the "in" field gives the Lorentz condition quite a simple form, that is, the same experession as that of the free field, which enables us to investigate the general properties of the gauge transformation.

The second purpose of this paper is to assert that the Gupta theory is not invariant for the Lorentz transformation. Bleuler[2] and Belinfante[9] already discussed this problem and gave a conclusion that the theory is invariant. Their proof of invariance of the theory seems to be incorrect. As will be mentioned in detail in § 5, the "rule of norm" of the Gupta theory depends on a special Lorentz system. We are, therefore, compelled to conclude that the Gupta theory is not a satisfactory formulation of the quantum electrodynamics.

§ 1. The indefinite metric in the Schrödinger and interaction representations

We start with the Schrödinger representation and consider a system of photons and electrons in interaction with the Hamiltonian

$$H=H_0+V=H_0-\int d^3\boldsymbol{x} j_\mu^s(\boldsymbol{x})\, A_\mu^s(\boldsymbol{x}), \tag{1·1}$$

$$j_\mu^s(\boldsymbol{x})=ie\psi_s^*(\boldsymbol{x})\beta\gamma_\mu\psi_s(\boldsymbol{x}), \tag{1·2}$$

$\gamma_0=-i\beta$, $\psi_s^*=$hermite conjugate of ψ_s

and

$$j_\mu A_\mu=j_1A_1+j_2A_2+j_3A_3-j_0A_0.$$

In the present paper, we shall use the natural unit $\hbar=c=1$, and the real time $x_0=t$. The Greek and Latin indices take the values 0, 1, 2, 3 and 1, 2, 3, respectively.

By virtue of the indefinite metric η defined by

$$\eta^2=1 \qquad \text{and} \qquad \eta^*=\eta, \tag{1·3}$$

every hermitian operator B_s in the current theory should be replaced by a self-adjoint operator

$$B_s=B_s^+\equiv\eta B_s^*\eta, \tag{1·4}$$

and the expectation value of any operator C_s should be defined by

$$\Psi_s^*(t)\,\eta C_s\Psi_s(t), \tag{1·5}$$

which is real if C_s is self-adjoint. The four potential A_μ is assumed to have the property

$$A_i^s=A_i^{s*} \qquad \text{and} \qquad A_0^s=-A_0^{s*}, \tag{1·6}$$

in addition to

$$A_i^s=A_i^{s+} \qquad \text{and} \qquad A_0^s=A_0^{s+}. \tag{1·7}$$

From (1·4) and (1·6), we have

$$[A_i^s,\ \eta]_-=[A_0^s,\ \eta]_+=0, \tag{1·8}$$

$$H_0=\eta H_0\eta, \qquad H_0^*=H_0 \tag{1·9}$$

and

$$H=\eta H^*\eta \quad (H\neq H^*), \tag{1·10}$$

which gives the Schrödinger equations

$$i\frac{d\Psi_s}{dt}=H\Psi_s \quad \text{and} \quad -i\frac{d\Psi_s^*}{dt}=\Psi_s^*H^*. \tag{1·11}$$

Let us now transform the whole quantities into those of the interaction representation. It must be remarked that many relations as (1·3) and (1·5), (1·9) are conserved because of the hermitian character of H_0. Quantities in the interaction representation are given by

$$\Psi_I(t)=e^{iH_0t}\Psi_s(t), \quad \Psi_I^*(t)=\Psi_s^*(t)\,e^{-iH_0t} \tag{1·12}$$

and

$$A_\mu(x)=e^{iH_0t}A_\mu^s(\boldsymbol{x})\,e^{-iH_0t}, \tag{1·13}$$

from which we see that both representations agree with each other at $t=0$. The commutation relations of $A_\mu(x)$'s are given by

$$[A_\mu(x),\ A_\nu(x')]_-=i\delta_{\mu\nu}\Delta(x-x'), \tag{1·14}$$

where

$$\delta_{00}=-1 \quad \text{and} \quad \partial\Delta(x)/\partial t|_{t=0}=-\delta^3(\boldsymbol{x}). \tag{1·15}$$

The Fourier expansions of $A_\mu(x)$ are expressed as

$$\begin{aligned} A_i(x)&=\sum_{j=1}^{3}\sum_{k}\frac{1}{\sqrt{2Vk_0}}e_i^j(\boldsymbol{k})\,(a_j(\boldsymbol{k})\,e^{ikx}+a_j^+(\boldsymbol{k})\,e^{-ikx}),\\ A_0(x)&=\sum_{k}\frac{1}{\sqrt{2Vk_0}}(a_0(\boldsymbol{k})\,e^{ikx}+a_0^+(\boldsymbol{k})\,e^{-ikx}), \end{aligned} \tag{1·16}$$

where a_j^- and a_0^- should be the adjoint operators of a_j and a_0 respectively because of (1·7), and

$$\sum_{m=1}^{3}e_i^me_j^m=\delta_{ij}, \quad \sum_{i=1}^{3}e_i^me_i^n=\delta_{mn}, \quad \sum_{i=1}^{3}e_i^jk_i=\delta_{j\beta}k_0 \quad \text{and} \quad k_0=|\boldsymbol{k}|. \tag{1·17}$$

From (1·16) and (1·17), we have

$$[a_\mu(\boldsymbol{k}),\ a_\nu^+(\boldsymbol{k})]_-=\delta_{\mu\nu}. \tag{1·18}$$

With the aid of (1·6) and (1·16), we may obtain

$$a_i^+(\boldsymbol{k})=a_i^*(\boldsymbol{k}), \quad [a_i^+(\boldsymbol{k}),\ \eta]_-=[a_i(\boldsymbol{k}),\ \eta]_-=0, \tag{1·19}$$

$$a_0^+(\boldsymbol{k})=-a_0^*(\boldsymbol{k}) \quad \text{and} \quad [a_0^+(\boldsymbol{k}),\ \eta]_+=[a_0(\boldsymbol{k}),\ \eta]_+=0. \tag{1·20}$$

On the basis of (1·18), (1·19) and (1·20), we can get Gupta's number representation.

§ 2. The Heisenberg representation

In the present section, we shall introduce many expressions or relations which will be of use for the discussions in the succeeding sections.

The state vector Ψ_H in the Heisenberg representation is defined by

$$\Psi_H = e^{iHt}\Psi_S(t) \quad \text{and} \quad \Psi_H^* = \Psi_S^*(t)e^{-iH^*t}. \tag{2·1}$$

On account of the non-hermite property of the total Hamiltonian H, almost all the formulas in the Heisenberg representation turn out to have unusual characters in contrast to the case of the interaction representation. Noting (1·10), we get

$$\Psi_S^*(t)\eta A_\mu^S(x)\Psi_S(t) = \Psi_H^*\eta e^{iHt}A_\mu^S(x)e^{-iHt}\Psi_H, \tag{2·2}$$

and the Heisenberg operator A_μ is given in the usual form

$$A_\mu(x) = e^{iHt}A_\mu^S(x)e^{-iHt}.$$

From (1·13) and (2·2), we can see that

$$A_\mu(x) = U(0, t)A_\mu(x)U(t, 0), \tag{2·3}$$

where

$$U(0, t) = e^{iHt}e^{-iH_0t} \quad \text{and} \quad U(t, 0) = e^{iH_0t}e^{-iHt}. \tag{2·4}$$

Let us now define $\bar{U}(0, t)$ and $\bar{U}(t, 0)$ as

$$\bar{U}(0, t) = e^{iH^*t}e^{-iH_0t} \quad \text{and} \quad \bar{U}(t, 0) = e^{iH_0t}e^{-iH^*t} \tag{2·4′}$$

It is seen that these four kinds of U's satisfy the following equations:

$$\begin{aligned}
U(t, 0) &= 1 - i\int_0^t dt' V(t')U(t', 0), \\
U(0, t) &= 1 + i\int_0^t dt' U(0, t')V(t'), \\
\bar{U}(t, 0) &= 1 - i\int_0^t dt' V^*(t')\bar{U}(t', 0)
\end{aligned} \tag{2·5}$$

and

$$\bar{U}(0, t) = 1 + i\int_0^t dt' \bar{U}(0, t')V^*(t'),$$

where

$$V(t) = e^{iH_0t}Ve^{-iH_0t} \quad \text{and} \quad V^*(t) = e^{iH_0t}V^*e^{-iH_0t}.$$

In addition, we shall introduce the following types of U's;

$$\begin{aligned}
U(-\infty, 0) &\equiv 1 - i\int_0^{-\infty} dt' V(t')U(t', 0), \\
U(0, -\infty) &\equiv 1 + i\int_0^{-\infty} dt' U(0, t')V(t'), \\
\bar{U}(-\infty, 0) &\equiv 1 - i\int_0^{-\infty} dt' V^*(t')\bar{U}(t', 0), \\
\bar{U}(0, -\infty) &\equiv 1 + i\int_0^{-\infty} dt' \bar{U}(0, t')V^*(t'),
\end{aligned} \tag{2·6}$$

where the integrals with respect to t' should be regarded as being defined by an appropriate limit in order to give the usual meaning to the oscillatory integrals.[10]

From (2·4), (2·5) and (2·6), we can readily prove the following important relations:

$$U(0,t)^*=\bar{U}(t,0),\quad \bar{U}(0,t)\eta=\eta U(0,t),$$
$$U(t,0)^*=\bar{U}(0,t),\quad \bar{U}(t,0)\eta=\eta U(t,0), \tag{2·7}$$

$$U(0,-\infty)^*=\bar{U}(-\infty,0),\quad \bar{U}(0,-\infty)\eta=\eta U(0,-\infty),$$
$$U(-\infty,0)^*=\bar{U}(-\infty,0) \text{ and } \bar{U}(-\infty,0)\eta=\eta U(-\infty,0). \tag{2·8}$$

In view of (2·4) and (2·7), it can be seen that

$$U^{-1}(0,t)=U(t,0)=\eta\bar{U}(t,0)\eta=\eta U(0,t)^*\eta=U(0,t)^+. \tag{2·9}$$

By means of the adiabatic theorem in the formal scattering theory, and assuming that there is no bound state*) in the eigenstates of the total Hamiltonian H, we can easily prove the fact that

$$U^{-1}(0,-\infty)=U(-\infty,0)=U(0,-\infty)^+ \tag{2·10}$$

which corresponds to (2·9).

Finally, we shall derive the useful formulas which relate the "in" field operators to the interaction operators.[8] The Heisenberg operarators $\boldsymbol{A}_\mu$ satisfy the equations of motion

$$\Box\boldsymbol{A}_\mu(x)=-\boldsymbol{j}_\mu(x).$$

Integration of these equations gives the integral equations

$$\boldsymbol{A}_\mu(x)=\boldsymbol{A}^{\text{in}}_\mu(x)+\int_{-\infty}^{\infty}\Delta_R(x-x')\boldsymbol{j}_\mu(x')\,d^4x', \tag{2·11}$$

where

$$d^4x=dx\,dx_0,\quad \Box\Delta_R(x)=-\delta^4(x),\quad \Delta_R(x)=-\theta(x)\Delta(x)$$

and

$$\theta(x)=\begin{cases}1 & \text{for } t>0,\\ 0 & \text{for } t<0.\end{cases}$$

From (2·3) and (2·11), we can derive the equations

$$\boldsymbol{A}^{\text{in}}_\mu(x)=\boldsymbol{A}_\mu(x)+\int_{-\infty}^{0}\Delta(x-x')\boldsymbol{j}_\mu(x')\,d^4x' \tag{2·12}$$

and

$$\boldsymbol{A}^{\text{in}}_\mu(x)=U(0,-\infty)\boldsymbol{A}_\mu(x)U(-\infty,0). \tag{2·13}$$

The reduction of (2·12) and (2·13) will be given in the appendix and the formula (2·13) will play an important role in the following discussions.

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* Dr. T. Imamura kindly suggested that this assumption seems to be superfluous in the practical cases.

§ 3. The indefinite metric in the Heisenberg representation

Utilizing the formulas given in § 2, let us investigate the property of the metric operator in the Heisenberg representation.

In § 2, we have seen that the expectation value of the Heisenberg operator $A_\mu(x)$ is correctly given by

$$\Psi_H^* \eta A_\mu(x) \Psi_H, \tag{3·1}$$

where η is the same one as that defined in the Schrödinger representation. As was pointed out in the introduction, however, η has a simple number representation only in the case of the free field, and does not satisfy

$$\eta \Psi_H \neq (-1)^{n_0} \Psi_H$$

but

$$\eta \Phi = (-1)^{n_0} \Phi,$$

where Ψ_H and Φ are the eigenstates of the Hamiltonian H and H_0, respectively. This situation makes us difficult to evaluate (3·1).

The adiabatic theorem

$$\Psi_H(E) = U(0, -\infty)\, \Phi(E)$$

gives (3·1) the following alternative expression:

$$\begin{aligned} \Psi_H^* \eta A_\mu \Psi_H &= \Phi^* U(0, -\infty)^* \eta A_\mu \Psi_H \\ &= \Phi^* \eta U(-\infty, 0)\, A_\mu \Psi_H \\ &= \Phi^* U(-\infty, 0)\, U(0, -\infty)\, \eta U(-\infty, 0)\, A_\mu \Psi_H \\ &= \Psi_H^{\star} \eta^{\mathrm{in}} A_\mu \Psi_H, \end{aligned} \tag{3·2}$$

where $\Psi_H^{\star}$, η^{in} together with $A_\mu^{\mathrm{in}\star}$ are defined as

$$\begin{aligned} \eta^{\mathrm{in}} &\equiv U(0, -\infty)\, \eta U(-\infty, 0), \\ \Psi_H^{\star} &\equiv \Phi^* U(-\infty, 0), \\ A_\mu^{\mathrm{in}\star} &\equiv U(0, -\infty)\, A_\mu^* U(-\infty, 0). \end{aligned} \tag{3·3}$$

It is easily seen that this η^{in} has the similar property as the original one, i.e.

$$\eta^{\mathrm{in}2} = U(0, -\infty)\, \eta^2 U(-\infty, 0) = 1$$

and

$$\eta^{\mathrm{in}\star} \equiv U(0, -\infty)\, \eta^* U(-\infty, 0) = \eta^{\mathrm{in}}. \tag{3·4}$$

By means of this new η^{in}, the definition of the adjoint and the self-adjoint operators should be modified in the following way;*)

* In what follows, we shall use the adjective "modified"; for example, $A^{\star}$, $A^{\#}$ and $B = B^{\#}$ are called modified hermite conjugate, modified adjoint and modified self-adjoint, respectively.

$$A_\mu^{\text{in}\#} \equiv \eta^{\text{in}} A_\mu^{\text{in}\star} \eta^{\text{in}} \tag{3·5}$$

and

$$A_\mu^{\text{in}\#} = A_\mu^{\text{in}}. \tag{3·6}$$

A_μ^{in} is, of course, neither hermitian nor anti-hermitian in the current sense, but has the modified hermitian character as

$$A_i^{\text{in}}(x) = A_i^{\text{in}\star}(x) \quad \text{and} \quad A_0^{\text{in}}(x) = -A_0^{\text{in}\star}(x), \tag{3·7}$$

owing to its definition and (1·6). (3·5), (3·6) and (3·7) give the commutation characters :

$$[A_i^{\text{in}}(x), \eta^{\text{in}}]_- = [A_0^{\text{in}}(x), \eta^{\text{in}}]_+ = 0, \tag{3·8}$$

which give η^{in} a simple representation similar to the case of the free field as we intended. The Fourier expansions of A_μ^{in}'s turn out to be

$$\begin{aligned} A_i^{\text{in}}(x) &= \sum_{j=1}^{3} \sum_{k} \frac{1}{\sqrt{2Vk_0}} e_i^j(k) \left(a_j(k) e^{ikx} + a_j^\#(k) e^{-ikx}\right), \\ A_0^{\text{in}}(x) &= \sum_{k} \frac{1}{\sqrt{2Vk_0}} \left(a_0(k) e^{ikx} + a_0^\#(k) e^{-ikx}\right), \end{aligned} \tag{3·9}$$

where

$$\begin{aligned} a_\mu(k) &= U(0, -\infty)\, a_\mu(k)\, U(-\infty, 0), \\ a_\mu^\star(k) &= U(0, -\infty)\, a_\mu^*(k)\, U(-\infty, 0), \\ a_\mu^\#(k) &= \eta^{\text{in}} a_\mu^\star(k) \eta^{\text{in}}, \end{aligned} \tag{3·9}'$$

with the commutation relations

$$[a_\mu(k), a_\nu^\#(k)]_- = U(0, -\infty)[a_\mu(k), a_\nu^+(k)]_- U(-\infty, 0) = \delta_{\mu\nu}. \tag{3·10}$$

In terms of a_μ and $a_\mu^\#$, (3·7) and (3·8) are written as

$$\begin{aligned} &a_i^\# = a_i^\star, \quad a_0^\# = -a_0^\star, \\ &[a_i^\#, \eta^{\text{in}}]_- = [a_i, \eta^{\text{in}}]_- = 0 \end{aligned} \tag{3·11}$$

and

$$[a_0^\#, \eta^{\text{in}}]_+ = [a_0, \eta^{\text{in}}]_+ = 0. \tag{3·12}$$

The "modified" number operators of the "in-field" are now defined by

$$N_i^{\text{in}} \equiv +a_i^\# a_i \quad \text{and} \quad N_0^{\text{in}} \equiv -a_0^\# a_0, \tag{3·13}$$

which have non-negative integral eigenvalues as will be seen in the following way. Our modified number operator can be written, for example, as

$$N_0^{\text{in}} \equiv -a_0^\# a_0 = a_0^\star a_0 = U(0, -\infty)\, a_0^* a_0 U(-\infty, 0) = U(0, -\infty) N_0 U(-\infty, 0),$$

from which we have

$$\begin{aligned} N_0^{\text{in}} \Psi(n_0) &= U(0, -\infty) N_0 U(-\infty, 0) U(0, -\infty) \Psi(n_0) \\ &= U(0, -\infty) N_0 \Psi(n_0) = n_0 \Psi_H(n_0), \end{aligned}$$

where

$$N_0\Phi(n_0)=n_0\Phi(n_0) \quad \text{and} \quad n_0\geq 0.$$

The relations (3·10), (3·12) together with the definitions of the number operators (3·13) give the familiar representation to a_μ, $a_\mu^\#$ and η^{in};

$$\begin{aligned}
&a_i\Psi_H(n_i)=n_i^{1/2}\Psi_H(n_i-1),\\
&a_i^\#\Psi_H(n_i)=(n_i+1)^{1/2}\Psi_H(n_i+1),\\
&a_0\Psi_H(n_0)=-n_0^{1/2}\Psi_H(n_0-1),\\
&a_0^\#\Psi_H(n_0)=(n_0+1)^{1/2}\Psi_H(n_0+1)
\end{aligned} \tag{3·14}$$

and

$$\eta^{\text{in}}\Psi_H(n_0)=(-1)^{n_0}\Psi_H(n_0); \tag{3·15}$$

which show that a_μ and $a_\mu^\#$ (including the case of $\mu=0$) are the annihilation and the creation operators of the "physical photons" in contrast to the case of the interaction representation where the "bare photons" are concerned. This interpretation of the operators gives the method of constructing the n-physical photon state from the true vacuum state Ψ_0;

$$\begin{aligned}
\Psi_H(n_i)&=\frac{1}{\sqrt{n_i!}}a_i^\#a_i^\#\cdots a_i^\#\Psi_0,\\
\Psi_H(n_0)&=\frac{1}{\sqrt{n_0!}}a_0^\#a_0^\#\cdots a_0^\#\Psi_0,\\
\Psi_H^\star(n_i)&=\frac{1}{\sqrt{n_i!}}\Psi_0^\star\eta^{\text{in}}a_i\cdots a_i\eta^{\text{in}},\\
\Psi_H^\star(n_0)&=\frac{1}{\sqrt{n_0!}}\Psi_0^\star\eta^{\text{in}}a_0\cdots a_0\eta^{\text{in}}.
\end{aligned} \tag{3·16}$$

Here the vacuum state Ψ_0 is tentatively defined as that with no photons of the four kinds. It will be shown in the next section that this definition of the vacuum is compatible with the Lorentz condition. The norms and the relation of completeness of the set of the states introduced above have the forms as

$$\begin{aligned}
\Psi_H^*(m_i)\eta\Psi_H(n_i)&=\Psi_H^\star(m_i)\eta^{\text{in}}\Psi_H(n_i)=\delta_{m_in_i},\\
\Psi_H^*(m_0)\eta\Psi_H(n_0)&=\Psi_H^\star(m_0)\eta^{\text{in}}\Psi_H(n_0)=\delta_{m_0n_0},
\end{aligned} \tag{3·17}$$

and

$$\sum_{n_i,\,n_0}\Psi_H(n_i,\,n_0)\,(-1)^{n_0}\Psi_H^*(n_i,\,n_0)\,\eta=\sum_{n_i,\,n_0}\Psi_H(n_i,\,n_0)\,(-1)^{n_0}\Psi_H^\star(n_i,\,n_0)\,\eta^{\text{in}}=1. \tag{3·18}$$

In view of the above arguments, we are now in a position to answer the question stated in the beginning of this section. It is now possible to evaluate (3·1) in an elegant way thanks to our "in-field" representation, especially to the relation (3·15), in spite of the fact that

$$\eta\Psi_H\neq(-1)^{n_0}\Psi_H.$$

In the expression

$$\pi(p^2)=\frac{V}{p_x^2-p^2}\sum_{p(z)=p}|\Psi_0^* j_x \Psi_H(z)|^2(-1)^{n_0(z)} \tag{3·19}$$

derived by Källen (Helv. Phys. Acta **25** (1952), 417), the last factor $(-1)^{n_0(z)}$ is just due to the same factor in our relation (3·18).

§ 4. Lorentz condition and gauge transformation

In the interaction representation, the exact treatment of the Lorentz condition becomes difficult on account of occurrence of the charge dependent term.[11] On the other hand, the " in-field " representation is quite preferable, since in this representation the above mentioned cumbersome term disappears and the Lorentz condition takes the same expression as that in the case of the free field.

As was proposed by Gupta,[11] the Lorentz condition in the Heisenberg representation is given by

$$(\partial A_\mu(x)/\partial x_\mu)^{(+)}\Psi_H=0 \quad \text{for all } t, \tag{4·1}$$

where the superscript $(+)$ means the positive frequency part of $(\partial A_\mu/\partial x_\mu)$. If we note the equation

$$A_\mu(x)=A_\mu^{\text{in}}(x)+\int_{-\infty}^{\infty}\Delta_R(x-x')\,j_\mu(x')\,d^4x',$$

we get the following operator equation by making use of the charge-conservation law;

$$\partial A_\mu/\partial x_\mu=\partial A_\mu^{\text{in}}/\partial x_\mu, \tag{4·2}$$

which yields

$$(\partial A_\mu/\partial x_\mu)^{(+)}=(\partial A_\mu^{\text{in}}/\partial x_\mu)^{(+)}=(\partial A_\mu^{\text{in}(+)}/\partial x_\mu).$$

With the aid of (4·3), we can replace (4·1) with the simple one[3] (4·3)

$$\frac{\partial A_\mu^{\text{in}}(x)^{(+)}}{\partial x_\mu}\Psi_H=0 \quad \text{for all } t. \tag{4·4}$$

Noting the relations (2·7), we have

$$\Psi_H^*\eta(\partial A_\mu^{\text{in}(-)}/\partial x_\mu)=\left\{\frac{\partial A_\mu^{\text{in}(+)}}{\partial x_\mu}\Psi_H\right\}^*\eta=0. \tag{4·5}$$

The supplementary condition (4·4) can be regarded as the quantum theoretical analogue of the classical condition $\partial A_\mu/\partial x_\mu=0$, because of the fact that

$$\Psi_H^*\eta(\partial A_\mu/\partial x_\mu)\Psi_H=\Psi_H^*\eta(\partial A_\mu^{\text{in}(-)}/\partial x_\mu+\partial A_\mu^{\text{in}(+)}/\partial x_\mu)\Psi_H=0, \tag{4·6}$$

where we have used (4·2), (4·4) and (4·5).

From (4·4), (4·5) and (3·10), we obtain

$$\boldsymbol{L}_k\Psi_H \equiv (\boldsymbol{a}_3(\boldsymbol{k}) - \boldsymbol{a}_0(\boldsymbol{k}))\Psi_H = 0, \tag{4·7a}$$

$$\Psi^*_H\eta\boldsymbol{L}^\#_k \equiv \Psi^*_H\eta(\boldsymbol{a}^\#_3(\boldsymbol{k}) - \boldsymbol{a}^\#_0(\boldsymbol{k})) = 0 \tag{4·7b}$$

and

$$[\boldsymbol{L}_k, \boldsymbol{L}^\#_k]_- = 0. \tag{4·8}$$

The state $\Psi_{H,n}$ which satisfies the condition (4·7a) can be written as

$$\Psi_{H,n} = \sum_{r=0}^{n} (-1)^r \sqrt{n!/r!(n-r)!}\,\Psi_H(n-r, r), \tag{4·9}$$

where $\Psi_H(n, m)$ is state which has the n-longitudinal and m-scalar photons. We can easily prove the relation

$$\boldsymbol{L}^\#\Psi_{H,n} = \sqrt{n+1}\,\Psi_{H,n+1} \tag{4·10}$$

by using the mathematical induction. The most general state which satisfies (4·7a) is given by a linear combination of (4·9) and is expressed as

$$\Psi = \boldsymbol{J}'\Psi_0 \tag{4·11}$$

and

$$\boldsymbol{J}' = 1 + \frac{1}{\sqrt{1!}}\sum_k C^{(1)}(\boldsymbol{k})\boldsymbol{L}^\#_k + \frac{1}{2!}\sum_{k,k'} C^{(2)}(\boldsymbol{k}, \boldsymbol{k}')\boldsymbol{L}^\#_k\boldsymbol{L}^\#_{k'} + \cdots, \tag{4·12}$$

where $\Psi_0 = \Psi_H(0, 0)$ and $C^{(n)}$'s are arbitrary functions of $\boldsymbol{k}$'s. By virtue of the Lorentz condition (4·7a), we can add any functional of $\boldsymbol{L}_k$ to the right-hand side of (4·12). Thus, in what follows, we shall use the alternative definition of $\boldsymbol{J}'$ insted of (4·12), which is given by

$$\boldsymbol{J}' = \frac{1 + \frac{1}{\sqrt{1!}}\sum_k C^{(1)}\boldsymbol{L}^\#_k + \frac{1}{\sqrt{2!}}\sum C^{(2)}(\boldsymbol{k}, \boldsymbol{k}')\boldsymbol{L}^\#_k\boldsymbol{L}^\#_{k'} + \cdots}{1 + \frac{1}{\sqrt{1!}}\sum_k C^{(1)*}\boldsymbol{L}_k + \frac{1}{\sqrt{2!}}\sum C^{(2)*}(\boldsymbol{k}, \boldsymbol{k}')\boldsymbol{L}_k\boldsymbol{L}_{k'} + \cdots}, \tag{4·13}$$

where the denominator has been assumed to be non-singular. This new $\boldsymbol{J}'$ has the inverse $\boldsymbol{J}'^+$, i.e.,

$$\boldsymbol{J}'^+\boldsymbol{J}' = \boldsymbol{J}'\boldsymbol{J}'^+ = 1,$$

where we have used the fact that

$$(\boldsymbol{L}^\#_k)^+ = \eta(\boldsymbol{L}^\#_k)^*\eta = \boldsymbol{L}_k.$$

In the special case where $C^{(n)}$'s are defined as

$$\begin{aligned} C^{(1)}(\boldsymbol{k}) &\equiv C(\boldsymbol{k}), \\ C^{(n)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots, \boldsymbol{k}_n) &\equiv \frac{1}{\sqrt{n!}} C(\boldsymbol{k}_1)C(\boldsymbol{k}_1)\cdots C(\boldsymbol{k}_n), \end{aligned} \tag{4·14}$$

$\boldsymbol{J}'$ and $\boldsymbol{J}'^+$ take the simple forms as

$$\begin{aligned} \boldsymbol{J} &= \exp \sum (C(\boldsymbol{k})\boldsymbol{L}_k^{\#} - C^*(\boldsymbol{k})\boldsymbol{L}_k), \\ \boldsymbol{J}^+ &= \exp \sum (C^*(\boldsymbol{k})\boldsymbol{L}_k - C(\boldsymbol{k})\boldsymbol{L}_k^{\#}). \end{aligned} \tag{4·15}$$

The operator $\boldsymbol{J}'$ defined by (4·13) is called the generator of the "generalized" gauge transformation. On the other hand, the operator $\boldsymbol{J}$ given by (4·15) can be written as

$$\boldsymbol{J} = \exp\left[-i\int d^3\boldsymbol{x}\left\{\frac{\partial \boldsymbol{A}_\mu}{\partial x_\mu}\frac{\partial \Lambda}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{A}_\mu}{\partial x_\mu}\right)\cdot\Lambda\right\}\right], \tag{4·16}$$

where

$$\Lambda(x) = \sum \frac{-i}{\sqrt{2Vk_0^3}}(C(\boldsymbol{k})e^{ikx} - C^*(\boldsymbol{k})e^{-ikx}).$$

The operator $\boldsymbol{J}$ is a constant of motion and is the generator of the "usual" gauge transformation as will be seen below.

Consider a transformation

$$\Psi \to \Psi' = \boldsymbol{J}'\Psi$$

which corresponds to that of the operators of the following type:

$$\begin{aligned} (\Psi^* \eta \boldsymbol{A}_\mu^{\text{in}} \Psi) \to (\Psi'^* \eta \boldsymbol{A}_\mu^{\text{in}} \Psi') &= (\Psi^* \eta \boldsymbol{A}_\mu^{\text{in}'} \Psi), \\ \boldsymbol{A}_\mu^{\text{in}'} &= \boldsymbol{J}'^+ \boldsymbol{A}_\mu^{\text{in}} \boldsymbol{J}'. \end{aligned} \tag{4·17}$$

In the special case where $\boldsymbol{J}'$ is replaced by $\boldsymbol{J}$, we can easily obtain

$$\boldsymbol{A}_\mu^{\text{in}'}(x) = \boldsymbol{A}_\mu^{\text{in}}(x) + \partial\Lambda(x)/\partial x_\mu, \tag{4·18}$$

whereas in the general case the right-hand side of (4·7) does not have a simple form as (4·18). The only case where

$$\boldsymbol{J}^+\boldsymbol{O}\boldsymbol{J} = \boldsymbol{J}'^+\boldsymbol{O}\boldsymbol{J}'$$

is that of the gauge invariant operator $\boldsymbol{O}$ which is defined to be commutable with $\boldsymbol{L}$ and $\boldsymbol{L}^{\#}$.

From (4·11), we see that any physical state can be derived from the "vacuum" state (namely, no longitudinal and no scalar photons present) by the generalized gauge transformation. It is also seen that any matrix element of any gauge invariant operator $\boldsymbol{O}$ can be reduced to that of the "vacuum" case in the following way;

$$\Psi^* \eta \boldsymbol{O} \Psi = \Psi_0^* \eta \boldsymbol{J}'^+ \boldsymbol{O} \boldsymbol{J}' \Psi_0 = \Psi_0^* \eta \boldsymbol{O} \Psi_0.$$

This fact allows us to use Ψ_0 exclusively in place of any Ψ without loss of generality. Before concluding this section, it must be noted that a set of states which can be generated from Ψ_0 by the usual gauge transformation is a sub-set of the larger one which consists of the states derived from Ψ_0 by the general $\boldsymbol{J}'$.

§ 5. Question of the Lorentz-invariance[2)9)]

Gupta's theory has an asymmetric appearance with respect to space and time. Many works[2)9)] have been done in order to prove the Lorentz-invariance of his formalism, but

according to the opinion of the present author these proofs seem to be unsatisfactory. We shall make a detailed investigation on this problem in the case of the free electromagnetic field. It will be seen that the similar conclusion will also be reached in the case of the interacting fields.

Let us consider a state $\Phi(n_0)$ where n_0 scalar photons are present in a frame of reference A; namely

$$N_0\Phi(n_0)=n_0\Phi(n_0), \tag{5·1}$$

which is related with $\Phi(n_0-1)$ by the destruction operator a_0 as

$$a_0\Phi(n_0)=-\sqrt{n_0}\,\Phi(n_0-1). \tag{5·2}$$

The metric operator η in this state has the representation as

$$\eta\Phi(n_0)=(-1)^{n_0}\Phi(n_0). \tag{5·3}$$

Besides the above mentioned frame of reference $\mathcal{A}$, consider another Lorentz-frame $\mathcal{B}$ and another state $\Phi'(n_0)$ in the latter frame of reference, which is defined by*

$$N_0'\Phi'(n_0)=n_0\Phi'(n_0). \tag{5·4}$$

(Note that the eigenvalue of the operator N_0' has the same value n_0 as that of N_0 in $\mathcal{A}$.)

Corresponding to (5·2) and (5·3), we should have the following relations owing to the requirement of the principle of relativity:

$$a_0'\Phi'(n_0)=-\sqrt{n_0}\,\Phi'(n_0-1) \tag{5·5}$$

and

$$\eta'\Phi'(n_0)=(-1)^{n_0}\Phi'(n_0-1). \tag{5·6}$$

It must be remarked that the observer in $\mathcal{B}$ should make use of the metric operator $\eta'=(-1)^{N_0'}$ instead of $\eta=(-1)^{N_0}$, if the commutation relations (1·8) and (1·14) in the Lorentz system $\mathcal{A}$ retain their forms even in the system $\mathcal{B}$ according to the principle of relativity.

The norm of Φ and Φ' are respectively defined as

$$\Phi^*(n_0)\eta\Phi(n_0)=(-1)^{n_0} \tag{5·7}$$

and

$$\Phi'^*(n_0)\eta'\Phi'(n_0)=(-1)^{n_0}, \tag{5·7}'$$

where the metric operator η' should be used in the system $\mathcal{B}$ from the reason just emphasized above. The equations (5·7) and (5·7)′ give an important requirement

$$\Phi^*(n_0)\eta\Phi(n_0)=\Phi'^*(n_0)\eta'\Phi'(n_0). \tag{5·7}''$$

Let it be assumed that $\Phi'(n_0)$ is connected with $\Phi(n_0)$ by a Lorentz transformation $\mathcal{L}$ as

---

* All the primed quantities are referred to the frame of reference $\mathcal{B}$, but the unprimed ones to $\mathcal{A}$.

$$\Psi'(n_0) = \mathcal{L}^{-1}\Psi(n_0). \tag{5·8}$$

Then (5·4) and (5·1) give the relationship between any pair of primed and unprimed operators :

$$F' = \mathcal{L}^{-1}F\mathcal{L}. \tag{5·9}$$

It is seen from (5·9) that the existence of the nonsingular operator $\mathcal{L}$ guarantees the validity of the commutation relation (1·14) and of the field equations in both systems (the Lorentz invariance of the whole system of the q-number relations except for (1·8)).

The operator $\mathcal{L}$ can be determined from the postulate that, under the Lorentz transformation

$$x_\mu \to x_\mu' = a_{\mu\nu}x_\nu \equiv x_\mu + \delta x_\mu,$$

A_μ should be transformed as

$$\begin{aligned} A_\mu'(x) = A_\mu'(x' - \delta x) &= A_\mu'(x') - \frac{\partial A_\mu(x)}{\partial x_\nu}\delta x_\nu \\ &= a_{\mu\nu}A_\nu(x) - \frac{\partial A_\mu(x)}{\partial x_\nu}\delta x_\nu \\ &= \mathcal{L}^{-1}A_\mu(x)\mathcal{L}. \end{aligned}$$

By considering the infinitesimal Lorentz transformation

$$x_1' = x_1 + \epsilon x_0, \quad x_0' = x_0 + \epsilon x_1, \quad x_2' = x_2 \text{ and } x_3' = x_3.$$

The operator $\mathcal{L}$ is given by[12)13)14)]

$$\mathcal{L} = 1 + G, \quad \mathcal{L}^{-1} = 1 - G \tag{5·10}$$

and

$$G = i\epsilon(\alpha + \beta),$$

where

$$\alpha = \int d^3x \left(\frac{\partial A_0}{\partial x_0}A_1 - \frac{\partial A_1}{\partial x_0}A_0\right) = i\sum_{\substack{k \\ j=1}}^{3} e_1{}^j(k)\,(a_j(k)a_0\ (k) - a_j^{\dagger}(k)a_0(k)) \tag{5·10}'$$

and

$$\beta = \int d^3x\,(T_{00}x_1 - T_{01}x_0).$$

The fact that $\mathcal{L}$ is a constant of motion as easily verified is necessary for the covariance of the field equations.

Contrary to the usual case, the present $\mathcal{L}$ is not unitary because of the following relations

$$\alpha^* = -\alpha, \quad \beta^* = \beta \quad \text{and} \quad [\alpha, \eta]_+ = [\beta, \eta]_- = 0, \tag{5·11}$$

accordingly

$$\mathcal{L}^{-1} = \eta\mathcal{L}^*\eta. \tag{5·12}$$

This non-unitary character of $\mathcal{L}$ gives rise to two alternatives in defining the transformation property of η. The first view-point is due to the important requirement $(5\cdot7)''$, which gives the relationship

$$\eta' = \mathcal{L}'^{*}\eta\mathcal{L}'. \tag{5·13}$$

This relationship violates the validity of the commutation relations $(1\cdot8)$ in the frame of reference $\mathcal{C}$, and the representation $(5\cdot6)$ of η' becomes not allowable in contradiction to the starting assumption $(5\cdot4)$, i.e., the postulate of the principle of relativity. In order to be able to use the representation $(5\cdot6)$ of η', or to retain the commutation relations $(1\cdot8)$, we are compelled to adopt the second standpoint:

$$\eta' = \mathcal{L}^{-1}\eta\mathcal{L}. \tag{5·14}$$

In this case, the norm of a state is, of course, not Lorentz-invariant, i.e.,

$$\Phi'^{*}\eta'\Phi' = \Phi^{*}(\mathcal{L}^{-1})^{*}\mathcal{L}^{-1}\eta\Phi \neq \Phi^{*}\eta\Phi,$$

because of the non-unitary character of $\mathcal{L}$.

Although we have shown the formal non-invariance of the norm of a state under the Lorentz transformation, there may retain a slight prospect of recovering the invariance by taking into account the Lorentz condition, because in the above argument the state concerned has not been assumed to be the physical one satisfying the Lorentz condition. Consider a simple example

$$\Phi_b'^{*}\eta'\Phi_a'$$

where

$$\Phi_b' = \chi'(0,0)\,(\Phi'(0,1) - \Phi'(1,0))$$

and

$$\Phi_a' = \chi'(1,0)\,\Phi'(0,0).$$

$\chi(n_1, n_2)$ means the state with n_1 and n_2 transversal photons of the 1st and 2nd polarization, while $\Phi(n_3, n_0)$ the state with n_3 longitudinal and n_0 scalar photons. The both states clearly satisfy the Lorentz condition. Inserting the expressions $(5\cdot10)$ into the above, we have

$$\Phi_b'^{*}\eta'\Phi_a' = \Phi_b^{*}\eta\Phi_a + 2i\epsilon[\chi^{*}(0,0)\,(\Phi^{*}(0,1) - \Phi^{*}(1,0))\,\eta\alpha\chi(1,0)\,\Phi(0,0)].$$

The left-hand side vanishes, while the second term of the right-hand side turns out to be

$$2\epsilon e_1^{\,1} \neq 0.$$

Thus we see that our final effort was in vain.

The similar situations can be expected to occur in evaluating the matrix elements of the S-matrix by using the interaction representation.

$$\Phi_f'^{*}\eta'S'\Phi_i' = \Phi_f^{*}\eta S\Phi_i + 2i\epsilon\,\Phi_f^{*}\eta\alpha S\Phi_i$$
$$\neq \Phi_f^{*}\eta S\Phi_i$$

even though S is commutative with $\mathcal{L}$ and both Φ_i and Φ_f are the physical states.

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Appendix: Proofs of (2·12) and (2·13)

The proofs of (2·12) and (2·13) were already given by S. Schweber.[8] He used, however, the power series expansion concerning the coupling constant in the course of his proof. It seems not preferable to employ such an expansion, when we investigate the properties of the theory without using any approximation. Therefore, we shall give the alternative proofs of (2·12) and (2·13) without expansions.

From (2·3), we see that

$$A_\mu(x)=U(0,t)A_\mu(x)U(t,0)=A_\mu(x)+U(0,t)[A_\mu(x),U(t,0)]_-. \tag{a·1}$$

Let us calculate $[A_\mu(x),U(t,0)]_-$ in the following way;

$$\begin{aligned}[A_\mu(x),U(t,0)]_-&=[A_\mu(x),1-i\int_0^t dt'V(t')U(t',0)]_-\\&=(-i)^2\int_0^t d^4x'\Delta(x-x')j_\mu(x')U(t',0)-i\int_0^t dt'V(t')[A_\mu(x),U(t',0)]_-,\end{aligned} \tag{a·2}$$

with the aid of (2·5), (1·1) and (1·14). The equation (a·2) can be regarded as an integral equation for $[A_\mu(x),U(t,0)]_-$, and the solution is given by

$$[A_\mu(x),U(t,0)]_-=-\int_0^t d^4x'\Delta(x-x')U(t,t')j_\mu(x')U(t',0). \tag{a·3}$$

Proof: Introducing (a·3) into the right-hand side of (a·2), we have

$$\begin{aligned}&[\textit{the second term of } (a\cdot 2)]=-i\int_0^t dt'V(t')[A_\mu(x),U(t',0)]_-\\&=i\int_0^t dt'V(t')\int_0^t d^4x''\theta(t'-t'')\Delta(x-x'')U(t',t'')j_\mu(x'')U(t'',0)\\&=i\int_0^t d^4x'\Delta(x-x')\int_0^t dt''V(t'')\theta(t''-t')U(t'',t')j_\mu(x')U(t',0)\\&=i\int_0^t d^4x'\Delta(x-x')\int_{t'}^t V(t'')U(t'',t')j_\mu(x')U(t',0).\end{aligned}$$

Thus we obtain

$$\begin{aligned}&[\textit{the right-hand side of } (a\cdot 2)]\\&=-\int_0^t d^4x'\Delta(x-x')[1-i\int_{t'}^t dt''V(t'')U(t'',t')]j_\mu(x')U(t',0)\\&=-\int_0^t d^4x'\Delta(x-x')U(t,t')j_\mu(x')U(t',0).\end{aligned}$$

Q.E.D.

From (a·1) and (a·3), we can easily see that

$$\boldsymbol{A}_\mu(x) = A_\mu(x) - \int_0^t d^4x' \Delta(x-x') \boldsymbol{j}_\mu(x'). \tag{a·4}$$

In view of the equation

$$\boldsymbol{A}_\mu(x) = \boldsymbol{A}_\mu^{\text{in}}(x) - \int_{-\infty}^t d^4x' \Delta(x-x') \boldsymbol{j}_\mu(x'), \tag{a·5}$$

we get the equation (2·12) :

$$\boldsymbol{A}_\mu^{\text{in}}(x) = A_\mu(x) + \int_{-\infty}^0 \Delta(x-x') \boldsymbol{j}_\mu(x') d^4x'. \tag{a·6}$$

Next, we shall give the proof of (2·13). From the fact that

$$U(0, -\infty) A_\mu(x) U(-\infty, 0) = A_\mu(x) + U(0, -\infty)[A_\mu(x), U(-\infty, 0)]_-, \tag{a·7}$$

we must calculate $[A_\mu(x), U(-\infty, 0)]_-$.

From (2·6) and (a·3), we have

$$[A_\mu(x), U(-\infty, 0)]_- = [A_\mu(x), 1 - i\int_0^{-\infty} dt' V(t') U(t', 0)]_-$$

$$= -\int_0^{-\infty} d^4x' \Delta(x-x') j_\mu(x') U(t', 0) + i\int_0^{-\infty} dt' V(t') \int_0^{t'} d^4x'' \Delta(x-x'') U(t', t'') \cdot j_\mu(x'') U(t'', 0).$$

The second term of the right-hand side of the above equation leads to

$$[\textit{the second term}] = +i\int_0^{-\infty} dt' \int_0^{-\infty} d^4x'' \bar{\theta}(t'-t'') \Delta(x-x'') U(t', t'') j_\mu(x'') U(t'', 0)$$

$$= i\int_0^{-\infty} d^4x' \Delta(x-x') \int_0^{-\infty} dt'' \bar{\theta}(t''-t') V(t'') U(t'', t') j_\mu(x') U(t', 0)$$

$$= i\int_0^{-\infty} d^4x' \Delta(x-x') \int_{t'}^{-\infty} dt'' V(t'') U(t'', t') j_\mu(x') U(t', 0),$$

where

$$\bar{\theta}(t) = \begin{cases} 0 & \text{for} \quad t > 0, \\ 1 & \text{for} \quad t < 0. \end{cases}$$

Thus we obtain

$$[A_\mu(x), U(-\infty, 0)]_- = -\int_0^{-\infty} d^4x' \Delta(x-x') [1 - i\int_{t'}^{-\infty} dt'' V(t'') U(t'', t')] j_\mu(x') U(t', 0)$$

$$= -\int_0^{-\infty} d^4x' \Delta(x-x') U(-\infty, t') j_\mu(x') U(t', 0). \tag{a·8}$$

Introducing (a·8) into (a·7), and using (a·6), we get the result

$$U(0, -\infty) A_\mu(x) U(-\infty, 0) = A_\mu(x) - \int_0^{-\infty} d^4x' \Delta(x-x') U(0, -\infty) U(-\infty, t') \cdot j_\mu(x') U(t', 0)$$

$$= A_\mu(x) - \int_0^{-\infty} d^4x' \Delta(x-x') \boldsymbol{j}_\mu(x')$$

$$= \boldsymbol{A}_\mu^{\text{in}}(x) .$$

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Dispersion Relations for K Meson-Nucleon Scattering

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The dispersion relations for K meson-nucleon scattering are derived on the analogy of the case of pion-nucleon scattering. Assuming the spin of K meson to be zero, it will be pointed out that there is an essential difference between the dispersion relations for K mesons with both types of interactions: scalar and pseudoscalar. The contributions to the scattering amplitude arising from the unphysical region, due to the exothermic reactions $K+P\rightarrow\pi+\Lambda$ and $K+P\rightarrow\pi+\Sigma$, are discussed.

§ 1. Introduction

Even though there have been many attempts[1] to determine the spin and parity of Λ, Σ and K, we have not yet found anything definite. This fact stimulated us to investigate a new method to determine them.

Assuming that we can describe the K mesons within the framework of the present quantum field theory, we derive the exact dispersion relations for K-nucleon scattering with both types of interactions: scalar and pseudoscalar, on the analogy of pion-nucleon scattering.[2),3)] The assumptions made here are as follows: the parity of K is definite, the spins of Λ, Σ are $1/2$ and that of K is 0. To determine the type of the interaction the method due to Haber-Schaim[4] is more suitable than that due to Anderson et al.[3]

In § 2, § 3 and § 5 we shall express the dispersion relations in the form as Haber-Schaim did. The interaction type as well as the coupling constant can be determined if an appropriate cross section measurement is performed. Discussions about the contribution from the unphysical region will be given in § 4.

Independently D. Amati and B. Vitale[5] derived dispersion relations for K-nucleon interaction in the fixed source theory, but did not discuss the contribution to the scattering amplitude from the unphysical region. There is a possibility that $A(\omega)$ has a cusp or kink at the threshold energy where a new channel opens. If $A(\omega)$ has a cusp at m_k, their work is incomplete because $D_-(m_k)$ diverges. If it is continuous, their results are consistent with ours.

§ 2. Dispersion relations for K-P scattering

The dispersion relations for K-P scattering are derived on the analogy of π-P scattering. We assume that the nucleon charge state does not change and that it is a proton. We assume:

I) the spins of Λ, Σ to be $1/2$ and that of K to be 0.

II) the interactions to be Nishijima-Gell-Mann's.

III) the parity of K to be definite.

Remarmakable differences between K-P and π-P scattering are:

(i) K-P interaction might be scalar, while π-P interaction is pseudoscalar.

(ii) the isotopic spin of K is $1/2$, while that of π is 1.

(iii) strangeness of Λ, $\sum$ and K being non-zero, the structure of the Born term is restricted by the "conservation of strangeness".

(iv) the contributions to the scattering amplitude from the unphysical region arise not only from the discrete but also from the continuous spectrum, the former corresponding to "the bound states" Λ, $\sum$, the latter to the exothermic reactions.

Writing the forward amplitude for the coherent $K^{\pm}$-P scattering $T_{\pm}(\omega)$, we devide it into dispersive part $D_{\pm}(\omega)$ and absorptive part $A_{\pm}(\omega)$,

$$T_{\pm}(\omega) = D_{\pm}(\omega) + i\epsilon(\omega)\, A_{\pm}(\omega), \tag{2·1}$$

where $\epsilon(\omega)$ is the sign function, 1 for $\omega_0 > 0$, -1 for $\omega_0 < 0$.

Putting
$$T^{(1)}(\omega) \equiv \tfrac{1}{2}[T_-(\omega) + T_+(\omega)] \equiv D^{(1)}(\omega) + i\epsilon(\omega) A^{(1)}(\omega), \tag{2·2}$$

$$T^{(2)}(\omega) \equiv \tfrac{1}{2}[T_-(\omega) - T_+(\omega)] \equiv D^{(2)}(\omega) + i\epsilon(\omega) A^{(2)}(\omega), \tag{2·3}$$

$D^{(i)}(\omega)$ and $A^{(i)}(\omega)$ have simple behavior when $\omega \to -\omega$; $D^{(1)}$ and $A^{(2)}$ are even functions whereas $D^{(2)}(\omega)$ and $A^{(1)}(\omega)$ are odd. Hence the dispersion relations can be written

$$\tfrac{1}{2}[D_-(\omega) + D_+(\omega)] = \frac{1}{\pi}\int_0^{\infty} d\omega' \frac{2\omega' \tfrac{1}{2}[A_-(\omega') + A_+(\omega')]}{\omega'^2 - \omega^2}, \tag{2·4}$$

$$\tfrac{1}{2}[D_-(\omega) - D_+(\omega)] = \frac{2\omega}{\pi}\int_0^{\infty} d\omega' \frac{\tfrac{1}{2}[A_-(\omega') - A_-(\omega')]}{\omega'^2 - \omega^2}. \tag{2·5}$$

When the convergence of the integral (2·4), (2·5) are not guaranteed, we may increase the power in the denominators in the following way:

$$\tfrac{1}{2}[D_-(\omega) + D_+(\omega)] - \tfrac{1}{2}[D_-(\omega_k) + D_+(\omega_k)]$$
$$= \frac{2(\omega^2 - \omega_k^2)}{\pi}\int_0^{\infty} d\omega' \frac{\omega' \tfrac{1}{2}[A_-(\omega') + A_+(\omega')]}{(\omega'^2 - \omega_k^2)(\omega'^2 - \omega^2)}, \tag{2·6}$$

$$\tfrac{1}{2}[D_-(\omega) - D_+(\omega)] - (\omega/\omega_k)\tfrac{1}{2}[D_-(\omega_k) - D_+(\omega_k)]$$
$$= \frac{2\omega(\omega^2 - \omega_k^2)}{\pi}\int_0^{\infty} d\omega' \frac{\tfrac{1}{2}[A_-(\omega') - A_+(\omega')]}{(\omega'^2 - \omega_k^2)(\omega'^2 - \omega^2)} \tag{2·7}$$

where ω_k is an arbitrary energy larger than m_k, the mass of K meson. The behavior of $A_-(\omega)$ is known only for $\omega > m_k$. So in evaluating the right-hand side of (2·6) and (2·7) we prefer ω_k to m_k in order that the contribution from the unphysical region

becomes small. It is possible to take ω_k as m_k, if $A_-(\omega)$ is known to be continuous.

To find the contributions from $0<\omega<m_k$, we write out the amplitudes $A_\pm(\omega)$ as sums over a complete set of states following reference 2 :

$$A_\pm(\omega)=\pi\sum_n|<n|j_\pm(0)|p>|^2\delta(\boldsymbol{p}-\boldsymbol{n}+\boldsymbol{k})\,(E_p-E_n+\omega)$$
$$-\pi\sum_n|<n|j_\mp(0)|p>|^2\delta(\boldsymbol{p}-\boldsymbol{n}-\boldsymbol{k})\,(E_p-E_n-\omega)\,. \qquad (2\cdot8)$$

For $0\leq\omega<m_k$, only the first term in $(2\cdot8)$ can contribute. By the strangeness conservation only $<n|j_\mp(0)|p>$ is different from zero in this region, corresponding to the absorption of K meson leading to Λ, Σ, $\Lambda+\pi$ or $\Sigma+\pi$ intermediate states. The energy of these states are

$$\omega=\omega_\Lambda=(M_\Lambda{}^2-M_p{}^2-m_k{}^2)/2M_p=103\,m_e=0.107\,m_k \qquad (2\cdot8a)'$$

corresponding to the bound state Λ,

$$\omega=\omega_\Sigma=(M_\Sigma{}^2-M_p{}^2-m_k{}^2)/2M_p=259\,m_e=0.268\,m_k \qquad (2\cdot8b)'$$

corresponding to the bound state Σ, and $\omega_{\Lambda\pi}\leq\omega<m_k$ due to the exothermic reactions $K^-+P\rightarrow\pi+\Lambda$, $\pi+\Sigma$

where
$$\omega_{\Lambda\pi}=\{(M_\Lambda+m_\pi)^2-M_p{}^2-m_k{}^2\}/2M_p=0.404\,m_k\,. \qquad (2\cdot8c)'$$

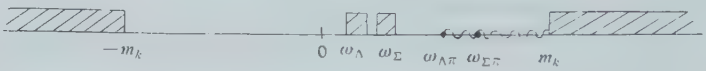

Fig. 1a. K^-P scattering

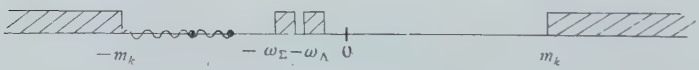

Fig. 1b. K^+P scattering
The energy spectrum of the absorptive part.

For the region of integration extending from m_k to ∞, we have the relations

$$\sigma_\pm(\omega)=(4\pi/k)\,A_\pm(\omega) \qquad (2\cdot9)$$

where $k=(\omega^2-m_k{}^2)^{1/2}$ and $\sigma_\pm(\omega)$ is the total cross sections for $K^\pm$-P scattering.

Referring to Fig. 1a, Fig. 1b, substitution of $(2\cdot9)$ in $(2\cdot4)$ and $(2\cdot5)$ leeds us to

$$D_+(\omega)=\frac{1}{4\pi^2}\int_{m_k}^{\infty}d\omega'k'\left[\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-(\omega')}{\omega'+\omega}\right]+\frac{1}{\pi}\int_0^{\omega_{\Lambda\pi}}d\omega'\frac{A_-(\omega')}{\omega'+\omega}$$
$$+\frac{1}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{\omega'+\omega}\,, \qquad (2\cdot10)$$

$$D_-(\omega)=\frac{1}{4\pi^2}\int_{m_k}^{\infty}d\omega' k'\left[\frac{\sigma_-(\omega')}{\omega'-\omega}+\frac{\sigma_+(\omega')}{\omega'+\omega}\right]+\frac{1}{\pi}\int_{0}^{\omega_{\Lambda\pi}}d\omega'\frac{A_-(\omega')}{\omega'-\omega}$$

$$+\frac{1}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{\omega'-\omega}. \qquad (2\cdot11)$$

The first integrals are already expressed by experimentally measurable quantities. As to the second and third terms, the connection with physical quantities will be investigated in the following sections. (§ 3, § 4)

When the convergence is not satisfactory, the same procedure as in (2·6) and (2·7) must be applied;

$$D_+(\omega)-\frac{1}{2}\left(1+\frac{\omega}{\omega_k}\right)D_+(\omega_k)-\frac{1}{2}\left(1-\frac{\omega}{\omega_k}\right)D_-(\omega_k)$$

$$=\frac{(\omega^2-\omega_k^2)}{4\pi^2}\int_{0}^{\omega_{\Lambda\pi}}d\omega'\frac{k'}{(\omega'^2-\omega_k^2)}\left[\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-(\omega')}{\omega'+\omega}\right]$$

$$+\frac{(\omega^2-\omega_k^2)}{\pi}\int_{0}^{\omega_{\Lambda\pi}}d\omega'\frac{A_-(\omega')}{(\omega'^2-\omega_k^2)(\omega'+\omega)}$$

$$+\frac{(\omega^2-\omega_k^2)}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{(\omega'^2-\omega_k^2)(\omega'+\omega)}, \qquad (2\cdot12)$$

$$D_-(\omega)-\frac{1}{2}\left(1+\frac{\omega}{\omega_k}\right)D_-(\omega_k)-\frac{1}{2}\left(1-\frac{\omega}{\omega_k}\right)D_+(\omega_k)$$

$$=\frac{(\omega^2-\omega_k^2)}{4\pi^2}\int_{m_k}^{\infty}d\omega'\frac{k'}{(\omega'^2-\omega_k^2)}\left[\frac{\sigma_-(\omega')}{\omega'-\omega}+\frac{\sigma_+(\omega')}{\omega'+\omega}\right]$$

$$+\frac{(\omega^2-\omega_k^2)}{\pi}\int_{0}^{\omega_{\Lambda\pi}}d\omega'\frac{A_-(\omega')}{(\omega'^2-\omega_k^2)(\omega'-\omega)}$$

$$+\frac{(\omega^2-\omega_k^2)}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{(\omega'^2-\omega_k^2)(\omega'-\omega)}. \qquad (2\cdot13)$$

§ 3. Bound state contribution

As was mentioned in § 2 (see Fig. 1), we have the bound state contributions; namely $A_-(\omega)$ exhibits the δ type singularities at ω_Λ and ω_Σ. We do not know whether

the interaction type is scalar or pseudoscalar. For simplicity, we consider the case in which the parities of Λ and $\sum$ are the same.

Case I). Scalar interaction—the relative parity of K to Λ and $\sum$ is positive. We can write,

$$A_-(\omega) = |F_s(\omega)|^2 \delta(\omega-\omega_\Lambda) + |G_s(\omega)|^2 \delta(\omega-\omega_\Sigma) \tag{3·1}$$

for $0 \leq \omega < \omega_{\Lambda\pi}$.

Define X and X' by

$$\frac{1}{\pi}\int_0^{\omega_{\Lambda\pi}} d\omega' \frac{A_-(\omega')}{\omega'+\omega} \quad \text{and} \quad \frac{1}{\pi}\int_0^{\omega_{\Lambda\pi}} d\omega' \frac{A_-(\omega')}{\omega'-\omega} \tag{3·2}$$

respectively.

Substituting (3·1) in (3·2), we obtain

$$X = \frac{2f_s^2}{\omega+\omega_\Lambda} + \frac{2g_s^2}{\omega+\omega_\Sigma}. \tag{3·3}$$

Here $2f_s^2 \equiv 1/\pi |F_s(\omega_\Lambda)|^2$, $2g_s^2 \equiv 1/\pi |G_s(\omega_\Sigma)|^2$ are renormalized coupling constants in $N\Lambda K$ and $N\sum K$ respectively. Similarly

$$X' = -\frac{2f_s^2}{\omega-\omega_\Lambda} - \frac{2g_s^2}{\omega-\omega_\Sigma}. \tag{3·4}$$

Case II). Pseudoscalar interaction—the relative parity of K to Λ and $\sum$ is negative. According to conservation of parity and angular momentum p-wave K mesons must be involved. So we can write

$$A_-(\omega) = |F_{ps}(\omega)|^2 k^2 \delta(\omega-\omega_\Lambda) + |G_{ps}(\omega)|^2 k^2 \delta(\omega-\omega_\Sigma). \tag{3·5}$$

Then,

$$X = -\frac{m_k^2-\omega_\Lambda^2}{\pi}\frac{|F_{ps}(\omega_\Lambda)|^2}{\omega+\omega_\Lambda} - \frac{m_k^2-\omega_\Sigma^2}{\pi}\frac{|G_{ps}(\omega_\Sigma)|^2}{\omega+\omega_\Sigma}.$$

Remembering ω_Λ^2, $\omega_\Sigma^2 \ll m_k^2$ (see (2·8)′) and putting

$$\frac{m_k^2}{\pi}|F_{ps}(\omega_\Lambda)|^2 = 2f_{ps}^2, \quad \frac{m_k^2}{\pi}|G_{ps}(\omega_\Sigma)|^2 = 2g_{ps}^2$$

we obtain

$$X = -\frac{2f_{ps}^2}{\omega+\omega_\Lambda} - \frac{2g_{ps}^2}{\omega+\omega_\Sigma}. \tag{3·6}$$

Similarly,

$$X' = \frac{2f_{ps}^2}{\omega-\omega_\Lambda} + \frac{2g_{ps}^2}{\omega-\omega_\Sigma}. \tag{3·7}$$

Notice the difference in sign between scalar interaction and pseudoscalar interaction.

§ 4. Contributions from the unphysical region $m_k > \omega \geq \omega_{\Lambda\pi}$

As stated in § 2 (see Fig. 1), we have a contribution from the unphysical region $m_k > \omega \geq \omega_{\Lambda\pi}$ due to the exothermic reactions. (last terms of (2·10) and (2·11)). The scattering amplitude $A(\omega)$ for $\omega < m_k$ does not correspond to a directly measurable quantity and the connection of $A(\omega)$ with physical quantities must be investigated. In principle, the behavior of $A(\omega)$ in the unphysical region $(m_k \geq \omega > \omega_{\Lambda\pi})$ can be obtained by the analytic continuation from the physical region $(\omega > m_k)$. But practically it is not so easy. If we know the behavior of $A(\omega)$ near $\omega_{\Lambda\pi}$, depending on which partial wave of π is involved, a smooth extrapolation from $\omega > m_k$ to this point will be a good approximation.

However, it was pointed out by Wigner[6] that sometimes a nuclear cross section shows a cusp-like energy dependence at a threshold energy for a certain type of nuclear reaction. Applying his method in the unphysical region, we investigate the energy dependence of the K-matrix to find a possible cusp or kink in collision matrix $T(\omega)$, total cross section $\sigma(\omega)$ and then the imaginary part of the scattering amplitude $A(\omega)$ near the threshold energy where a new channel opens.[7]

Putting total Hamiltonian of the system $H \equiv H_0 + V$, and introducing

$$T = V + V \frac{1}{\omega + M + i\eta - H} V \tag{4·1}$$

$$K = V + V \mathrm{P} \frac{1}{\omega + M - H} V \tag{4·2}$$

we obtain the following relation between them

$$T = K - i\pi K \delta(\omega + M - H_0) T. \tag{4·3}$$

Now the reaction matrix element from the channel a to a is, from (4·2),

$$K_{aa} = V_{aa} + \sum_c V_{ac} \frac{1}{\omega - \omega_c} V_{ca} \tag{4·4}$$

where ω_c is the spectrum of $H - M$ (In the case of KP scattering, see Fig. 1). Put channel $P+K$ to be a, and $\Lambda+\pi$ (l-wave), $P+K$ to be c. In order to investigate the behavior of $K(\omega)$ near $\omega_{\Lambda\pi}$, we replace the summation in (4·4) by an integration. In the following arguments only singular behaviors near $\omega_{\Lambda\pi}$ are important and we shall keep only those terms which are singular at $\omega_{\Lambda\pi}$. Then we get

$$K_{aa}(\omega) \begin{cases} \text{no cut line} & \text{for } \omega > \omega_{\Lambda\pi} \\ \propto (\omega_{\Lambda\pi} - \omega)^{l+1/2} & \text{for } \omega < \omega_{\Lambda\pi} \end{cases} \tag{4·5}$$

so that

$$T_{aa}(\omega) \sim K_{aa}(\omega) \propto (\omega_{\Lambda\pi} - \omega)^{l+1/2} \text{ for } \omega < \omega_{\Lambda\pi}. \tag{4·6a}$$

Using analyticity of $T_{aa}(\omega)$ in the upper half ω-plane,

$$T_{aa}(\omega) \propto (-1)^{l} i(\omega-\omega_{\Lambda\pi})^{l+1/2}. \tag{4·6b}$$

Let us investigate the behavior of $T(\omega)$ near $\omega_{\Lambda\pi}$ in more detail. From (4·3) we obtain,

$$T_{aa}(\omega) = K_{aa}(\omega) - i\pi \sum_{c} K_{ac}(\omega)\, \rho_c(\omega)\, T_{ca}(\omega) \tag{4·7}$$

where ρ_c is the intermediate state density. Let us consider the case that a new channel $x(\equiv \Lambda\pi)$ opens from $\omega_{\Lambda\pi}$. Using eq. (4·7) above the threshold energy $\omega_{\Lambda\pi}$, we can derive the following approximate relation.

$$\begin{aligned}
T_{aa}(\omega) &= K_{aa}(\omega) - i\pi \sum_{c}{}' K_{ac}(\omega)\, \rho_c(\omega)\, T_{ca}(\omega) - i\pi K_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega) \\
&= K_{aa}(\omega) - i\pi \sum_{c}{}' K_{ac}(\omega)\, \rho_c(\omega) \{ (K_{ca}(\omega) - i\pi \sum_{d}{}' K_{cd}(\omega)\, \rho_d(\omega)\, T_{da}(\omega)) \\
&\qquad - i\pi K_{cx}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega) \} \\
&\qquad - i\pi K_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega) \\
&= K_{aa}(\omega) - i\pi \sum_{c}{}' K_{ac}(\omega)\, \rho_c(\omega) \{ K_{ca}(\omega) - i\pi \sum_{d}{}' K_{cd}(\omega)\, \rho_d(\omega)\, T_{da}(\omega) \} \\
&\qquad - i\pi \{ K_{ax}(\omega) - \sum_{c}{}' K_{ac}(\omega)\, \rho_c(\omega)\, T_{cx}(\omega) \} \rho_x(\omega)\, T_{xa}(\omega) \\
&\approx T_{aa}(\omega_{\Lambda\pi}) - i\pi T_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega),
\end{aligned}$$

where $$\rho_x(\omega) \sim \sqrt{\omega-\omega_{\Lambda\pi}} \quad \text{for } \omega > \omega_{\Lambda\pi}, \tag{4·8}$$

and $\sum'$ is a summation over the different old channels.

From the analyticity of $T(\omega)$ in the upper half-plane of ω, (4·8) can be used also for $\omega < \omega_{\Lambda\pi}$, if we take $\rho_x(\omega)$ as $\rho_x(\omega) \sim i\sqrt{\omega_{\Lambda\pi}-\omega}$.

So we can obtain the behavior of $T(\omega)$ near $\omega_{\Lambda\pi}$ if the phase of $T_{ax}(\omega)$, $T_{xa}(\omega)$ near $\omega_{\Lambda\pi}$ is known.

By requiring $Im\, T_{aa}(\omega) = 0$ for $\omega < \omega_{\Lambda\pi}$, (see, energy spectrum of Fig. 1), we get

$$Im\, T_{aa}(\omega) = -\pi Re\, (T_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega)) \tag{4·9}$$

from (4·8), when $\omega > \omega_{\Lambda\pi}$.

Using the well-known unitarity property of the S matrix,

$$\begin{aligned}
\sigma(\omega) &= -\frac{2(2\pi)^3}{v_a} Im\, T_{aa}(\omega) \\
&= \frac{(2\pi)^4}{v_a} Re\, (T_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega))
\end{aligned} \tag{4·10}$$

where v_a is the relative velocity of the colliding particles in the initial state a. Now

$$\begin{aligned}
A(\omega) &= \frac{k_a}{4\pi}\, \sigma(\omega) \\
&= \frac{(2\pi)^3 \omega}{2} Re\, (T_{ax}(\omega)\, \rho_x(\omega)\, T_{xa}(\omega)) \propto (\omega-\omega_{\Lambda\pi})^{l+1/2}.
\end{aligned} \tag{4·11}$$

If we assume s-wave pion to be dominant, we have

$$A(\omega) \propto (\omega - \omega_{\Lambda\pi})^{1/2}. \quad \text{(see Fig. 2)} \tag{4·12}$$

If we should have a new channel $\gamma (\equiv \Sigma\pi)$ from $\omega_{\Sigma\pi}$, we can derive the following approximate relation.

$$T_{\alpha\alpha}(\omega) \approx T_{\alpha\alpha}(\omega_{\Sigma\pi}) - i\pi T_{\alpha\gamma}(\omega)\rho_\gamma(\omega) T_{\gamma\alpha}(\omega) \tag{4·13}$$

where $$\rho_\gamma(\omega) = (\omega - \omega_{\Sigma\pi})^{1/2}.$$

Putting channel $P+K=a$, $\Lambda+\pi=x$ and $\Sigma+\pi=y$, and solving $T(\omega)$ from (4·7) explicitly,

$$T_{y\alpha}(\omega_{\Sigma\pi}) = \frac{K_{y\alpha} + i\pi (K_{xx}K_{y\alpha} - K_{yx}K_{x\alpha})\rho_x}{(1+i\pi K_{\alpha\alpha}\rho_\alpha)(1+i\pi K_{xx}\rho_x) + \pi^2 K_{x\alpha}K_{\alpha x}\rho_\alpha\rho_x} \tag{4·14a}$$

$$T_{\alpha y}(\omega_{\Sigma\pi}) = \frac{K_{\alpha y} + i\pi (K_{xx}K_{\alpha y} - K_{\alpha x}K_{xy})\rho_x}{(1+i\pi K_{\alpha\alpha}\rho_\alpha)(1+i\pi K_{xx}\rho_x) + \pi^2 K_{x\alpha}K_{\alpha x}\rho_\alpha\rho_x} \tag{4·14b}$$

$$T_{\alpha\alpha}(\omega_{\Sigma\pi}) = \frac{K_{\alpha\alpha}(1+i\pi K_{xx}\rho_x) - i\pi K_{x\alpha}K_{\alpha x}\rho_x}{(1+i\pi K_{\alpha\alpha}\rho_\alpha)(1+i\pi K_{xx}\rho_x) + \pi^2 K_{x\alpha}K_{\alpha x}\rho_\alpha\rho_x}. \tag{4·14c}$$

Hence it can be known that $T_{\alpha y}(\omega_{\Sigma\pi})$ and $T_{y\alpha}(\omega_{\Sigma\pi})$ are complex. If we assume *s*-wave pion to be dominant, we have

$$A(\omega) \propto \begin{cases} (\omega_{\Sigma\pi} - \omega)^{1/2} \\ \text{or } -(\omega_{\Sigma\pi} - \omega)^{1/2} \end{cases} \quad \text{for } \omega < \omega_{\Sigma\pi}$$

$$A(\omega) \propto \begin{cases} (\omega - \omega_{\Sigma\pi})^{1/2} \\ \text{or } -(\omega - \omega_{\Sigma\pi})^{1/2} \end{cases} \quad \text{for } \omega > \omega_{\Sigma\pi} \quad \text{(see Fig. 2)}$$

Similar argument is applied to the behavior at $\omega = m_k$, where a new channel opens, giving two possibilities for each cases, when *s*-wave K meson is dominant or when *p*-wave is dominant (see Fig. 2). Thus we may obtain general behavior of $A(\omega)$ in the unphysical region.

If the kinetic energy of the incident K meson is small this contribution may be large. For higher energies, however, it contributes little.

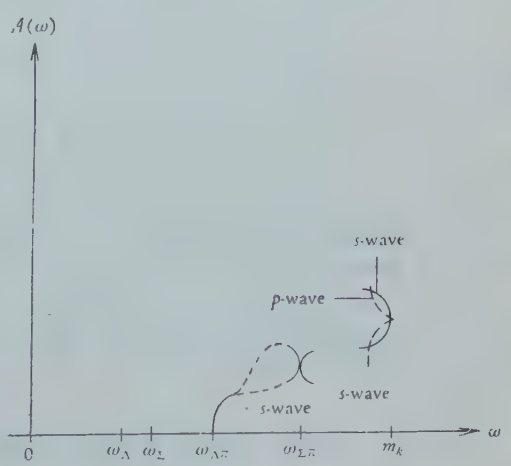

Fig. 2

§ 5. Application

The dispersion relations for *K-P* scattering can be applied to determine the type of their interactions. Following Haber-Schaim[4] let us rewrite the dispersion rela-

tions (2·10), (2·11) with (3·3), (3·4), (3·6) and (3·7) into a form more convenient for this purpose.

From the above relations we obtain

$$\omega\frac{D_-(\omega)-D_+(\omega)}{2}=\frac{2f^2\omega^2}{\omega^2-\omega_\Lambda{}^2}+\frac{2g^2\omega^2}{\omega^2-\omega_\Sigma{}^2}+\frac{\omega^2}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{\omega'^2-\omega^2}$$

$$+\frac{\omega^2}{4\pi^2}\int_{m_k}^{\infty}d\omega'k'\left[\frac{\sigma_-(\omega')-\sigma_+(\omega')}{\omega'^2-\omega^2}\right] \qquad (5\cdot1)$$

where

$$f^2=\begin{cases} f_{ps}{}^2 & ps.\ \text{case} \\ -f_s{}^2 & s.\ \text{case}. \end{cases}$$

Now making use of the following identity on the one hand,

$$\frac{1}{\omega'^2-\omega^2}=\frac{1}{\omega'^2}+\frac{\omega^2}{\omega'^2(\omega'^2-\omega^2)}, \qquad (5\cdot2)$$

and neglecting $\omega_\Lambda{}^2$ and $\omega_\Sigma{}^2$ (since $\omega_\Lambda{}^2/m_k{}^2=1.14\times10^{-2}$, $\omega_\Sigma{}^2/m_k{}^2=7.18\times10^{-2}$) on the other hand, we get a simplified expression.

$$\omega\frac{D_-(\omega)-D_+(\omega)}{2}-\frac{\omega^2}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{\omega'^2-\omega^2}$$

$$-\frac{\omega^4}{4\pi^2}\int_{m_k}^{\infty}d\omega'\frac{k'[\sigma_-(\omega')-\sigma_+(\omega')]}{\omega'^2(\omega'^2-\omega^2)}$$

$$=2F+\frac{\omega^2}{4\pi^2}\int_{m_k}^{\infty}d\omega'\frac{k'[\sigma_-(\omega')-\sigma_+(\omega')]}{\omega'^2} \qquad (5\cdot3)$$

where

$$2F=\begin{cases} 2(f_{ps}{}^2+g_{ps}{}^2) & ps.\ \text{case} \\ -2(f_s{}^2+g_s{}^2) & s.\ \text{case}. \end{cases}$$

Then the apparently complicated ω dependence indicated by the left-hand side of (5·3) boils down to a linear function of ω^2. And its value at $\omega=0$ is equal to $2F$.

The forward scattering amplitudes $D_\pm(\omega)$ may be determined from the forward differential cross sections except for their sign. From (5·3) it is expected that the experimental plot of the left-hand side of (5·3) against ω^2 will constitute a straight line. See Fig. 3.

The point at which this line crosses y-axis, $2F$, gives the value of the coupling constant.

If F is negative, the interaction is found to be of a pseudoscalar type, and the positive F means scalar interaction.

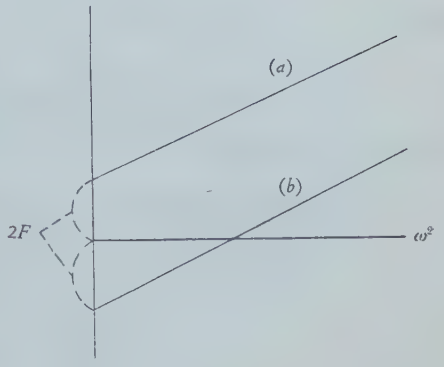

Fig. 3 The left-hand side of (5·3) plotted against ω^2.

§ 6. Discussions

Assuming the parity of K to be definite (which may mean that the parity conservation does not hold in weak interactions), we can obtain some information about the type of interactions—namely relative parity between K and Λ, Σ.

On the contrary we may adopt the parity doublet theory for K meson and assume the parity conservation to hold. In this case the situation is rather complicated.

When the parity of K is not changed in the scattering (Fig. 4a, Fig. 4b), we shall be able to determine the proportion of the type of interactions by taking linear combinations of the scattering amplitude in both interactions.

When the parity is changed in the scattering (Fig. 4c, Fig. 4d), we must derive the dispersion relations for spin-flip amplitude, which are rather cumbersome.

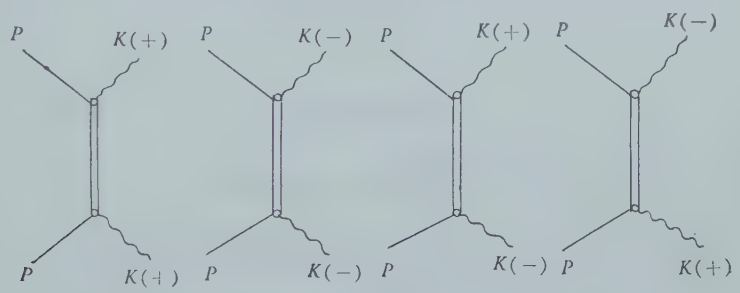

Fig. 4a. Fig. 4b. Fig. 4c. Fig. 4d.

(∓) on the right hand is parity.

We did not take K to be spin one, which seems very improbable. Derivation of the dispersion relation for this case, though, is quite analogous.

In this occasion, we derive here the dispersion relations for $T^{(1)}(\omega)$. If we increase the power in the denominators owing to bad convergence in this case, we obtain

$$D^{(1)}(\omega)-D^{(1)}(\omega_k)$$

$$=-\frac{2F(\omega^2-\omega_k{}^2)}{\omega^2}+\frac{2(\omega^2-\omega_k{}^2)}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{\omega'\frac{1}{2}A_-(\omega')}{(\omega'^2-\omega_k{}^2)(\omega'^2-\omega^2)}$$

$$+\frac{(\omega^2-\omega_k{}^2)}{4\pi}\int_0^\infty dk'\ \frac{\sigma_-(k')+\sigma_+(k')}{k'^2-k^2} \qquad (6\cdot1)$$

where

$$2F=\begin{cases}2(f_{ps}^{\,2}+g_{ps}^{\,2})/\omega_k^{\,2} & ps.\ \text{case}\\ -2(f_s^{\,2}+g_s^{\,2})/\omega_k^{\,2} & s.\ \text{case}\end{cases}$$

which can be rewritten as

$$\frac{\omega^2 D^{(1)}(\omega)-\omega_k^{\,2}D^{(1)}(\omega_k)}{(\omega^2-\omega_k^{\,2})}-\frac{\omega^2}{\pi}\int_{\omega_\Lambda\pi}^{m_k}d\omega'\,\frac{\omega' A_-(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\omega_k^{\,2})}$$

$$-\frac{\omega^4}{4\pi^2}\int_{m_k}^{\infty}d\omega'\,\frac{\sigma_-(\omega')+\sigma_+(\omega')}{k'\omega'(\omega'^2-\omega^2)}$$

$$=[-2F+D^{(1)}(\omega_k)]+\frac{\omega^2}{4\pi^2}\int_{m_k}^{\infty}d\omega'\,\frac{\sigma_-(\omega')+\sigma_+(\omega')}{k'\omega'}\,. \qquad (6\cdot 2)$$

The dispersion relations derived in this paper are useful for many purposes. When we find the spin of K mesons, through whatever means, these dispersion relations will tell us something about their parity and then about the coupling constants.

The author would like to express his deep gratitude to Dr. H. Miyazawa for his guidance and kind encouragement. Especially he owes the discussions in § 4 to Prof. G. Takeda.

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On the Longitudinal and Transverse Polarization of Positrons in μ^+-Meson Decay†

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We have calculated the density matrix which characterizes the polarization state of positrons emitted from μ^+-meson decays at rest, using general parity non-conserving interactions, the mass of electron being neglected. From the density matrix, we can tell the degree and direction of polarization of the positrons. Detection of transverse polarization perpendicular to the plane spanned by the polarization vector of the initial μ^+-meson and the propagation vector of the positron, will provide a test for whether invariance under Wigner's time reversal is valid or not.

§ 1. Introduction

So far the two component neutrino theory proposed by T. D. Lee and C. N. Yang[1] does not seem to be inconsistent with the observed asymmetries in β-decay[2] and π-μ-e decay[3] interactions. However we have as yet no clear-cut information about relative magnitude and phase of the two coupling constants contained in the μ-e decay interaction. The latter is of particular importance concerning invariance under Wigner's time reversal. To determine them, it is advisable to measure the transverse polarization of positrons emitted from μ^+-meson decay.

We have calculated the density matrix which characterizes the polarization state of the positron emitted in μ -decay at rest. We use here the general parity non-conserving interactions in the four component neutrino theory, to get also some information about the validity of the two component neutrino theory since the question does not seem to be completely settled.

From the density matrix, we can tell the direction and degree of polarization of the positrons and the probability of the positrons being emitted with a certain direction of polarization, as will be stated in § 2. In § 3 we shall describe the method of calculation of the density matrix and the results of the calculation will be stated in § 4. In § 5 the corresponding polarization in μ^--decay will be briefly discussed using SLP-theorem[4].

† Note added in proof—After the manuscript was completed, the authors were informed that H. Überall (Nuovo Cimento **6** (1957), 376) calculated the probability of electrons with a certain polarization being emitted from μ-decay in the two component neutrino theory and T. Kinoshita and A. Sirlin calculated the longitudinal polarization of electrons in muon decay with general parity non-conserving interactions. Both results are special cases of ours. Since detailed discussion about longitudinal polarization was made by T. Kinoshita and A. Sirlin, we left out that part from the manuscript.

Here we summarize the notations used in this paper. The interaction Hamiltonian density used is

$$H=\sum_i [g_i(\bar{\mu}O^i e)(\bar{\nu}O^i)+g_i'(\bar{\mu}O^i e)(\bar{\nu}O^i\gamma_5\nu)]+\text{h.c.}. \tag{1·1}$$

i	S	V	T	A	P
O^i	1	γ_μ	$\sigma_{\mu\nu}=(2i)^{-1}\gamma_{[\mu}\gamma_{\nu]}(\mu<\nu)$	$i\gamma_5\gamma_\mu$	γ_5

μ, e, ν : the quantized wave functions of a muon, an electron and a neutrino, respectively.

μ, e, ν : the corresponding c-number wave functions.

$*$: Hermite conjugate or complex conjugate.

$\boldsymbol{S}^\mu$: the polarization vector of the initial μ^+-meson, the direction of which we choose as the polar axis (z-axis). $\eta=S_z^\mu$.

λ : the degree of polarization of the μ^+-meson. When the two component neutrino theory is valid, $\lambda=1$.

$\boldsymbol{S}^e$: the polarization vector of the decay positron, i.e., the spin of the positron looked at from the rest system of the particle.

$\xi=S_z^e$: the z-component of the vector $\boldsymbol{S}^e$.

$\boldsymbol{p}$: the momentum of the positron. $p=|\boldsymbol{p}|$.

W : the maximum energy of the positron $(=m_\mu/2)$.

m_μ : the mass of the muon.

$d\Omega_p$: an element of solid angle along the direction of $\boldsymbol{p}$.

$\boldsymbol{z}$: the unit vector along the direction of the z-axis.

θ : the angle between $\boldsymbol{p}$ and $\boldsymbol{z}$.

$\boldsymbol{n}_1=\boldsymbol{p}/p$: the unit vector with the direction $\boldsymbol{p}$.

$\boldsymbol{n}_2$: the unit vector perpendicular to $\boldsymbol{p}$ but lying in the plane $(\boldsymbol{p}, \boldsymbol{z})$. $\boldsymbol{n}_2=(\boldsymbol{z}-\boldsymbol{n}_1\cos\theta)/\sin\theta$.

$\boldsymbol{n}_3$: the unit vector perpendicular to the plane $(\boldsymbol{p}, \boldsymbol{z})$. $\boldsymbol{n}_3=\boldsymbol{p}\times\boldsymbol{z}/p\sin\theta$.

$\langle\zeta|D|\zeta'\rangle$: the density matrix. We used the notation D rather than ρ, to avoid confusion with the Michel parameter.

σ : Pauli's spin matrix.

$a_{ij}=g_ig_j^*+g_i'g_j'^*$, $a_i=a_{ii}$, $a_{ji}=a_{ij}^*$.

$b_{ij}=g_ig_j'^*+g_i'g_j^*$, $b_i=b_{ii}$, $b_{ji}=b_{ij}^*$.

$x=p/W$.

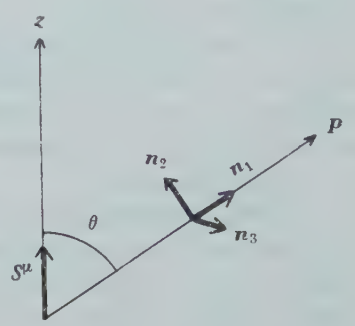

Fig. 1.

§ 2. Information derivable from the density matrix

In general the 2×2 matrix D takes the form

$$\langle\zeta|D|\zeta'\rangle=\langle\zeta|A+\boldsymbol{\sigma}\cdot\boldsymbol{B}|\zeta'\rangle. \tag{2·1}$$

With the density matrix, we can infer that the direction of polarization of the positron coincide with the direction $\boldsymbol{B}$, and the degree of polarization is given by $|\boldsymbol{B}|/A$. Next suppose that we are detecting the emitted positron through a polarizer with the direction $\boldsymbol{n}$, then the probability of the positron being emitted with the direction $\boldsymbol{n}$ of polarization is $A+\boldsymbol{n}\cdot\boldsymbol{B}$.

To prove these, we notice that if $\boldsymbol{n}$ is an arbitrary unit vector with polar angles θ_n, φ_n and $\nu=\boldsymbol{S}^e\cdot\boldsymbol{n}$, then the density matrix $\langle\nu|D|\nu'\rangle$ with reference to the direction $\boldsymbol{n}$ can be expressed in terms of that refering to a fixed direction, say the z-axis:

$$\langle\nu|D|\nu'\rangle=\sum_{\zeta,\zeta'}\langle\nu|\zeta\rangle\langle\zeta|D|\zeta'\rangle\langle\zeta'|\nu'\rangle, \tag{2·2}$$

where

$$\langle\nu|\zeta\rangle=\tfrac{1}{2}\langle\zeta|\nu\rangle=\boldsymbol{s}\cdot\boldsymbol{\sigma}, \tag{2·3}$$

and

$$\boldsymbol{s}=(\boldsymbol{n}+\boldsymbol{z})/|\boldsymbol{n}+\boldsymbol{z}| \quad \text{or} \quad (\sin\tfrac{1}{2}\theta_n\cos\varphi_n,\ \sin\tfrac{1}{2}\theta_n\sin\varphi_n,\ \cos\tfrac{1}{2}\theta_n), \tag{2·4}$$

so that

$$\langle\nu|D|\nu'\rangle=\langle\nu|A+\sigma[2\boldsymbol{s}(\boldsymbol{B}\cdot\boldsymbol{s})-\boldsymbol{B}]|\nu'\rangle. \tag{2·5}$$

The probability of observing the positron emitted with polarization $\pm\boldsymbol{n}$ is given by $w(\nu)=\langle\nu|D|\nu\rangle$, where

$$\begin{aligned}\langle\nu|D|\nu\rangle&=A\pm[2s_z(\boldsymbol{B}\cdot\boldsymbol{s})-B_z]\\&=A\pm\boldsymbol{n}\cdot\boldsymbol{B}\qquad\text{for}\quad\nu=\pm\tfrac{1}{2},\end{aligned} \tag{2·6}$$

where use is made of eq. (2·4).

According to H. A. Tolhoek and S. R. de Groot[5], the direction of polarization is defined to be that of $\boldsymbol{n}$ for which the density matrix $\langle\nu|D|\nu'\rangle$ becomes diagonal: $\langle\nu|D|\nu'\rangle=w(\nu)\,\delta(\nu,\nu')$. Then the coefficients of σ_x and σ_y in eq. (2·5) should vanish from which we obtain $2s_x(\boldsymbol{B}\cdot\boldsymbol{s})=B_x$, $2s_y(\boldsymbol{B}\cdot\boldsymbol{s})=B_y$ so that

$$\boldsymbol{n}=\boldsymbol{B}/B. \tag{2·7}$$

The degree X of polarization is defined by

$$X=[w(\tfrac{1}{2})-w(-\tfrac{1}{2})]/[w(\tfrac{1}{2})+w(-\tfrac{1}{2})]. \tag{2·8}$$

Inserting eq. (2·6) into (2·8) and using eq. (2·7), we have[6]

$$X=|\boldsymbol{B}|/A. \tag{2·9}$$

§ 3. Method of calculation

The density matrix is defined by the equation[7],

$$\langle\zeta|D|\zeta'\rangle=\sum_{\eta}\langle\zeta|T|\eta\rangle P_\eta\langle\eta|T^*|\zeta'\rangle, \tag{3·1}$$

where T denotes the transition matrix and P_η is the probability of the muon having the polarization η in the initial state so that

$$P_{1/2}+P_{-1/2}=1 \quad \text{and} \quad P_{1/2}-P_{-1/2}=\lambda. \tag{3·2}$$

We may replace $\sum_{\eta}|\eta\rangle P_{\eta}\langle\eta|$ by

$$\begin{aligned}\sum_{\eta,\eta'}|\eta\rangle\langle\eta|[\tfrac{1}{2}(1+\sigma_3)P_{1/2}+\tfrac{1}{2}(1-\sigma_3)P_{-1/2}]|\eta'\rangle\langle\eta'| \\ =\sum_{\eta,\eta'}|\eta\rangle\langle\eta|\tfrac{1}{2}(1+\lambda\sigma_3)|\eta'\rangle\langle\eta'|,\end{aligned} \tag{3·3}$$

in eq. (3·1). We have further

$$\begin{aligned}\langle\zeta|T|\eta\rangle=\sum_i\overline{e^{+(\xi)}}(\boldsymbol{p})O^i\mu^{+(\eta)}(0)\times \\ [g_i{}^*\bar{\nu}(\boldsymbol{l})O^i\nu(\boldsymbol{k})-g_i{}'^*\bar{\nu}(\boldsymbol{l})\gamma_5O^i\nu(\boldsymbol{k})](m_em_\nu{}^2/p_0kl)^{1/2}\end{aligned} \tag{3·4}$$

and

$$\begin{aligned}\langle\eta|T^*|\zeta\rangle=\sum_i\overline{\mu^{+(\eta)}}(0)O^ie^{+(\xi)}(\boldsymbol{p})\times \\ [g_i\bar{\nu}(\boldsymbol{k})O^i\nu(\boldsymbol{l})+g_i{}'\bar{\nu}(\boldsymbol{k})O^i\gamma_5\nu(\boldsymbol{l})](m_em_\nu{}^2/p_0kl)^{1/2},\end{aligned} \tag{3·5}$$

using the standard perturbation theory, where m_ν is the fictitious neutrino mass and $\boldsymbol{k}$, $\boldsymbol{l}$ are the momenta of $\bar{\nu}$ and ν respectively and the superscript $+$ means negative energy solution. We shall put $m_\nu=m_e=0$ and $p_0=p$ later on. Inserting (3·3) to (3·5) into (3·1), we are left with

$$\langle\zeta|D|\zeta'\rangle=\sum_{\eta\eta'ij}\langle\zeta|\bar{M}_j|\eta\rangle\langle\eta|\tfrac{1}{2}(1+\lambda\sigma_3)|\eta'\rangle\langle\eta'|M_i|\zeta'\rangle S_{ji}(\text{f.s.d.}), \tag{3·6}$$

after the summation and integration are made with respect to the spins and the momenta of ν and $\bar{\nu}$ which we do not observe. In eq. (3·6), we put

$$\begin{aligned}S_{ij}=\sum_{\text{spins}}\bar{\nu}(\boldsymbol{k})[g_iO^i+g_i{}'O^i\gamma_5]\nu(\boldsymbol{l})\times \\ \times\bar{\nu}(\boldsymbol{l})[g_j{}^*O^j-g_j{}'^*\gamma_5O^j]\nu(\boldsymbol{k})\,(m_\nu{}^2/kl), \\ =-(1/4kl)\,\mathrm{Tr}\,[a_{ij}\gamma kO^i\gamma lO^j-b_{ij}O^j\gamma kO^i\gamma l\gamma_5], \quad \text{where} \quad \gamma k=\gamma_\mu k_\mu,\end{aligned} \tag{3·7}$$

and

$$\begin{aligned}\langle\eta|M_i|\zeta\rangle=\bar{\mu}^{(\eta)}(0)O^ie^{(\xi)}(\boldsymbol{p})\,(m_e/p_0)^{1/2}, \\ \langle\zeta|\bar{M}_i|\eta\rangle=\bar{e}^{(\xi)}(\boldsymbol{p})O^i\mu^{(\eta)}(0)\,(m_e/p_0)^{1/2}.\end{aligned} \tag{3·8}$$

(f.s.d.) stands for the final state density,

$$(\text{f.s.d.})=\int d^3\boldsymbol{k}\int d^3\boldsymbol{l}\,\delta(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{l})\,\delta(m_\mu-p-k-l)\,d^3\boldsymbol{p}. \tag{3·9}$$

If we put $d^3\boldsymbol{p}=p^2dpd\Omega_p$ and $d^3\boldsymbol{k}=k^2dkd\Omega_k$ and perform integrations with respect to $\boldsymbol{l}$ and $k=|\boldsymbol{k}|$, we have

$$(\text{f.s.d.})=\int d\Omega_kk^2l(m_\mu-p+p\cos\theta_k)^{-1}\,p^2dpd\Omega_p, \tag{3·10}$$

where $\theta_k=<(\boldsymbol{k},\boldsymbol{p})$.

We can express $e^{+(\xi)}(\boldsymbol{p})$ as[8]

$$e^{+(\xi)}(\boldsymbol{p}) = -[2m_e(m_e+p_0)]^{-1/2}(i\gamma p+m_e)e^{+(\xi)}(0), \tag{3·11}$$

where

$$e^{+(\xi)}(0)=\mu^{+(\xi)}(0)=|\tilde{\xi}\rangle, \tag{3·12}$$

and

$$\begin{aligned}|\tilde{\xi}\rangle &= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{for} \quad \tilde{\xi}=+\tfrac{1}{2} \\ &= \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad \text{for} \quad \tilde{\xi}=-\tfrac{1}{2}.\end{aligned} \tag{3·13}$$

Inserting (3·11) into eq. (3·8) and letting $m_e\to 0$, we obtain

$$\begin{aligned}\langle\eta|M_i|\zeta\rangle &= (\sqrt{2}\,p)^{-1}\langle\eta|O^i i\gamma p|\zeta\rangle, \\ \langle\zeta|\bar{M}_i|\eta\rangle &= (\sqrt{2}\,p)^{-1}\langle\zeta|i\gamma pO^i|\eta\rangle.\end{aligned} \tag{3·14}$$

If we replace γ_k, γ_4 by $\rho_2\sigma_k$, ρ_3 and use $\langle\eta|\rho_1|\zeta\rangle=\langle\eta|\rho_2|\zeta\rangle=0$, $\langle\eta|\rho_3|\zeta\rangle=\langle\eta|-1|\zeta\rangle$ in Eq. (3·14), we may regard $|\eta\rangle$ and $|\zeta\rangle$ as two component wave functions, i.e., we have only to retain the third and fourth components of (3·13). Then we may put $\sum|\eta\rangle\langle\eta|=1$ in eq. (3·6).

It is, however, often more convenient to use (3·14) just as they stand. In this case we must put $\sum_\eta|\eta\rangle\langle\eta|=(1-\gamma_4)/2$. Thus we get

$$\langle\zeta|D|\zeta'\rangle = -(8p^2)^{-1}\langle\zeta|\gamma pO^j(1+i\lambda\gamma_3\gamma_5)(1-\gamma_4)O^i\gamma p|\zeta'\rangle S_{ij}(\text{f.s.d.}), \tag{3·15}$$

where use has been made of the relation,

$$\tfrac{1}{2}(1-\gamma_4)(1+\lambda\sigma_3)\tfrac{1}{2}(1-\gamma_4)=(1+i\lambda\gamma_3\gamma_5)\tfrac{1}{2}(1-\gamma_4).$$

However, $\langle\zeta|$ and $|\zeta'\rangle$ may be regarded as two component in the above-mentioned sense.

The subsequent calculation will be somewhat tedious but straightforward. So we shall immediately write down the result obtained, in the next section.

§ 4. Result

The result splits into three terms (S, P), (V, A) and T. Interference terms between $(S, P$ or $T)$ and $(V$ or $A)$ appear only if polarization of ν and/or $\bar{\nu}$ are observed. Interference terms between $(S$ or $P)$ and T vanish unless we observe momenta of ν and/or $\bar{\nu}$.

We express the density matrix as

$$D=(\pi W^5/3)[A+\boldsymbol{\sigma}\cdot\boldsymbol{B}]x^2dxd\Omega_p, \tag{4·1}$$

and

$$\boldsymbol{B}=\alpha\boldsymbol{n}_1+\beta\boldsymbol{n}_2+\gamma\boldsymbol{n}_3. \tag{4·2}$$

Then we have*

$$A=c[3(1-x)+2\rho(\tfrac{4}{3}x-1)-\lambda\{\rho_1(1-x)-2\rho_2(\tfrac{4}{3}x-1)\}\cos\theta], \tag{4·3}$$

$$c=a_S+a_P+4(a_V+a_A)+6a_T,$$

$$\rho=3[a_V+a_A+2a_T]/c, \quad \text{the Michel parameter},$$

$$\rho_1=[3(b_{SP}+b_{PS})-4(b_{VA}+b_{AV})-14b_T]/c,$$

$$\rho_2=3[b_{VA}+b_{AV}+2b_T]/c,$$

$$\alpha=c'[-3(1-x)+2\rho'(\tfrac{4}{3}x-1)+\lambda\{\rho_1'(1-x)+2\rho_2'(\tfrac{4}{3}x-1)\}\cos\theta], \tag{4·4}$$

$$c'=b_{SP}+b_{PS}-4(b_{VA}+b_{AV})+6b_T,$$

$$\rho'=3[b_{VA}+b_{AV}-2b_T]/c',$$

$$\rho_1'=[3(a_S+a_P)+4(a_V+a_A)-14a_T]/c',$$

$$\rho_2'=3[a_V+a_A-2a_T]/c',$$

$$\beta=\lambda[3(a_S-a_P)(1-x)+2(a_V-a_A)]\sin\theta, \tag{4·5}$$

$$\gamma=i\lambda[3(b_{SP}-b_{PS})(1-x)+2(b_{AV}-b_{VA})]\sin\theta, \tag{4·6}$$

in the right handed coordinate system.

If $g_i'=0$ $(i=S,V,T,A,P)$, the probability of the positron being emitted with longitudinal polarization will become

$$(\pi W^5/3)(A+\alpha)x^2dxd\Omega_p$$

$$=(\pi W^2/3)[3(|g_S|^2+|g_P|^2)(1-x)+2(|g_V|^2+|g_A|^2)(3-2x)$$

$$+2|g_T|^2(3-x)+\lambda\{3(|g_S|^2+|g_P|^2)(1-x)-2(|g_V|^2+|g_A|^2)(1-2x)$$

$$-2|g_T|^2(1+x)\}\cos\theta]x^2dxd\Omega_p,$$

using (2·6). This coincides with the calculation made by S. Tanaka and K. Matumoto[9] apart from the difference of ordering of the wave functions in the interaction Hamiltonian.

In the two component neutrino theory we have

$$A=(|g_V|^2+|g_A|^2)(3-2x)-\lambda(g_Vg_A^*+g_Ag_V^*)(1-2x)\cos\theta,$$

$$\boldsymbol{B}=\{(g_Vg_A^*+g_Ag_V^*)(3-2x)-\lambda(|g_V|^2+|g_A|^2)(1-2x)\cos\theta\}\boldsymbol{n}_1+$$

$$+\lambda\{(|g_V|^2-|g_A|^2)\boldsymbol{n}_2-i(g_Vg_A^*-g_Ag_V^*)\boldsymbol{n}_3\}\sin\theta.$$

In particular the probability of the positron being longitudinally polarized in this theory becomes

$$\frac{\pi W^5}{3}[A+\alpha]x^2dxd\Omega_p=(4\pi W^5/3)|g_V+g_A|^2\{(3-2x)-\lambda(1-2x)\cos\theta\}x^2dxd\Omega_p$$

---

* The result refers to emission of non-identical neutrinos. For the case of emission of identical neutrinos we have only to put $g_T=g_T'=g_V=g_A'=0$.

The same result was already obtained by T. Kinoshita and A. Sirlin[10] and H. Segawa and S. Hori[11]. If the transverse polarization turns out to be energy independent apart from the factor x^2dx, the fact will provide the two component neutrino theory with another experimental support.*

The interaction Hamiltonian permitted under the hypothesis of unobservability of sign of mass[12] is

$$H=\sum_{i=V,A} f_i(\bar{\mu}(1\pm\gamma_5)O^i e)\,(\bar{\nu}O^i\nu),$$

or

$$\sum_{i=S,T,P} f_i(\bar{\mu}(1\pm\gamma_5)O^i e)\,(\bar{\nu}O^i\nu).$$

Transforming this Hamiltonian back into the form (1·1), we obtain

$$g_i=f_i \quad (i=S,\cdots,P)\ ;\ g_S'=\pm f_P,\ g_V'=\pm f_A,\ g_T'=\pm f_T,$$
$$g_A'=\pm f_V,\ g_P'=\pm f_S,$$

so that

$$a_S=a_P=|f_S|^2+|f_P|^2,\quad a_T=2|f_T|^2,\quad a_V=a_A=|f_V|^2+|f_A|^2;$$
$$b_{SP}=b_{PS}=\pm(|f_S|^2+|f_P|^2),\quad b_T=\pm 2|f_T|^2,\quad b_{VA}=b_{AV}=\pm(|f_V|^2+|f_A|^2).$$

Therefore we see $\beta=\gamma=0$ and the polarization is 100% longitudinal in this case. The transverse polarization now vanishes, whether the theory is invariant under the time reversal or not.

§ 5. Discussions

The transverse polarization along the direction $\boldsymbol{n}_3$ appears only when invariance under the time reversal does not hold, although the converse does not necessarily hold as stated at the end of the previous section. This can easily be understood since the triple product $\boldsymbol{S}^e\cdot\boldsymbol{p}\times\boldsymbol{S}^\mu$ changes its sign under the time reversal.

We treated here μ^+-decay rather than μ^--decay taking into account facility of the experiment. The result in the latter case can readily be deduced using SLP-theorem[4]. According to the theorem, operation of the charge conjugation C is equivalent to that of the product of the space inversion P and the time reversal T. To apply the theorem we decompose the vector $\boldsymbol{B}$ into three components; one along $\boldsymbol{p}$, one along $\boldsymbol{z}$ and one along $\boldsymbol{p}\times\boldsymbol{z}$. If we take the direction of $\boldsymbol{S}^\mu$ as the z-axis in both cases, then $\boldsymbol{S}^e\cdot\boldsymbol{S}^\mu$ and $\boldsymbol{S}^e\cdot\boldsymbol{p}\times\boldsymbol{S}^\mu$ remain unaltered under the operation of C or PT, while $\boldsymbol{S}^e\cdot\boldsymbol{p}$ changes its sign. So the behavior of $\boldsymbol{B}$ under the operation of C will be rather complicated and the anisotropy $\boldsymbol{S}^\mu\cdot\boldsymbol{p}$ changes too, i.e., $\theta\to\pi-\theta$, since $\boldsymbol{S}^\mu\cdot\boldsymbol{p}=S^\mu p\cos\theta$. However experimentalists will perhaps prefer to compare them choosing the direction of the momentum $\boldsymbol{p}_\mu$ of

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* Put $a_s=a_p=0$ in eq. (4·5) and $b_{SP}=b_{PS}=0$ in eq. (4·6). Then β and γ become independent of x.

the muon to be the same in both cases.* If we choose the direction of $\boldsymbol{p}_\mu$ as the z-axis, then $\boldsymbol{S}^e \cdot \boldsymbol{p}$, $\boldsymbol{S}^e \cdot \boldsymbol{p}_\mu$ and $\boldsymbol{S}^e \cdot \boldsymbol{p} \times \boldsymbol{p}_\mu$ all change their signs under the operation of C or PT, while the anisotropy $\boldsymbol{p} \cdot \boldsymbol{p}_\mu$ remains the same. So we have

$$A \to A, \quad \boldsymbol{B} \to -\boldsymbol{B}$$

in the case of μ^--meson decay.

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* In the following discussion we neglect the effect of Coulomb fields which would actually be important in the case of μ^--decay, especially in μ^--decay at rest. And we note that $\boldsymbol{S}^\mu \cdot \boldsymbol{p}$ changes the sign under the operation of C, so the direction of $\boldsymbol{S}^\mu$ relative to $\boldsymbol{p}_\mu$ in μ^--decay is opposite to that of $\boldsymbol{S}^\mu$ in μ^+-decay.

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Theory of Cohesive Energy of LiH Crystal

—The Method of Semi-Localized Crystalline Orbital (SLCO), I—

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The method of semi-localized crystalline orbital (SLCO) is presented to investigate the influence of homopolar binding on the cohesive energies of ionic crystals. The computations are performed for LiH crystal. As a result, it is shown that the discrepancy between the theoretical value of the cohesive energy of LiH crystal, which was calculated by Lundqvist, and the observed one is removed by taking homopolar binding into account.

Furthermore, the relation between the energy band structure and SLCO is discussed and it is shown that SLCO corresponds to Wannier's functions of the valence and conduction bands for the case of LiH crystal where the valence and conduction bands are non-degenerate.

§ 1. Introduction

Since the calculation of Hylleraas[1] on LiH crystal, the quantum mechanical computations of the cohesive energies of ionic crystals have been carried out by many authors[2] by using the method of Heitler-London. In these computations, only ionic states have been taken into account. (We call this approximation "ionic approximation" hereafter.) On the other hand, Kanda's experiment[3] of quadrupole moment on alkali-halides which are the prototypes of ionic crystals shows that these crystals contain some amount of valence bonding. Lundqvist[4] has carried out the detailed computation of the cohesive energy of LiH crystal by the use of ionic approximation and has shown that there is a considerable discrepancy between the theoretical value (199.1 kcal/mol) and the observed one (217 ± 7 kcal/mol). This fact seems to indicate the importance of the influence of homopolar binding on the cohesive mechanism of LiH crystal. Furthermore, in cases of PbS, ZnS, CaF_2, Cu_2O and other crystals having the same structres as these crystals, the homopolar binding may play a role comparable with or more important than ionic binding in their cohesive mechanisms. The crystal structures of these crystals may easily be understood by taking account of homopolar binding.

A quantum mechanical treatment of the cohesive energies of valence crystals has been carried out by Schmid[5] on diamond. He has shown that the usual formulation based on the Slater-Pauling theory of directional valency fails to yield any stable binding but satisfactory results are achieved when this theory is reformulated to allow for ionic states in addition to the electron-pair bonds. Asano and Tomishima[6], furthermore, have applied Schmid's method to the computation of the cohesive energy of ZnS. The influence of the valence states upon the ionic binding has usually been worked through the

procedure of configuration interaction. However, in order to take into account the effect mentioned above within the frame of the method of Heitler-London, in which the total wave function is a single determinant consisting of one-electron functions, the method of semi-localized crystalline orbital (which is called "SLCO" hereafter) presented in the present paper is useful.* The aim of the present paper is to formulate the method of SLCO and to apply it to the computation of the cohesive energy of LiH crystal** which has the simplest electronic structure among all ionic crystals and the cohesive energy of which has been already calculated in detail through the method of ionic approximation by Lundqvist.[4)]

As is mentioned above, the first computation of the cohesive energy of LiH crystal has been carried out by Hylleraas and then detailed computations have been carried out by Lundqvist.[4)] As a result, it has been shown that the theoretical value of the cohesive energy is smaller than the observed one by about 18 kcal/mol. In the present paper, making use of SLCO, we shall show that the discrepancy can be almost removed by taking into account the influence of homopolar binding.

On the basis of the considerations on the character of the homopolar binding which has to be taken into account as configuration interaction, a set of SLCO is introduced in § 2. The set of SLCO is defined by the use of a variational matrix ε. A trial form for the matrix ε which contains only a single variational parameter is proposed. In § 3, the expression of the lattice energy containing the influence of the homopolar binding is given. In § 4, the relationship between the set of SLCO and the energy band structure of LiH crystal is discussed and it is shown that the set of SLCO is equal to the (approximate) Wannier functions of the valence and conduction bands. In § 5, the numerical result for the lattice energy obtained in § 3 is given and discussed.

§ 2. SLCO

In their computations of the cohesive energy of LiH crystal, Hylleraas and Lundqvist have used a single Slater determinantal function consisting of 1s spin-orbitals of free H^- and Li^+ ions. The discrepancy between the theoretical value obtained by Lundqvist through such an ionic approximation and the observed one seems to indicate the importance of homopolar binding in the cohesion of LiH. In the ionic structure, each 1s orbital on the H^- ions is occupied by two electrons having anti-parallel spins with each other. In the valence structure, one of the two electrons is excited into a 2s orbital on one of six Li^+ ions around the H^- ion, forming an electron-pair (homopolar) bond with the remaining one. Such an electron-pair bond may resonate on the six Li^+ ions. In order to take account of such a valence structure within the framework of the method of

* After our computations had been almost completed, we have noticed that T. B. Grimley (Proc. Phys. Soc. **A 70** (1957), 123) discussed the effect of homopolar bindings in LiF crystal by the use of the second-order perturbation theory.

** Applications of SLCO to other crystals will be published in the form of separated papers in the near future.

Heitler-London using a single Slater determinant, we need to take, in place of a 1s spin-orbital of each H^- ion, a linear combination of it and 2s spin-orbitals of the six Li^+ ions around it. The coefficients of the linear combination have to be determined so as to minimize the expectation value of the lattice energy. This procedure can be carried out, without any difficulty, by making use of SLCO.*

a). Construction of SLCO

Let us write the set of $2N$ atomic 1s spin-orbitals on the H^- ions as $\boldsymbol{u}^0=\{u_\alpha^0, u_b^0,\cdots\}$ and the set of $2N$ atomic 2s spin-orbitals on the Li^+ ions as $\boldsymbol{v}^0=\{v_\alpha^0, v_\beta^0,\cdots\}$ where $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ are row matrices,** and the indices $a, b,\cdots$ refer to the H^- ions and the indices $\alpha, \beta,\cdots$ to the Li^+ ions. The sets of atomic spin-orbitals $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ are not orthogonal at the observed lattice constant, on account of their appreciable overlap. Since it is convenient to use an orthogonalized set of one-electron functions, we introduce an orthonormal set $\boldsymbol{\varphi}=\{\boldsymbol{u}, \boldsymbol{v}\}$ of atomic orbitals, of which the first $2N$ elements are $\boldsymbol{u}$ and the last $2N$ elements are $\boldsymbol{v}$, as follows. First we orthogonalize $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ separately, that is,

$$\left.\begin{aligned}\boldsymbol{u}&=\boldsymbol{u}^0\boldsymbol{\Delta}_1^{-1/2}\\ \boldsymbol{v}'&=\boldsymbol{v}^0\boldsymbol{\Delta}_2^{-1/2}\end{aligned}\right\} \qquad (2\cdot1)$$

where

$$\left.\begin{aligned}&\boldsymbol{\Delta}_1=(\mathbf{1}+\mathbf{S}_1),\qquad \boldsymbol{\Delta}_2=(\mathbf{1}+\mathbf{S}_2),\\ &\mathbf{S}_{1,ab}=\int u_a^0(\boldsymbol{x})\,u_b^0(\boldsymbol{x})\,d\tau-\delta_{ab},\quad \mathbf{S}_{2,\alpha\beta}=\int v_\alpha^0(\boldsymbol{x})\,v_\beta^0(\boldsymbol{x})\,d\tau-\delta_{\alpha\beta}\end{aligned}\right\} \qquad (2\cdot2)$$

and $\boldsymbol{x}$ is a combination of space coordinate $\boldsymbol{r}$ and a spin coordinate. Then, orthogonalizing $\boldsymbol{v}'$ to $\boldsymbol{u}$, we get $\boldsymbol{v}$, that is

$$\boldsymbol{v}=\boldsymbol{u}\boldsymbol{A}+\boldsymbol{v}'\boldsymbol{B} \qquad (2\cdot3)$$

where

$$\left.\begin{aligned}&\boldsymbol{A}=-\mathfrak{S}(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2},\qquad \boldsymbol{B}=(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2},\\ &\mathfrak{S}=\boldsymbol{\Delta}_1^{-1/2}\mathbf{s}\boldsymbol{\Delta}_2^{-1/2},\end{aligned}\right\} \qquad (2\cdot4)$$

$$\mathbf{s}_{a\alpha}=\int u_a^0(\boldsymbol{x})\,v_\alpha^0(\boldsymbol{x})\,d\tau. \qquad (2\cdot5)$$

Now let us introduce the unitary $(4N\times4N)$ matrix with the block form

$$\mathbf{X}=\begin{bmatrix}(\mathbf{1}-\mathcal{E}\mathcal{E}^+)^{1/2} & \mathcal{E}\\ \mathcal{E}^+ & -(\mathbf{1}-\mathcal{E}^+\mathcal{E})^{1/2}\end{bmatrix}, \qquad (2\cdot6)$$

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* Another example of such a treatment is " an extention of the valence bond method " proposed by J. C. Slater, Quarterly Prog. Rep. Solid-State and Molecular Theory Group, MIT. Jan. 15 (1955).

** In what follows, we ignore for simplicity the wave mechanical behaviors of ls orbitals of the Li^+ ions and treat them as classical charge clouds. This approximation may be allowed as far as the core-distortion of Li^+ ion is negligibly small.

and define SLCO $\boldsymbol{\phi}$ by

$$\boldsymbol{\phi} = \boldsymbol{\varphi} \boldsymbol{X}. \tag{2·7}$$

The $2N \times 2N$ matrix $\mathcal{E}$ appearing in (2·6), which has elements $\mathcal{E}_{\alpha\alpha}$, is determined by the following variation principle. We express the total wave function of the crystal by

$$\Psi_0(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2N}) = \frac{1}{\sqrt{(2N)!}} \sum_P (-)^P P \prod_{\alpha}^{2N} \phi_\alpha(\boldsymbol{x}_\alpha), \tag{2·8}$$

by using the first $2N$ elements of $\boldsymbol{\phi}$. Then we set up the variation principle

$$\delta E = 0 \tag{2·9}$$

for the expectation value of the lattice energy

$$E = \int \Psi_0^*(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2N}) \mathcal{H} \Psi_0(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2N}) d\tau_{(2N)}, \tag{2·10}$$

where $\mathcal{H}$ is the Hamiltonian of the crystal.

When $\mathcal{E}=0$, ϕ_α is equal to u_α and ϕ_α to v_α. In this case, therefore, Ψ_0 is reduced to the total wave function used by Lundqvist.* Hence all the effects of the valence states appear through $\mathcal{E}$. ϕ_α (or ϕ_α) is located around the atom, on which $u_\alpha{}^0$ (or $v^0{}_\alpha$) is centered, and its form is determined so as to minimize the expectation value of the crystal energy. This is the reason for calling $\boldsymbol{\phi}$ "**SLCO**". The method of SLCO may be applicable to other ionic and valence crystals. However, how to select $\boldsymbol{u}$ and $\boldsymbol{v}$ in such cases has to be determined case by case.

b) Trial form for $\mathcal{E}$

Though the matrix $\mathcal{E}$ has to be determined by the variation principle (2·9), it is difficult to perform it exactly. So we set up a trial form for $\mathcal{E}$. Here we assume that $\boldsymbol{X}$ and $\mathcal{E}$ have the same translational and rotational symmetry as the crystal. This assumption does not mean any restriction on the variation principle (2·9). It may be understood by remembering that a complete set of Bloch functions of a filled band is transformed by a unitary transformation to the complete set of Wannier functions of the band which has the same translational and rotational symmetry as the lattice.

ϕ_α corresponds to one electron state, in which an electron spends a large part of its time on the spin orbital u_α and the rest on the spin-orbitals v_α's around u_α. One of the wave functions describing such a behavior of the electron may be expressed as

$$\phi \sim (1+6A^2)^{-1/2} \{u_\alpha + A \sum_{\alpha}^{b} v_\alpha\}, \tag{2·11}$$

where the summation $\sum_{\alpha}^{b}$ is taken over the six v_α's neighbouring u_α and having the same

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* In our wave function Ψ_0 expressed by (2.8), 1s spin-orbitals of the Li^+ ions are dropped.

spin state as u_α. Comparing (2·11) with (2·7) and remembering the symmetric property of $\mathcal{E}$, we can take a trial form for $\mathcal{E}$ as

$$\left.\begin{aligned} \mathcal{E}_{a\alpha} &= A \quad \text{when } \alpha \text{ is subject to the same spin-state as } a \text{ and locates on one of the six nearest neighbour sites of } a, \\ &= 0 \quad \text{otherwise,} \end{aligned}\right\} \tag{2·12}$$

up to second order by treating A as a first-order small quantity. Thus

$$\left.\begin{aligned} (\mathcal{E}\mathcal{E}^+)_{ab} &= 6A^2 \quad \text{when } a=b, \\ &= 2A^2 \quad \text{when } b \text{ is subject to the same spin-state as } a \text{ and locates on one of the second-order neighbours of } a, \\ &= A^2 \quad \text{when } b \text{ is subject to the same spin-state as } a \text{ and locates on one of the 4th-order neighbours of } a, \\ &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{2·13}$$

Use of our trial form for $\mathcal{E}$ reduces the variation problem (2·9) to that of the single variation parameter A.

§ 3. Expression of the crystal energy

The Hamiltonian* of LiH crystal is

$$\mathcal{H} = \sum_{\mu}^{2N} O_1(\mu) + \sum_{\mu \neq \nu}^{2N}\sum^{2N} O_2(\mu, \nu), \tag{3·1}$$

where

$$O_1(\mu) = -\frac{1}{2}\Delta_\mu - \sum_{i}^{R^-}\frac{1}{|\boldsymbol{r}_\mu - \boldsymbol{R}_i|} - \sum_{j}^{M^+} v(\boldsymbol{r}_\mu - \boldsymbol{R}_j) \tag{3·2}$$

$$O_2(\mu, \nu) = \frac{1}{2}\frac{1}{|\boldsymbol{r}_\mu - \boldsymbol{r}_\nu|}. \tag{3·3}$$

Substituting (2·8), (3·1), (3·2) and (3·3) into (2·10) and performing some mathematical manipulations, we have

$$\begin{aligned} E = E_0 &+ 2\sum_{a}^{2N}\sum_{\alpha}^{2N}\{(\mathbf{1}-\mathcal{E}\mathcal{E}^+)\mathcal{E}\}^{(1)}_{a\alpha}[a|H_F(\boldsymbol{u})|\alpha] \\ &- \sum_{a}^{2N}\sum_{b}^{2N}(\mathcal{E}\mathcal{E}^+)^{(2)}_{ab}[a|H_F(\boldsymbol{u})|b] + \sum_{\alpha}^{2N}\sum_{\beta}^{2N}(\mathcal{E}^+\mathcal{E})^{(2')}_{\alpha\beta}[\alpha|H_F(\boldsymbol{u})|\beta] \\ &- \sum_{a}^{2N}\sum_{b}^{2N}\sum_{\alpha}^{2N}\sum_{\beta}^{2N}\{(\mathbf{1}-\mathcal{E}\mathcal{E}^+)^{1/2}\mathcal{E}\}_{a\alpha}\{(\mathbf{1}-\mathcal{E}\mathcal{E}^+)^{1/2}\mathcal{E}\}_{b\beta} \end{aligned}$$

* Atomic units are used throughout.

$$\times (2[ab|\alpha\beta]^{(3)}-[ab|\beta\alpha]^{(3')}-[a\alpha|b\beta]^{(3')})$$
$$-\sum_{a}^{2N}\sum_{b}^{2N}\sum_{c}^{2N}\sum_{\alpha}^{2N}\{(\mathbf{1}-\varepsilon\varepsilon^{+})^{1/2}\varepsilon\}_{a\alpha}(\varepsilon\varepsilon^{+})_{bc}(2[ab|\alpha c]-[ab|c\alpha]-[a\alpha|bc])$$
$$-\sum_{a}^{2N}\sum_{\alpha}^{2N}\sum_{\beta}^{2N}\sum_{\gamma}^{2N}\{(\mathbf{1}-\varepsilon\varepsilon^{+})\varepsilon\}_{a\alpha}(\varepsilon^{+}\varepsilon)_{\beta\gamma}(2[a\beta|\alpha\gamma]-[a\beta|\gamma\alpha]-[a\gamma|\beta\alpha])$$
$$+\frac{1}{2}\sum_{a}^{2N}\sum_{b}^{2N}\sum_{c}^{2N}\sum_{d}^{2N}(\varepsilon\varepsilon^{+})_{ab}(\varepsilon\varepsilon^{+})_{cd}([ac|bd]-[ac|db])$$
$$-\sum_{a}^{2N}\sum_{\beta}^{2N}\sum_{\alpha}^{2N}\sum_{\beta}^{2N}(\varepsilon\varepsilon^{+})_{ab}(\varepsilon^{+}\varepsilon)_{\alpha\beta}([a\alpha|b\beta]-[a\alpha|\beta b])$$
$$+\frac{1}{2}\sum_{\alpha}^{2N}\sum_{\beta}^{2N}\sum_{\gamma}^{2N}\sum_{\delta}^{2N}(\varepsilon^{+}\varepsilon)_{\alpha\beta}(\varepsilon^{+}\varepsilon)_{\gamma\delta}([\alpha\gamma|\beta\delta]-[\alpha\gamma|\delta\beta]), \qquad (3\cdot4)$$

where E_0 is the crystal energy at $\varepsilon=0$, which was calculated by Lundqvist, $H_F(\boldsymbol{u})$ is the effective Hartree-Fock Hamiltonian :

$$H_F(\boldsymbol{u})\psi(\boldsymbol{x})=\Big\{-\frac{1}{2}\Delta-\sum_{i}^{N'}\frac{1}{|\boldsymbol{r}-\boldsymbol{R}_i|}+\sum_{j}^{N'}r\cdot(\boldsymbol{r}-\boldsymbol{R}_j)+\sum_{a}^{2N}\int\frac{|u_a(\boldsymbol{x}_1)|^2}{|\boldsymbol{r}-\boldsymbol{r}_1|}d\tau_1$$
$$-\sum_{a}^{2N}\frac{u_a(\boldsymbol{x})}{\psi(\boldsymbol{x})}\int\frac{u_a^{*}(\boldsymbol{x}_1)\psi(\boldsymbol{x}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|}d\tau_1\Big\}\psi(\boldsymbol{x}), \qquad (3\cdot5)$$

and the bracket notations [ ]'s are used to express the following quantities :

$$\left.\begin{aligned}&[a|H_F(\boldsymbol{u})|b]=\int u_a^{*}(\boldsymbol{x})H_F(\boldsymbol{u})u_b(\boldsymbol{x})\,d\tau\\&[ab|cd]=\int u_a^{*}(\boldsymbol{x}_1)u_b^{*}(\boldsymbol{x}_2)\frac{1}{r_{12}}u_c(\boldsymbol{x}_1)u_d(\boldsymbol{x}_2)\,d\tau_1 d\tau_2\\&\text{and so on.}\end{aligned}\right\} \qquad (3\cdot6)$$

We shall utilize the bracket notation ( ) for quantities which are obtained by replacing orthogonalized wave functions $\boldsymbol{u}$ and $\boldsymbol{v}$ with non-orthogonalized atomic wave functions $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ in the bracket notations [ ]'s defined by (3·6).

It is convenient for numerical calculations to express (3·4) by using the free atomic spin-orbitals $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$. Here we introduce the following approximations. (1) Regarding A as the first-order small quantity, we neglect all small terms of at least the third order. (2) We reduce all many-center Coulomb integrals with respect to $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ to one- or two-center integrals by using the so-called Mulliken approximation,[7] for example

$$(ab|\alpha\beta)=\tfrac{1}{4}\boldsymbol{s}_{a\alpha}\boldsymbol{s}_{b\beta}\{(ab|ab)+(a\beta|a\beta)+(\alpha b|\alpha b)+(\alpha\beta|\alpha\beta)\}. \qquad (3\cdot7)$$

The use of this approximation allows us to reduce the expression of the crystal energy to a somewhat compact form, though it may inevitably bring some amount of error in our calculations. Thus we get from (3·4) the expression of E expressed by $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$:

$$E=E_0+\sum_{a}^{2N}\{\mathcal{E}^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\}_{a\alpha}^{(\alpha)}((\alpha|H_F(\boldsymbol{u}^0)|\alpha)-(a|H_F(\boldsymbol{u}^0)|a))+F^{(1)}+G^{(1)}$$

$$+\sum_{a}^{2N}(\varepsilon\varepsilon^{+})_{a\alpha}\left((\alpha|\overset{(b)}{H_F}(\boldsymbol{u}^0)|\alpha)-(a|H_F(\boldsymbol{u}^0)|a)\right)$$
$$+F^{(2)}+G^{(2)}$$
$$+\frac{1}{2}\sum_{\alpha(\text{or }\alpha)}^{2N}\sum_{b(\text{or }\beta)}^{2N}\{\varepsilon^{-1}\xi(1-\xi^{+}\xi)^{-1/2}\}_{a\alpha}\underset{(\text{or }\alpha\alpha)}{\{\varepsilon^{-1}\xi(1-\xi^{+}\xi)^{-1/2}\}_{b\beta}}$$
$$\times\left((ab|ab)-(a\beta|a\overset{(c)}{\beta})+(\alpha\beta|\alpha\beta)-(\alpha b|\alpha b)\right)$$
$$+H^{(2)} \tag{3·8}$$

where $H_F(\boldsymbol{u}^0)$ is the effective Hamiltonian obtained by replacing $\boldsymbol{u}$ with $\boldsymbol{u}^0$ in the expression (3·5) for $H_F(\boldsymbol{u})$, and $F^{(1)}$, $G^{(1)}$, $F^{(2)}$, $G^{(2)}$ and $H^{(2)}$ have considerably complicated expressions which are listed in the Appendix.

The main part of the second term (1) on the right-hand side of (3·4) is transformed into (*a*) on the right-hand side of (3·8) and the remaining part into $F^{(1)}$ and $G^{(1)}$. The main parts of the third and fourth terms, (2) and (2′), in (3·4) are transformed into (*b*) and the remaining parts into $F^{(2)}$ and $G^{(2)}$. The terms denoted by (3) and (3′) are transformed into (*c*) and $H^{(2)}$, respectively.

§ 4. The energy-band structure and SLCO

Before proceeding to the numerical calculations of the lattice energy, we shall discuss the relation between the energy-band structure of LiH crystal and SLCO.

To start with, we prove the theorem :

$$\int\phi_\alpha^{*}(\boldsymbol{x})\,H_F(\boldsymbol{\phi})\,\phi_\alpha(\boldsymbol{x})\,d\tau=0, \tag{4·1}$$

where $H_F(\boldsymbol{\phi})$ is obtained from the expression (3·5) by replacing u_α's with ϕ_α's. The proof is easy. Let $\Psi_{\alpha\to\alpha}$ be the wave function obtained from Ψ_0 for the ground state by replacing ϕ_α with ϕ_α. Then, from the variation principle (2·9) it results that

$$\int\Psi_{\alpha\to\alpha}^{*}\mathcal{H}\Psi_0 d\tau_{(2N)}=0, \tag{4·2}$$

which is easily shown to be reduced to (4·1).

Now let us consider a system consisting of the perfect crystal plus an excess (conduction) electron introduced into the crystal. The state that the electron occupies a spin-orbital ϕ_α is expressed by

$$\Psi_{(2N+1),\alpha}=\frac{1}{\sqrt{(2N+1)!}}\sum_{P(2N+1)}(-)^{P}P\{(\overset{2N}{\underset{a}{\Pi}}\phi_a(\boldsymbol{x}_a)\}\cdot\phi_\alpha(\boldsymbol{x}_{2N+1}). \tag{4·3}$$

Here it has to be noted that $\Psi_{(2N+1),\alpha}$ is not an eigen-state of the system. The eigen-state, in which the conduction electron itinerates through the whole crystal with the wave-number vector $\boldsymbol{k}$, is given by

$$\Psi_{(2N+1),\boldsymbol{k}}=\frac{1}{\sqrt{N}}\sum_{\alpha}^{(2N/2)}e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\alpha}\Psi_{(2N+1),\alpha}, \tag{4·4}$$

where the upper symbol $(2N/2)$ of $\sum_{\alpha}^{(2N/2)}$ means the summation over only a specified electron-spin state and $\boldsymbol{R}_\alpha$ is the position vector of the nucleus on which ϕ_α is centered. The corresponding eigen-energy is

$$E(2N+1, \boldsymbol{k}) = \int \Psi^*_{(2N+1),\boldsymbol{k}} \mathcal{H}_{(2N+1)} \Psi_{(2N+1),\boldsymbol{k}} d\tau_{(2N+1)}, \tag{4·5}$$

where

$$\mathcal{H}_{(2N+1)} = \mathcal{H} + O_1(2N+1) + \sum_{\mu=1}^{2N} O_2(2N+1, \mu). \tag{4·6}$$

Making use of (4·1) and (4·2), we can reduce (4·5) to

$$\epsilon_c(\boldsymbol{k}) = E(2N+1, \boldsymbol{k}) - E = \sum_{\beta}^{(2N/2)} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta)} \{H_F(\boldsymbol{\phi})\}_{\alpha\beta}, \tag{4·7}$$

where E is the energy of the crystal given in the preceding section and

$$\{H_F(\boldsymbol{\phi})\}_{\alpha\beta} = \int \phi^*_\alpha(\boldsymbol{x}) H_F(\boldsymbol{\phi}) \phi_\beta(\boldsymbol{x}) d\tau. \tag{4·8}$$

We can also obtain $\epsilon_c(\boldsymbol{k})$ from the one-electron wave function

$$\psi_{c,\boldsymbol{k}} = \frac{1}{\sqrt{N}} \sum_{\alpha}^{(2N/2)} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\alpha} \phi_\alpha(\boldsymbol{x}), \tag{4·9}$$

instead of the wave function of the $(2N+1)$-electron system $\Psi_{(2N+1),\boldsymbol{k}}$, if we use $H_F(\boldsymbol{\phi})$ as the effective one-electron Hamiltonian.

$$\epsilon_c(\boldsymbol{k}) = \int \psi^*_{c,\boldsymbol{k}} H_F(\boldsymbol{\phi}) \psi_{c,\boldsymbol{k}} d\tau = \sum_{\beta}^{(2N/2)} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}_\alpha - \boldsymbol{R}_\beta)} \{H_F(\boldsymbol{\phi})\}_{\alpha\beta}. \tag{4·10}$$

Thus we can regard $\epsilon_c(\boldsymbol{k})$ as the energy level with the wave-number vector $\boldsymbol{k}$ in the conduction band and $\psi_{c,\boldsymbol{k}}$ as its wave function.

Next let us consider a system consisting of the perfect crystal plus a positive hole which is a defect of a valence electron. The state, in which the positive hole occupies a spin-orbital ϕ_b, is expressed by

$$\Psi_{(2N-1),b} = \frac{(-1)^b}{\sqrt{(2N-1)!}} \sum_{P_{(2N-1)}} (-)^P \prod_{\alpha\neq b}^{(2N-1)} \phi_\alpha(\boldsymbol{x}_\alpha). \tag{4·11}$$

In the same way as in the case mentioned just above, the eigen-state, in which the positive hole itinerates through the whole crystal with the wave-number vector $\boldsymbol{k}$, is expressed by

$$\Psi_{(2N-1),\boldsymbol{k}} = \frac{1}{\sqrt{N}} \sum_{b}^{(2N/2)} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_b} \Psi_{(2N-1),b} \tag{4·12}$$

and its eigen-energy is given by

$$E(2N-1, \boldsymbol{k}) = \int \Psi^*_{(2N-1),\boldsymbol{k}} \mathcal{H}_{(2N-1)} \Psi_{(2N-1),\boldsymbol{k}} d\tau_{(2N-1)} \tag{4·13}$$

or, making use of (4·1) and (4·2),

$$-\epsilon_v(\boldsymbol{k}) = E(2N-1,\ \boldsymbol{k}) - E = -\sum_{b}^{(2N/2)} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}_a-\boldsymbol{R}_b)} \{H_F(\phi)\}_{ab}, \tag{4·14}$$

where

$$\mathcal{H}_{(2N-1)} = \mathcal{H} - O_1(2N) - \sum_{\mu=1}^{2N-1} O_2(2N,\ \mu). \tag{4·15}$$

We can also obtain $\epsilon_v(\boldsymbol{k})$ from the one-electron wave function

$$\psi_{v,k} = \frac{1}{\sqrt{N}} \sum_{a}^{(2N/2)} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_a} \phi_a(\boldsymbol{x}) \tag{4·16}$$

instead of the wave function of the $(2N-1)$-electron system $\Psi_{(2N-1),k}$ if we use $H_F(\phi)$ for the effective one-electron Hamiltonian. Therefore, we can regard $\epsilon_v(\boldsymbol{k})$ as an energy level of the valence band and $\psi_{v,k}$ as its eigen-function. The energy gap at $k=0$ between the conduction and valence bands is

$$\Delta\epsilon = \{\epsilon_c(\boldsymbol{k}) - \epsilon_v(\boldsymbol{k})\}_{k=0}. \tag{4·17}$$

Since ϕ is the orthonormal set, it holds that

$$\int \psi^*_{v,k}(\boldsymbol{x})\, \psi_{c,k}(\boldsymbol{x})\, d\tau = 0. \tag{4·18}$$

Furthermore, from (4·1) we get

$$\int \psi^*_{v,k}(\boldsymbol{x})\, H_F(\phi)\, \psi_{c,k}(\boldsymbol{x})\, d\tau = 0. \tag{4·19}$$

Relations (4·18) and (4·19) indicate that ϕ_a's and ϕ_a's represent close approximations to the Wannier functions of the valence and conduction bands, respectively.

§ 5. Numerical results and discussions

Numerical calculations are carried out at the nearest neighbour distance $a=3.9$ a.u., which is very close to the observed distance (3.875 a.u.). Free atomic wave functions used in our numerical calculations are[8]

$$\text{1s orbital of } H^-: \quad u^0 = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r}, \quad \alpha = \frac{11}{16} \tag{5·1}$$

$$\text{1s orbital of } Li^+: \quad w^0 = \sqrt{\frac{\beta^3}{\pi}} e^{-\beta r}, \quad \beta = \frac{43}{16} \tag{5·2}$$

$$\text{2s orbital of Li}: \quad v^0 = \sqrt{\frac{\mu^5}{3\pi}} \frac{1}{\sqrt{N_1}} \left\{ r e^{-\mu r} - \frac{3N_2}{\mu} e^{-b\mu r} \right\}, \tag{5·3}$$

where

$$\mu = 0.658, \qquad b\mu = 2.26,$$

$$N_1 = 0.9844, \qquad N_2 = 0.6351.$$

Numerical values of overlap-integrals, Coulomb-integrals and kinetic integrals obtained from the above wave functions are tabulated in Table 1. In order to evaluate the lattice

energy E from (3·8), we need to know numerical values of matrix elements of Δ_1^n, Δ_2^n and $(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^n$ for $n=-1/2$ or -1. In order to calculate these matrix elements for $n=-1$ we use the elementary method proposed by Löwdin[9] and applied to LiH crystal by Fröman and Lundqvist[10], and, furthermore, we take account of the overlaps between neighbours up to the 8th order. For $n=\pm 1/2$, we make use of the multiplication law of matrices

$$\sum_j (X^{1/2})_{ij}(X^{1/2})_{jk}=X_{ik} \tag{5·4}$$

and determine the matrix elements $(X^{1/2})_{ij}$ by the method of trial and error, allowing for the contributions from the neighbours up to the sixth order. In these calculations, the slowness of the convergency of the overlaps between 2s orbitals of the Li^+ ions injures considerably the accuracy of the calculations for Δ_2^{-1} and $\Delta_2^{-1/2}$. Because of this circumstance, it is likely that the values of $F^{(2)}$, $G^{(2)}$ and especially $H^{(2)}$, which contain complicated multiplications of the matrices, contain considerable amount of error.

In calculating (c) in (3·8), we approximate Coulomb-integrals between neighbours separated further than the sixth order by Coulomb potential energies between point charges, and evaluate the lattice sum of them by using Evjen's method[11] for calculating Madelung's constant. As a result of it, we find that the value of (c) is very small. $H^{(2)}$ is the exchange term of (c), which we have neglected, anticipating that its magnitude will be small. The numerical value of each term appearing in (3·8) is listed in Table 2. The value of A for the lowest value of the crystal energy E, which we denote by A_0, is determined by $\partial E/\partial A=0$. A_0 and $E-E_0$ at $A=A_0$ are

$$A_0=-0.056, \quad E-E_0=-25 \text{ kcal/mol}. \tag{5·5}$$

The value of $E-E_0$, -25 kcal/mol, should be compared with the discrepancy between

Table 1

κ=order of neighbourhood	0	1	2	3	4	5	6	7	8
				overlap integrals					
$\Delta_{1\sigma\kappa}$	1.00	—	0.21655	—	0.07499	—	0.03094	—	0.01417
$\Delta_{2\sigma\kappa}$	1.00	—	0.51865	—	0.28514	—	0.16364	—	0.09606
$s_{\sigma\kappa}$	—	0.53776	—	0.23156	—	0.1106	—		
				Coulomb integrals					
$(u_0^0 u_\kappa^0\|O_2\|u_0^0 u_\kappa^0)$	0.42937	—	0.17890	—	0.12805	—	0.1047		
$(v_0^0 v_\kappa^0\|O_2\|v_0^0 v_\kappa^0)$	0.23069	—	0.15856	—	0.12235	—	0.1047		
$(u_0^0 v_\kappa^0\|O_2\|u_0^0 v_\kappa^0)$	—	0.21375	—	0.14454	—	0.11341	—		
				kinetic integrals					
$(u_0^0\|-1/2\cdot\Delta\|u_\kappa^0)$	0.23633	—	0.00002	—	−0.00357	—			
$(u_0^0\|-1/2\cdot\Delta\|v_\kappa^0)$	—	0.04072	—	0.00173	—				
$(v_0^0\|-1/2\cdot\Delta\|v_\kappa^0)$	0.09869	—	0.00464	—	−0.00175	—			

the theoretical value of the cohesive energy obtained by Lundqvist and the observed one, -18 ± 7 kcal/mol. That our value of $E-E_0$ is a little larger than that we expected may be due to the inaccuracies in evaluating $F^{(2)}$ and $G^{(2)}$. In view of the above result, however, it may be concluded that the cohesive energy of the LiH crystal is explained by taking account of the effect of homopolar binding upon the ionic binding.

As is seen from Table 2, the magnitude of the linear term with respect to A of (3·8) is almost determined by (a). The quantity involved in the curled bracket expression in (a) represents the energy required for transferring an electron from a negative ion to an infinitely separate positive ion in the crystal with $A=0$ and with rigidly fixed wave functions for other electrons. When it is denoted by $\Delta\epsilon$, we have

Table 2

(a) $=1.7144\,A$	(b) $=5.141\,A^2$
$F^{(1)}=-0.3081\,A$	$F^{(2)}=4.621\,A^2$
$G^{(1)}=0.0263\,A$	$G^{(2)}=2.418\,A^2$
	(c) $=-0.022\,A^2$

$$\Delta\epsilon=(\alpha|H_F(\boldsymbol{u}^0)|\alpha)-(a|H_F(\boldsymbol{u}^0)|a). \tag{5·6}$$

(b), $F^{(2)}$ and $G^{(2)}$ have order of magnitudes comparable with each other, but, $F^{(2)}$ and $G^{(2)}$ are expected to decrease their magnitudes faster than (b) as the overlap integrals decrease,* as will be seen from the expression of $F^{(2)}$ and $G^{(2)}$. In such a case, we get a very simple approximate expression for $E-E_0$, that is

$$E-E_0\doteqdot 2N\Delta\epsilon(\{\mathcal{E}^+\mathfrak{S}(1-\mathfrak{S}^+\mathfrak{S})^{-1/2}\}_{\alpha\alpha}-(\mathcal{E}\mathcal{E}^+)_{\alpha\alpha}). \tag{5·7}$$

Making use of this expression, one can discuss semi-quantitatively the effect of homopolar binding on the cohesive energies of various ionic crystals.

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Appendix

$$F^{(1)}=\sum_a[\mathcal{E}^+\mathfrak{S}(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2}]_{a\alpha}\{(a|-\tfrac{1}{2}\Delta|a)-(\alpha|-\tfrac{1}{2}\Delta|\alpha)\}$$
$$+2\sum_a\sum_\alpha[\mathcal{A}_1^{-1/2}\mathcal{E}^+(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2}\mathcal{A}_2^{-1/2}]_{a\alpha}(a|-\tfrac{1}{2}\Delta|\alpha)$$
$$-2\sum_a\sum_b[\mathcal{A}_1^{-1}\mathcal{E}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})]_{ab}(a|-\tfrac{1}{2}\Delta|b).$$

$$G^{(1)}=-\sum_a\sum_b[\mathcal{E}^+\mathfrak{S}(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2}]_{a\alpha}\{(ab|ab)-(\alpha b|b\alpha)\}$$
$$+\tfrac{1}{2}\sum_a\sum_b[\mathcal{A}_1^{-1}\mathcal{E}^+\mathfrak{S}(\mathbf{1}-\mathfrak{S}^+\mathfrak{S})^{-1/2}]_{ab}\cdot\mathcal{A}_{1ab}\cdot(ab|ab)$$

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* For examples, these circumstances can be expected for LiCl, LiBr and LiI which have their lattice constants considerably larger than that of LiH. It must be noted, however, that in these crystals the valence electrons of negative ions have the symmetry of p-character contrary to the case of LiH.

$$+\tfrac{1}{2}\sum_{a}\left[\varepsilon^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\right]_{aa}(aa|aa)$$

$$-\tfrac{1}{2}\sum_{a}\sum_{\alpha}\left[\Delta_1^{-1/2}\varepsilon^{+}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{-1/2}\right]_{a\alpha}\boldsymbol{s}_{a\alpha}(a\alpha|a\alpha)$$

$$-\tfrac{1}{4}\sum_{a}\sum_{\alpha}\left[\Delta_1^{1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{-1/2}\right]_{a\alpha}\left[\Delta_1^{-1}\boldsymbol{s}\right]_{a\alpha}(a\alpha|a\alpha)$$

$$F^{(2)}=-\sum_{a}\sum_{b}\left[\Delta_1^{-1/2}\varepsilon^{+}\varepsilon\Delta_1^{-1/2}\right]_{ab}(a|-\tfrac{1}{2}\Delta|b)$$

$$+\sum_{a}\left[\varepsilon\ \varepsilon\right]_{a\alpha}(a|-\tfrac{1}{2}\Delta|a)-\sum\left[\varepsilon^{+}\varepsilon\right]_{\alpha\alpha}(\alpha|-\tfrac{1}{2}\Delta|\alpha)$$

$$+\sum_{\alpha}\sum_{\beta}\left[\Delta_2^{-1}\varepsilon\varepsilon^{+}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\right]_{\alpha\beta}(\alpha\ -\tfrac{1}{2}\Delta|\beta)$$

$$-2\sum_{a}\sum_{\alpha}\left[\Delta_2^{-1/2}\varepsilon\varepsilon^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\Delta_1^{-1/2}\right]_{a\alpha}(a|-\tfrac{1}{2}\Delta|\alpha)$$

$$+\sum_{a}\sum_{b}\left[\Delta_1^{-1}\varepsilon^{+}\varepsilon\mathfrak{S}^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\right]_{ab}(a|-\tfrac{1}{2}\Delta|b)$$

$$G^{(2)}=-\sum_{a}\sum_{b}\left[\varepsilon^{+}\varepsilon\right]_{aa}\{(ab|ba)-(a\alpha|\alpha a)\}$$

$$+\tfrac{1}{2}\sum_{a}\left[\varepsilon^{+}\varepsilon\right]_{aa}(aa|aa)+\tfrac{1}{2}\sum_{a}\sum_{b}\left[\varepsilon^{+}\varepsilon\Delta_1\right]_{ab}\left[\Delta_1^{-1}\right]_{ab}(ab|ab)$$

$$+\tfrac{1}{2}\sum_{a}\sum_{\alpha}\left[\Delta_1^{-1/2}\varepsilon^{+}\varepsilon\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\Delta_2^{-1/2}\right]_{a\alpha}\boldsymbol{s}_{a\alpha}(a\alpha|a\alpha)$$

$$-\tfrac{1}{4}\sum_{\alpha}\sum_{\beta}\left[\Delta_2^{-1}\varepsilon^{+}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\right]_{\alpha\beta}\left[\boldsymbol{s}\Delta_1^{-1}\boldsymbol{s}\right]_{\alpha\beta}(\alpha\beta|\alpha\beta)$$

$$-\tfrac{1}{4}\sum_{a}\sum_{b}\left[\varepsilon^{+}\varepsilon\mathfrak{S}^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1}\Delta_1\right]_{ab}\Delta_{1ab}(ab|ab)$$

$$H^{(2)}=-\tfrac{1}{2}\sum_{a}\sum_{\alpha}\Big\{\left[\Delta_1^{-1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{1/2}\right]_{a\alpha}\left[\Delta_1^{1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{-1/2}\right]_{a\alpha}$$

$$+\left[\Delta_1^{1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{1/2}\right]_{a\alpha}\left[\Delta_1^{-1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{-1/2}\right]_{a\alpha}\Big\}(a\alpha|a\alpha)$$

$$+\tfrac{1}{4}\sum_{\alpha(\mathrm{or}\,a)}\sum_{b(\mathrm{or}\,\beta)}\left[\varepsilon\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\right]_{ab}\}^{2}\{(ab|ab)+(\alpha\beta|\alpha\beta)$$

$$-\tfrac{1}{2}\sum_{a}\sum_{\alpha}\left[\Delta_1^{-1/2}\varepsilon\mathfrak{S}^{+}\mathfrak{S}(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{1/2}\right]_{a\alpha}\left[\Delta_1^{1/2}\varepsilon(\mathbf{1}-\mathfrak{S}^{+}\mathfrak{S})^{-1/2}\Delta_2^{-1/2}\right]_{a\alpha}(a\alpha|a\alpha).$$

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Antinucleon-Nucleon Interactions

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A simple model for the nucleon-antinucleon interactions is presented, in which the nucleon or the antinucleon is regarded as composed of two parts, the pion cloud and the "core". The latter is characterized by two properties: i) it acts as a nearly black body for an incoming antiparticle wave, and ii) its characteristic time for annihilation is much shorter than the oscillation period of the outer pion cloud. These simple assumptions lead to natural semi-quantitative explanations of the salient features of the experimental information so far available concerning the nucleon-antinucleon reaction and capture processes. Some arguments are added to look for a justification of such an approach.

§ 1. Introduction

According to the current concept of nucleon-pion physics, it is reasonable to regard a nucleon or an antinucleon as having a compound structure, with its core in the center and the pion cloud in the outer part. Success in theoretical analysis of low energy pion-nucleon scattering and photopion production by nucleon[1] has shown that the static pion theory is capable of describing the main features of the pion cloud. Recent experiments[2] of high energy electron-proton and electron-deuteron scattering, on the other hand, have given for the first time some information about the nucleon core, distinguishing it from the effects of the pion cloud; they are concerned with the spatial distribution of the core and the cloud. Still we have very little understanding about the properties of the core. Theoretically it is expected to be a very complicated aggregate of strongly interacting virtual particles, i. e. nucleon, nucleon-antinucleon pairs, pions, K-mesons and hyperons. Analysis of high energy pion-nucleon and nucleon-nucleon scattering, the anomalous magnetic moment of the nucleon, and the mass difference between a proton and a neutron have not been able to give much insight into the nature of the core.

In this connection the behaviour of an antinucleon in its interaction with a nucleon deserves particular attention. In such an encounter the annihilation of nucleon-antinucleon pair takes place very frequently, leading to the emission of several pions. It is natural to assume that the pair annihilation occurs only when the core of the nucleon touches

on that of the antinucleon. So we can expect that some of the characteristic features of the core will be revealed in the annihilation process.

Indeed, recent experiments[3),4),5)] have shown the following peculiarities: i) The total antiproton-proton interaction cross section is much larger than that of the nucleon-nucleon scattering. Moreover, the annihilation cross section appears to occupy a very large fraction of the total cross section. ii) In antiproton capture by a nucleon a larger number of pions is emitted on the average than the number predicted by a simple statistical model.

Although a quantitative treatment of antiproton-proton interactions based on the field theory is at present far from being available, we can show that the very presence of two different parts, the pion cloud and the nucleon core, leads to a fairly natural explanation of the above mentioned two important features of the experimental results. For this purpose we have to attribute to the core two distinct properties: i) it acts as a nearly black body for an incoming antiparticle wave, and ii) the characteristic time for its annihilation to occur is very short ($\sim\hbar/2m_Nc^2$) compared with the characteristic time for oscillations of the pion cloud.

In nucleon-antinucleon interactions, then, the core of a target nucleon acts as a completely (or nearly completely) absorbing medium for the incoming antinucleon, that is to say, an annihilation occurs every time (or nearly every time) when the incoming wave hits the core surface. Therefore, we expect an annihilation cross section of the order of πa^2 (a=the core radius) at very high energies. When the mutual distance between the nucleon and the antinucleon is larger than a, there acts a potential between them due to pion exchanges. In § 2 a simple version of this model is adopted, which is essentially equal to the Feshbach-Weisskopf treatment[1)] in the theory of nuclear reactions. Since in the considered energy range the incoming wave has a wave length $\lambda\!\!\!^{-}$ comparable with a, it must be treated quantum-mechanically and a strong annihilation occurs even when the antinucleon wave passes through a region outside the core, but not farther than $\lambda\!\!\!^{-}$ from the core surface. Therefore the annihilation cross section exceeds the classical value πa^2 considerably and becomes approximately $\pi(a+\lambda\!\!\!^{-})^2$. Comparing this with the measured annihilation cross section,[3)] we find that $a\approx(2/3)\times(\hbar/m_\pi c)$ could give a reasonable fit to the experiments. Although this numerical value cannot be taken seriously, it might still seem to be too large, since we would expect $a\approx 2\hbar/m_N c$ from rough consideration of the field theory. But recent experiments[2)] of high energy electron-proton and electron-deuteron scattering have revealed a root mean square radius of the core $(0.77\pm 0.10)\cdot 10^{-13}$ cm, indicating that our value of a is not quite unreasonable.

The elastic scattering cross section depends not only on scattering due to the annihilation, but also on scattering by the outside potential. This obscures a possible connection between the elastic scattering and the core properties. But it will be shown in § 2 that the elastic scattering cross section can be made much smaller than the annihilation cross section.

For antinucleon-nucleon captures at rest simple statistical models have predicted a smaller value ($2.2\sim 3.6$) for the pion multiplicity and a larger number of events involving

K-meson emission ($\sim 65\%$) than the measured values 4.7 ± 0.4 and 6% respectively.[7] In these models the total rest energy ($\sim 2m_N c^2$) of an antinucleon-nucleon pair is released in a volume $\Omega_0 = (4\pi/3)(\hbar/m_N c)^3$, where statistical equilibrium among different modes is first established and then several pions of the corresponding states are emitted according to this statistical distribution. Any simple modification of the statistical model[8] seems to fail in explaining the observed multiplicity and composition. Indeed we have to take a volume, where the equilibrium is reached, about $3 \sim 8$ times larger than Ω_0 in order to get the observed multiplicity and composition.

The failure of the statistical models suggests us that not all of the possible final states are on equal footing; some characteristics of nucleon (antinucleon) structure are being reflected in the transition matrix elements leading to the large pion multiplicity. Actually we should expect two types of pions are coming out, pions from the pion clouds and those produced by violent annihilation of the core-anticore pair. Essential part of the core annihilation will end within a short period $\sim \hbar/2m_N c^2$, which is smaller than a period of the oscillation of the pion cloud, i. e., a period in which a pion in the cloud is absorbed and re-emitted by the core nucleon. The latter is of the order of $\hbar/\omega_\pi$, where ω_π is the average pion energy in the cloud.

During the short time in which the core annihilation occurs and finishes, therefore, the state of the pion clouds, the number of pions in the clouds in particular, will remain almost unchanged. These pions then will be emitted, because they have lost their centre to associate with. The number of pions thus emitted will be $2 \sim 3$ on the average. On the other hand in the violent annihilation of core-anticore pair one, two, three, — pions will be produced and then emitted. In § 3 we apply the static pion theory to the clouds and the statistical theory to the core annihilation. Thus we shall show that a pion multiplicity and composition, which are comparable with the experimental ones, can be obtained.

§ 2. Nucleon-antinucleon reaction*

As mentioned in the introduction, we shall assume that a physical nucleon can be regarded, as far as its interaction with an antinucleon is concerned, as consisting of two regions.** The inner region could be approximately represented by a black sphere, which absorbs all the waves that hit its surface. The radius of this sphere, a, say, is left as an adjustable parameter. The outer part, on the other hand, stands for the pion cloud and its effect is described by an ordinary potential. The latter is due to the exchange of a few pions and can, in principle, be worked out based on the meson theory—at least in the relatively lower energy region.[9),10)*]

* The main content of this section was distributed as Cosmotron Report of Brookhaven National Laboratory, Oct. 1956.

** Strictly speaking, the term "region" should be understood in a rather abstract way. We had better say that it appears reasonable, in the first approximation, to distnguish two kinds of interactions and treat them by different methods. We resort, as a matter of fact, to a fictitious boundary between inner and outer regions, just as a simple model that can be handled easily.

Here we shall treat, for orientation, the simplest case of absorption and "shadow" scattering by a black sphere without any surrounding potential, although, of course, this is not a realistic model since the effect of the outer potential can by no means be neglected. In the classical limit (wave length of the incident wave $\lambda\!\!\!^{-} \ll a$), we have the well-known results:

$$\sigma_{\text{abs.}} = \pi a^2,$$

$$\sigma_{\text{scatt.}} = \pi a^2,$$

and

$$\sigma_{\text{total}} = 2\pi a^2.$$

But the energy region, where the experiments have been performed so far, corresponds to the wave length in the center of mass system lying just between the Compton wave length of the pion and that of the nucleon, and so the classical approach fails. Thus we have to solve the Schrödinger equation with suitable boundary conditions at infinity and at the surface of the black sphere, and in this way take into account the quantum character of this collision.**

The quantum effect results in two remarkable features, which are in qualitative agreement with experimental information. First, the absorption cross section is increased considerably. This is almost obvious because the tails of the partial waves with higher angular momenta, that would classically never reach the central core, can penetrate the centrifugal potential barrier and be absorbed. In other words, we have approximately $\sigma_{\text{abs.}} = \pi (a + \lambda\!\!\!^{-})^2$ instead of πa^2.

In the second place, this partial absorption of higher order spherical waves also leads to the relative enhancement of the absorption cross section. This point could be made plausible by the following consideration. If the l-th partial wave of the solution of this collision process is written in the form,

$$(\text{ingoing wave}) \rightarrow \eta \ (\text{outgoing wave}),$$

its contributions to the absorption and scattering cross sections are given respectively by

$$\sigma_{\text{abs.}}^{(l)} = (\pi/k^2)\,(2l+1)\,(1-|\eta|^2), \tag{2·1}$$

and

$$\sigma_{\text{scatt.}}^{(l)} = (\pi/k^2)\,(2l+1)\,|1-\eta|^2. \tag{2·2}$$

In the classical treatment η is either equal to 1 or to 0, so that the two cross sections are always the same. In the quantum mechanical treatment, however, some waves are only partially absorbed. Let us take, for simplicity, a case of very small absorption and put $\eta = 1 + \varepsilon$ where ε is a small (in general complex) quantity. Then it turns out

* After completing our work we have received a paper by Ball and Chew.[10] There they treat the nucleon-antinucleon scattering as scattering by the pion exchange potential with an absorbing core inside. The energy region considered there is lower than ours and the W. K. B. method is employed. We are grateful to them for sending us their manuscript.

** We think that such a nonrelativistic treatment makes sense here, because the relativistic feature (annihilation) has been phenomenologically replaced by an absorbing medium.

$$\sigma_{\text{abs.}}=O(|\mathcal{E}|),\ \sigma_{\text{scatt.}}=O(|\mathcal{E}|^2),$$

indicating that absorption would, generally speaking, exceed scattering considerably.

It can further be shown, that we have to include higher order spherical waves in order to obtain a relatively inelastic collision ($\sigma_{\text{scatt.}}\ll\sigma_{\text{total}}$). In fact we can derive from (2·1) and (2·2) the relation

$$\sigma^l_{\text{total}}\leq(4\pi/k^2)\,(2l+1)\,\sigma^{(l)}_{\text{scatt.}}/\sigma^{(l)}_{\text{total}} \tag{2·3}$$

and from this we can conclude that, for example, for a constant (l-independent) value 0.2 of elasticity, we need at least the waves up to $l=4$ in order to obtain a 100 mb total cross section at $E_p=500$ Mev.*

Now the boundary condition at $r=a$ should be such as to represent a strong absorption. We have specified it, somewhat arbitrarily, in the following way. The logarithmic derivative of the wave function should be continuous at the surface of the core and further the radial dependence of the wave function inside the black sphere is assumed to be $e^{-\zeta kr}/r$, irrespective of the angular momentum of the waves.** This is an analogy to the continuum theory of Feshbach and Weisskopf in the nuclear reaction,[6] as was mentioned in § 1.

There is no convincing argument as to the choice of the virtual wave number K in our simple model. But we find that the results are not essentially changed, as far as the change ΔK in K remains small compared with $1/a$. We have therefore simply assumed that there is no change of the wave number through the surface. At very low energies this crude approximation fails and we must take into account a finite increase of the wave number inside; otherwise the $1/k$-dependence, which appears to hold according to the experiments,[3] cannot be obtained.

Thus the boundary condition at $r=a$ reads:

$$\frac{\eta_l h_l^{(1)\prime}(ka)+h_l^{(2)\prime}(ka)}{\eta_l h_l^{(1)}(ka)+h_l^{(2)}(ka)}=\frac{h_0^{(2)\prime}(ka)}{h_0^{(2)}(ka)}, \tag{2·4}$$

where

$$h_l^{(1,2)}(x)=\sqrt{\frac{\pi}{2x}}\,H^{(1,2)}_{l+1/2}(x), \tag{2·5}$$

* This energy in the laboratory system corresponds to the wave number $2.5\times10^{13}\,\text{cm}^{-1}$ in the center of mass system.

** We may assume that at the surface of the black sphere there should be ingoing waves only. It is quite ambiguous, however, how to define the outgoing and ingoing waves at a point close to the center. One might be inclined to indentify them with those represented by the Hankel functions $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ respectively (see (2·5)), and impose that there should be no $h_l^{(1)}(kr)$ component. But then we should have obviously no outgoing wave at remote places (wave zone); i. e. the absorption cross section is infinite and so is the shadow scattering. Physically this may mean a really strong sink that pulls in even the waves passing at an infinite distance. In such a case the results are independent of the radius a. Indeed it suffices to have a point sink in the center. This example, though not realistic, indicates the possibility of reducing the parameter a by a more complicated boundary condition than our simple model. Of course the existence of the outer potential can change the value further.

and

$$h_l^{(1,2)\prime}(x) = dh_l^{(1,2)}(x)/dx .$$

k is the wave number of the free antiproton in the center of mass system ; $\eta_l = \exp(-2i\delta_l)$, where δ_l is the complex phase shift for the l-th partial wave, from which we can calculate both absorption (annihilation) and scattering cross sections for each partial wave.

The results of the calculation are shown in Fig. 2·1, for three values, 1, 2/3 and 1/3 (in the pion Compton wave length) of the parameter, a. It appears that the choice

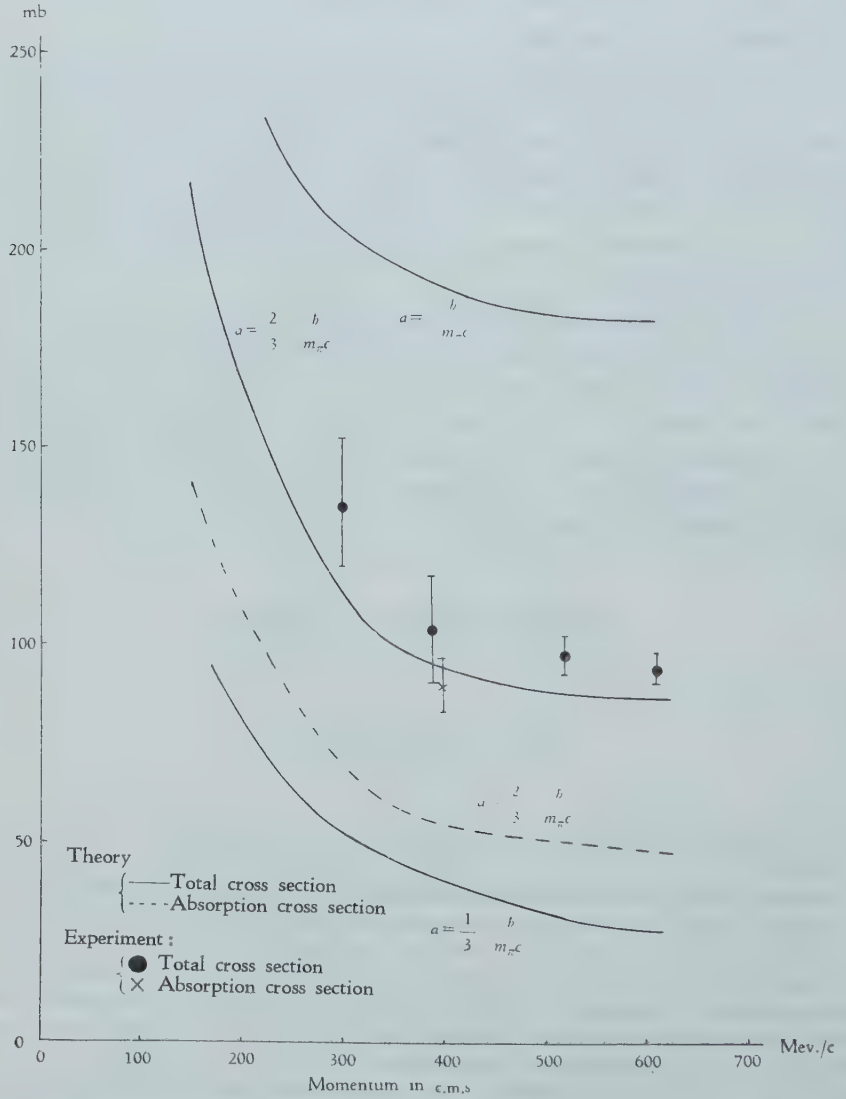


Fig. 2·1 The calculated values of the total and absorption cross section for the nucleon-antinucleon reaction.

of $a=(2/3)\cdot(\hbar/m_\pi c)$ fairly well reproduces the experimental data of the total cross section, although this numerical value cannot be taken seriously because our model is a very crude one. In the figure is also given the curve of absorption cross sections predicted by $a=(2\ 3)\cdot(\hbar/m_\pi c)$. Recent experiments seem to suggest the presence of more absorption than the result of our simple model.[4] The contributions from various partial waves to absorption and scattering are listed in Table 2·1. It is seen that relatively high angular

Table 2·1 Contributions from various partial waves to the total cross section ($a=2/3\cdot\hbar/m_\pi c$)

	E_p / l	190	300	500	700 (Mev in lab. system)
	0	12%	8%	5%	4%
	1	33	24	15	11
σ_{abs}	2	14	18	16	14
	3	2	8	18.5	20
	4	—	1	3.5	9
	0	12	8	5	4
	1	23	23	17	12
σ_{scatt}	2	3	10	15.5	18
	3	—	—	3.5	8
	4	—	—	—	1

momentum waves play essential roles in absorption, while to scattering cross section they contribute rather little, which agrees with our considerations given at the beginning of this section.

The simplest case described above is not a realistic one. Unfortunately, however, we have not been able so far to work out the case, which takes into account the effect of the surrounding pion cloud. So we shall here restrict ourselves to a rough discussion of the expected modification of the foregoing results owing to the existence of the outer real potential.[9),10)] Scattering will be more or less enhanced, especially when the edge of the potential is sharp (most notably in the case of a square well, see below). Since the experiments indicate that the scattering is relatively small the shape of the potential will perhaps turn out, at least for the main states of a nucleon-antinucleon system, to be a tapered one. It is almost evident that a repulsive potential will reduce the absorption. As for the attractive potential, it will have a two-fold effect on the absorption. First, it will pull in the waves and let more of the waves hit the surface of the black sphere, thus enhancing the absorption. Second, it will reflect a part of the incident wave and let less of the waves hit the surface, thus reducing the absorption. In the case of a smooth potential the first trend will perhaps predominate.

An example of the case of an attractive square well potential is given in the following. The employed parameters are :

V: depth of the square well potential $\sim$70 Mev,

R: range of the square well potential $\sim\hbar/m_\pi c\sim 1.4\times 10^{-13}$ cm

a: radius of the black sphere $\sim 1/3\,(\hbar/m_\pi c)\sim 0.47\times 10^{-13}$ cm.

For $E_p\sim 250$ Mev we obtained

$$\sigma_{\text{total}} \sim 120 \text{ mb} \quad (47 \text{ mb}),$$

$$\sigma_{\text{abs.}} \sim 30 \text{ mb} \quad (31 \text{ mb}),$$

and

$$\sigma_{\text{scatt.}} \sim 90 \text{ mb} \quad (16 \text{ mb}).$$

The values in the brackets are those obtained by the black sphere alone. We observe, thus, that in this particular case the annihilation cross section does not change appreciably, while the scattering and consequently the total cross section is considerably increased due to the potential scattering.

§ 3. Antinucleon capture by nucleons

Recent experiments in nuclear emulsions have given us some knowledge of antiproton captures by nucleons, particularly, average multiplicity of pions produced, multiplicity distribution and energy distribution of the pions, and average multiplicity of K meson pairs produced.

The observed multiplicity of the pions and K-meson pairs, $\langle n_\pi\rangle = 4.7\pm 0.4$ and $\langle n_{k-\bar{k}}\rangle \approx 0.06$,[5] are in contradiction with theoretical values obtained by a simple straightforward application of Fermi's statistical model. In this model the total rest energy of the antiproton-target nucleon system ($\sim 2m_N c^2$) is released in a volume $\Omega_0 (\approx 4\pi/3\cdot(\hbar/m_\pi c)^3)$, where a statistical equilibrium is reached within a very short interval. Then various number of pions and K meson pairs are emitted with a probability determined by the statistical equilibrium. The value of $\langle n_\pi\rangle$ calculated in this way[7]* (2.2$\sim$3.6) is much smaller than the observed one. In order to get agreement with the observed $\langle n_\pi\rangle$ $\langle n_{k-\bar{k}}\rangle$, we have to increase Ω_0 by a factor as large as 3$\sim$8.

Some attempts have been made for increasing $\langle n_\pi\rangle$, in which strong pion-pion interactions among emitted pions have been taken into account. The interactions have effectively increased Ω_0 to some extent, but the resultant increase in $\langle n_\pi\rangle$ is rather small.[7] It is our opinion that this effect may be present but not enough to explain the observed large $\langle n_\pi\rangle$.**

A statistical model like Fermi's is not expected to give a good fit to $\langle n_\pi\rangle$ and $\langle n_{k-\bar{k}}\rangle$; the discrepancy between the theoretical and observed $\langle n_\pi\rangle$ suggests us that characteristics of the nucleon and antinucleon structures are strongly reflected in the observed large $\langle n_\pi\rangle$ and small $\langle n_{k-\bar{k}}\rangle$, whereas in a statistical model any peculiarity of

* There has been a mistake of a factor of about 5 in the second paper of ref. (6), which was pointed out by Belenky et al. (the last paper of ref. (6)). The $\langle n_\pi\rangle$ calculated by the last authors is 3.6 when only pion emission is considered and 2.2 when both pions and K-mesons can be emitted.

** Recently D. Ito and S. Minami have proposed a model where strong pion-pion interactions are supposed to explain the large pion multiplicity. (Private communication.) In our opinion this model cannot be considered as being within the framework of the statistical theory because it is equivalent to drastically modifying certain matrix elements.

a system concerned does not appear in predictions for $\langle n_\pi \rangle$ and $\langle n_{k-\bar{k}} \rangle$ at all. Hence we look for some mechanism of production of pions based on our knowledge of the nucleon structure that would reproduce characteristic features shown by the experiments.

Here we shall pursue the consequences of the assumption that a nucleon is composed of the nucleon core and the pion cloud. These two parts will behave quite differently in the capture process, because firstly a nucleon-antinucleon pair annihilation will occur only when their cores collide and annihilate with each other and secondly the core annihilation will take place in a very short time $\sim\hbar/2m_N c^2$ compared with the characteristic time of the cloud oscillation $\sim\hbar/\omega_\pi$.*

In an antiproton capture by a nucleon at rest an annihilation will occur probably from an L or a K-orbit of the antinucleon. In this orbit the antinucleon wave function has small but finite probability for overlapping with the nucleon wave function. Therefore their cores can also overlap with each other and an annihilation proceeds. Pions in their clouds are oscillating around their respective cores, being absorbed and reemitted by the cores. Since this oscillation period is expected to be longer than the core annihilation time $\sim\hbar/2m_N c^2$, the pion cloud would not change much during the core annihilation process. Then the pions lose their respective centers to be bound to and are emitted as free pions. Therefore the number of pions thus emitted are nearly equal to the number of pions in the clouds.** Energy distribution of pions may reflect the energy distribution of pions in the clouds, but the impulse they receive during the core annihilation is appreciable and will disturb substantially their original energy or momentum distribution. In the core annihilation one, two,—pions and possibly one or two $K-\bar{K}$ pairs are newly produced. These and the pions coming out from the clouds will constitute the total observed pions in capture processes.

Since the static pion theory could give reasonable description of low energy pion-nucleon scattering, photo-pion productions by nucleons, and Compton scattering by nucleons, we use the static theory to study the properties of the pion cloud. The number of p-wave pions*** in the cloud is readily obtained from a dispersion-like relation[11] and the results are

$$\langle n_{\pi^+} \rangle \approx 0.66,\ \langle n_{\pi^0} \rangle \approx 0.43\ \langle n_{\pi^-} \rangle \approx 0.20, \tag{3·1}$$

and

$$\langle n_\pi \rangle \approx 1.3.$$

Therefore, 2.6 $(=1.3\times 2)$ pions are on the average coming out from the clouds of the nucleon-antinucleon pair.

Now we discuss the multiplicity of pions produced in the core annihilation. No

* The estimation for these times is made by a simple application of the uncertainty principle. Some field theoretical consideration will be presented in § 4.

** Possible strong interactions among produced pions may produce another pion and increase actual multiplicity of pions coming out. Even if this effect is present it will bring the theoretical $\langle n_\pi \rangle$ to the right direction.

*** Most of low energy pions are supposed to be in P-orbits. On the other hand most pions in S orbit have characteristic frequencies comparable to $\sim\hbar/2m_N c^2$ and should be considered as a part of the core.

justified field theoretical way for studying the core phenomena is yet available. So, tentatively we apply Fermi's statistical model to the core annihilation, that is to say, we neglect the detailed behaviour of the matrix elements. The average pion energy observed in annihilation processes is 346 Mev.[5] Hence we assume that an energy $0.96\, m_N c^2$ (≈ 346 Mev $\times 2.6$) is on the average ascribed to the clouds and the rest ($\approx 1.04\, m_N c^2$) is released in the core annihilation. If this energy is in statistical equilibrium in a volume Ω, which we take tentatively equal to the core volume $(4\pi/3)a^3 (a = (2/3)\hbar/m_\pi c)$ in § 2,* we obtain the pion multiplicity 2.2.

Adding this value to that due to the pion clouds, we obtain 4.8 as the average pion multiplicity in the antiproton-nucleon capture. Agreement of this with the observed multiplicity is satisfactory. Of course, we should not put much confidence in this numerical agreement, but we would like to emphasize that the mere distinction between the cloud and the core leads to an easy and reasonable explanation for the observed $\langle n_\pi \rangle$.

The situation concerning $K-\bar{K}$ production is also improved. Virtual $K-\bar{K}$ pairs in the nucleon had better be considered as a part of the core. A characteristic time of an oscillation associated with a virtual production and annihilation of $K-\bar{K}$ pairs in the nucleon is comparable with the short collision time necessary for the core annihilation and is much shorter than the characteristic time of the cloud oscillation. Therefore, we treat the observed $K-\bar{K}$ pairs as coming out from the core. If we apply Fermi's model to it, only a very small amount of $K-\bar{K}$ pairs could be produced, because the available energy for the core annihilation is only $1.04\, m_N c^2$ on the average, which just suffices to make a $K-\bar{K}$ pair. Thus a small probability for observing a $K-\bar{K}$ pair can be explained.**

Since the average pion and $K-\bar{K}$ pair multiplicities are reasonably explained in our model, we shall now study more detailed properties of the capture process. In particular, a distribution of pion multiplicities is of some interest to us, because it will more or less reflect a distribution of pion numbers in the cloud.

An application of a dispersion-like formula to problems of the pion cloud could predict only the average pion number (≈ 1.3) in the cloud but not the distribution of pion numbers in it. No calculation of the latter has been done either which gives the distribution around the average pion number 1.3. Therefore, we use the results from the intermediate coupling theory calculated by one of the authors.*[12] This unfortunately

* Strictly speaking, this is not the volume of the nucleon core itself, but that of a sphere, whose radius is equal to the channel radius (twice the core radius) of the pair annihilation. It would be also possible to assume a large volume for statistical equilibrium and supply it with a larger fraction of available energy, since we should expect secondary exchanges of energy between particles after the annihilation of the cores is over. (This remark is due to Prof. S. Takagi.) Our main point is, in any case, that the mesons that come from the cloud would behave differently, not fully participating in the statistical equilibrium.

** If K-interactions are smaller than pion-interactions with a nucleon, then the number of virtual $K-\bar{K}$ pairs to that of virtual pions in the nucleon is very small. Thus the small probability for seeing $K-\bar{K}$ pairs is evident.

*** Various constants used there are $f_0^2/4\pi \approx 0.3$ and $K_c \approx 4.7\, m_\pi c$, where f_0 is the unrenormalized coupling constant and K_c is the cut off momentum. They are to be compared with those used in the dispersion-like calculations for n_π; Miyazawa used $f_0^2/4\pi = 0.19$ and $K_c = 5.5\, m_\pi c$ and Fubini and Thirring used $f_0^2/4\pi = 0.22$ and $K_c = 4.7\, m_\pi c$. Both obtained the average pion number 1.3.

Table 3·1 Distribution of the pion numbers in the pion cloud.

Number of pions	Probability (%)
0	34.3
1	40.4
2	18.8
3	5.6
4	0.9
—	—
	$\langle n_\pi \rangle = 0.98$

Table 3·2 Probability for seeing various numbers of pions coming out from the nucleon and antinucleon clouds in the capture process.

Number of pions	Probability (%)
0	12
1	28
2	29
3	19
4	9
5	3
—	—
	$\langle n_\pi \rangle = 1.96$

gives 0.98 instead of 1.3 for the average pion number, but most of the qualitative results of the following arguments are not altered by this deviation.*

The probabilities for seeing various numbers of pions in the cloud are given in Table 3·1.

The probability for n pions coming out from the clouds in the capture process is equal to the probability for seeing in total n pions in either the antinucleon or the nucleon cloud. This is easily obtained from Table 3·1 and is given in Table 3·2.

The number of pions produced in the core annihilation may have some correlation with the number of pions present in the cloud. This correlation can be argued by using the statistical model. When n pions are present in the cloud, an available energy for the core annihilation is

$$2m_N c^2 - n(346 \text{ Mev}) = (2 - 0.37n) m_N c^2.$$

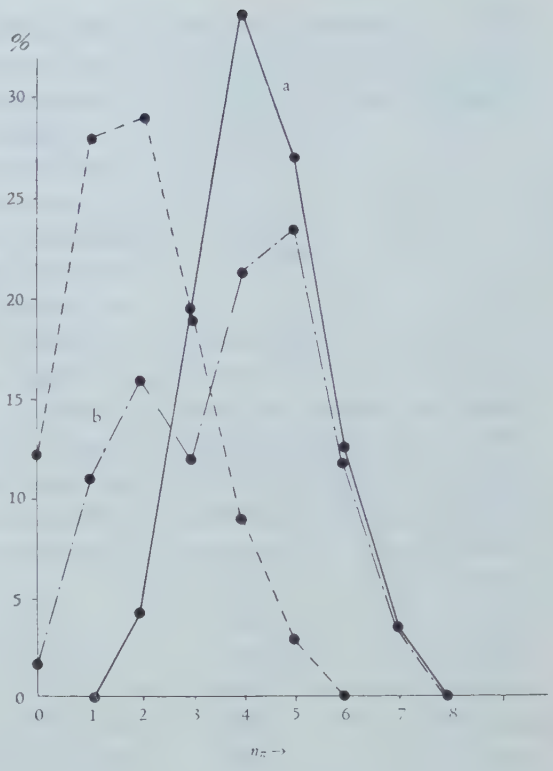
If this amount of energy is released in a volume $\Omega = (4\pi/3) \cdot (2\hbar/3 m_\pi c)^3$ and a statistical equilibrium is reached after a short time, we can evaluate probabilities for n' pions coming out from the small region. Calculations are made in the same way as Belenky et al.[7] for two different cases: in the case (a) only pion emission is considered and in the other case (b) $K-\bar{K}$ pair production is also taken into account.** From this we can easily obtain the probability for seeing N-mesons ($N = n + n'$). The resultant prong distribution of produced pions in the antinucleon-nucleon capture is given in Fig. 3·1 for the two cases. For making comparison we have also plotted the number distribution of pions in the clouds. The curve for the latter distribution is obtained from that of the former by shifting the pion number axis by nearly 2.2,*** where this number 2.2 is the average pion multiplicity due to the core annihilation. Thus it is hoped that further experimental study of the pion prong distributions will give us some knowledge about the number distribution of pions in the cloud.

* In the following, therefore, we distribute the numbers of pions derived from the dispersion relation according to the probabilities obtained by the intermediate coupling calculation.

** For both cases pions are treated as extremely relativistic. On the other hand K mesons are treated as non-relativistic for the case b).

*** In the case b) large $K-\bar{K}$ pair production (~27%) obscures this correlation. Since the observed $K-\bar{K}$ pair production is small (~6%), the actual situation seems to be nearer to the case a) than to the case b). The curves have been, of course, worked out by combining various numbers of pions in the cloud with corresponding energies available for the core annihilation. The result is roughly given by the pion distribution in the cloud plus the average number from the core (2.2), as described in the text.

The energy distribution of pions produced in the core annihilation may be different from that of pions coming out from the clouds. If we could find two groups of pions different in their energies, it will be a justification for the above conjecture. But a close correlation between the energy distribution of produced pions and that of virtual pions in the clouds are unlikely to exist because of the large impact they receive during the core annihilation.



——— the prong distribution for the case (a)
—·—·— the prong distribution for the case (b)
– – – – – – the number distribution of pions in the antinucleon-nucleon clouds

Fig. 3·1 The calculated prong distribution of produced pions in the antinucleon-nucleon capture.

§ 4. Discussion

In the previous sections we have presented a model of the antiproton-nucleon interaction which can explain the characteristic features of the experiments. Our model is a phenomenological and preliminary one. Justification for the underlying philosophy in the model should be brought from a more fundamental theory, and then a more quantitative treatment of the antiproton-nucleon interaction would become possible.

Here we shall make a few remarks on various points that have something to do with our model:

(1) We shall reconsider the mechanism of the pion emission in the capture processes from the field theoretical point of view. In our model the capture process is composed of two processes, i) the slow oscillation and emission of pions in the clouds and ii) the rapid core annihilation process; both of them lead to the pion emission, but in different ways.

Field theoretically the situation can be expressed by Feynman graphs like Fig. 4·1. The hatched circles correspond to a process for making the pion cloud around its respective cores and should be calculated in the static approximation. Therefore, only P-wave pions are included in the cloud. These pions π_1, π_2, π_3 make oscillations with a period $\sim \hbar/\omega_\pi$, being emitted and reabsorbed by their respective cores. The unhatched circle corresponds to the annihilation process of the pair of the cores, by which several pions π_4, π_5,—are emitted. Because of our ignorance of the detailed behaviour of the

matrix elements leading to different numbers of pions, we have applied Fermi's statistical method to this process.

The core annihilation process goes very fast and ends within a short interval $\sim\hbar/2m'_N c^2$,* producing about two pions on the average. The pions in the clouds, π_1, π_2, π_3, do not have enough time to move during the short interval, loose their respective cores to be bound to and then are emitted as free pions. The existence of this relatively slow process for pion emission is essential in our model and cannot be inferred from a simple application of the uncertainty principle to the antiproton-nucleon annihilation process as a whole, which would say that the entire process ends within a short interval.

Most of S-wave pions and virtual nucleon-antinucleon pairs should be considered as parts of the core, since the characteristic time associated with their oscillations is expected to be of the order of $\hbar/2m_N c^2$. Fig. 4·2–4·4 represent some different types of the core annihilation process. In Fig. 4·3 the nucleon core contains one virtual nucleon-antinucleon pair and in Fig. 4·4 two virtual S-wave pions are present in the core. Since distributions of S-wave pions and nucleon-antinucleon pairs in the cloud are not known to us, some field theoretical calculations of these core wave functions are highly desirable. Then it will become possible for us to obtain the pion multiplicity in the core annihilation without recourse to a conventional theory such as the Fermi model.

Interactions among the pions produced may distort the energy distribution and number distribution of pions. We have assumed that possible interactions among π_1, π_2, $\cdots\pi_4$, $\cdots$ can cause energy and momentum exchange between them, but that they do not alter the number of pions.

(2) In § 3 we have assumed that the annihilation of a nucleon and an antinucleon

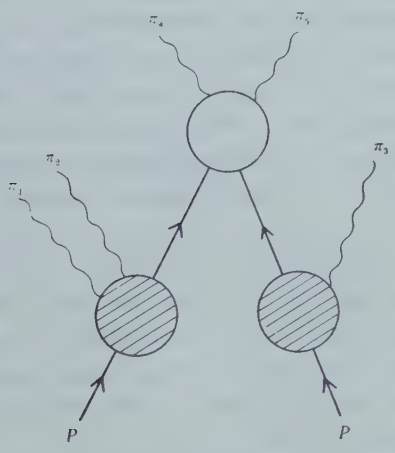


Fig. 4·1

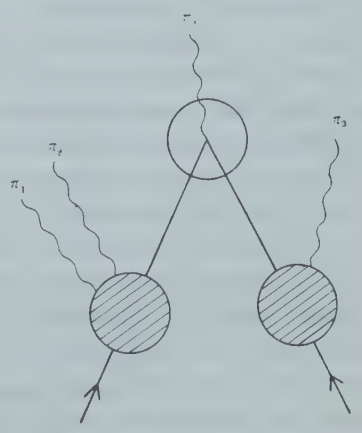


Fig. 4·2 Production of pions by a collision between the antinucleon and nucleon bare cores.

* $2m_N'c^2$ should be taken as the energy associated with the core annihilation, which is of the order of $2m_N c^2$.

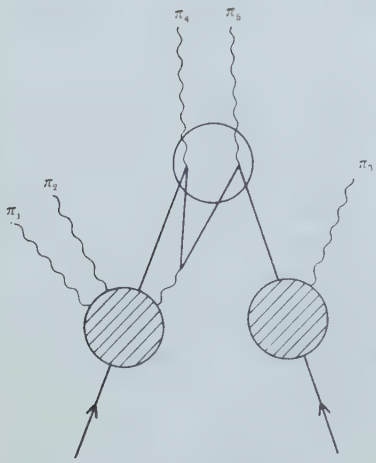


Fig. 4·3 Production of pions by the collision between the cores containing antinucleon-nucleon pairs.

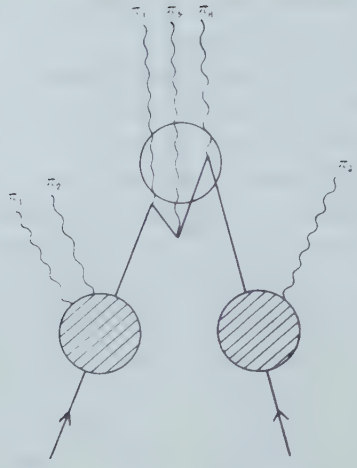


Fig. 4·4 Production of pions by a collision between the cores containing S-wave pions.

takes place in so small a time after they come into contact with each other that the pion clouds of the original particles will remain essentially unaltered. The extremely opposite point of view would be to regard the process as going on very slowly. Then the state of the pion cloud just before the annihilation occurs would be described approximately by a field surrounding a pair of nucleon and antinucleon, which lie apart from each other by a distance $2\hbar/m_N c$, say. It is not very simple to work out the general behaviour of such a field. If we restricted ourselves to π^0-mesons only, for instance, the field around a $\bar{P}N$ pair would be a "dipole" field for an odd number of pions and a twice field for an even number of them, while for a $\bar{P}P$ pair the result would be always a twice field.* The first case would be expected to give a more singular spatial distribution and a shift of the energy spectrum to the high side. The second case would be essentially the same as our treatment, if we could neglect the possible existence of pion-pion interaction. Thus we can hardly say at present anything definite concerning the mechanism of the core annihilation.

(3) Concerning the charge distribution of the so-called nucleon core, which has been deduced from the high energy electron scattering on proton and deuteron, Tamm[13] has proposed a model of nucleon structure. He admits that the interaction between nucleon and antinucleon is very large (which is, according to his opinion, "quite understandable from the point of view of meson theory"), and tries to explain the effective spread of the nucleon core to the order of pion Compton wave length.

Our treatment in § 2 of this note goes the same way with Tamm's, but in the opposite direction, so to speak: We want an explanation of the large cross section (strong interaction) of nucleon-antinucleon reaction and are led to a somewhat larger extension of the

* We are grateful to Prof. S. Takagi and Prof. T. Nakano for their helpful comments on this point.

nucleon core than is usually expected, although our model is too crude to be taken literally.

In any case these two attempts, Tamm's and ours, appear to share the point of view that the strong interaction between a nucleon-antinucleon pair, on one hand, and the unusual extension of the nucleon core suggested by high energy electron scattering, on the other, should have some intimate interrelations.

Concluding remarks

In conclusion we should like to emphasize again the preliminary character of the present work - at least in the following sense. First, the treatment of the nucleon-antinucleon interaction (§ 2) should be improved by taking into account the effects of the outer pion cloud; then we could make a more reliable guess concerning the function of the nucleon-core. Second, the model employed here should be applied to the analysis of other phenomena (at high energies) to see whether it leads to plausible explanation of the experiments. Third, this kind of approach should be justified from a more fundamental point of view; we should start, for example, from field theoretical interactions and examine whether it is possible to formulate the theory so as to correspond to our phenomenological model.

Acknowledgement

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Formulas in the Fermi Theory of Beta-Decay, I

—Beta-Spectrum—

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β-spectra are calculated by assuming the Fermi theory. All the effects due to the finiteness of the nuclear size are taken into account in an analytical form, and the screening effect by atomic electrons are included as additional factors though their values are not given explicitly. In § 2 the electron wave functions are calculated and in § 3 they are applied to get the correction factors for β-spectra. The mixed type interaction consisting of scalar, tensor and pseudoscalar is especially examined in detail. The final formulas are similar to those for light nuclei of Konopinski and Uhlenbeck. There is a difference, however, which lies in the fact that several "effective nuclear radii" are used in our formulas instead of the usual nuclear radius. Special attention is paid to the errors in our formulas, which seem to be inevitable from the practical view-point, and also to the insensitivity of the β-spectra to the nuclear charge distribution.

§ 1. Introduction

The β-decay interaction first proposed by Fermi[1] is a sum of products of the four spinor field operators for the proton, neutron, electron and neutrino. The theory with the above mentioned characteristics was extended afterwards to include all the possibilities within the restriction of invariance under the proper and improper Lorentz transformations. This theory, which has been called "the Fermi theory of β-decay", is not only simple but also very powerful. There seemed to be no evidence of contradiction in it, until recently the surprising experiment by Wu et al.[2] has revealed that the parity is not conserved in the β-decay. Although the original Fermi theory had to be abandoned in this respect, it is easy to extend it to explain the parity violation, still keeping the form of a sum of products of the four spinor field operators. Therefore, we still call this generalized theory the Fermi theory. Besides the Fermi theory a number of varieties were proposed on the theory of β-decay but they either failed to explain experiments or were too peculiar to be accepted widely. The Fermi theory is the ruling theory at present, although we cannot neglect other theories which may be successful too.

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In order to compare the Fermi theory with experiments many mathematical manipulations have been performed, and many formulas have been derived with which experimental results may be compared. These formulas are sometimes reliable but sometimes not. In the cases of the allowed and unique forbidden transitions (see § 3) the formulas are fairly precise. On the other hand in the case of the non-unique forbidden transition most of them are quite inaccurate. There are a few formulas which are precise even in the latter case, but they are too complicated to compare with experiments.

In this series of papers we try to obtain formulas for β-decay which are not only precise in all cases but also convenient for practical handling. In this first paper, I, we treat the β-ray spectrum.

Very useful formulas for β-spectra were first derived by Konopinski and Uhlenbeck[2] (abbreviated as $K-U$ hereafter). Their method is based on partial wave analysis. Although the wave functions of the electron and neutrino are explicitly taken into account, the wave functions of the nuclei are not specified because it is so difficult to know them in most cases. However, the effect of the nuclear wave functions is included in a selection rule and in a few adjustable parameters. This method of arranging the formulas is very convenient and has been adopted by most researchers. This paper follows this method too. One of the main approximations of $K-U$ is their use of the electron wave function whose radial part is proportional to $r^{l'}$ ($\hbar l'$: the orbital angular momentum) inside the nucleus and is continuously (not smoothly) connected with the Coulomb wave function at the nuclear surface. This is a generalization of Fermi's method for allowed decay.[1] We call this approximation the $K-U$ approximation hereafter. Several papers[4] which appeared thenceforth are based on this $K-U$ approximation. $K-U$ also proposed a further approximation which was supposed to be valid for light elements[3]. We particularly call the latter approximation the $K-U$ low-Z approximation.

On the other hand, the necessity of applying several corrections to the $K-U$ approximation was pointed out. Among them are 1) the screening effect[2], 2) the radiative correction,[6] 3) the finite nuclear size effect in a narrow sense[7], 4) the finite de Broglie wave length effect[8] and 5) the effect of the decay in the internal region of the nucleus[9]. Since the last three corrections all concern the finiteness of the nuclear size we call them simply "the finite nuclear size effect" in this paper. The above corrections were treated separately in most cases, and we have not known how to apply all of them together because the corrections are sometimes too large to add up independently, especially in the case of the so-called "large cancellation"[10]. There is also a paper by Takebe[11] which formally includes most corrections but is too complicated to allow practical calculations with sufficient precision.

In this paper we include both the screening effect and the finite nuclear size effect (in a broad sense) synthetically. We neglect the radiative correction because it is estimated to be rather small[6] and also this correction has quite a different character from the other two. Investigations made in the present paper were formerly attempted by one of the authors[12], but the argument was rather crude in spite of the strong restriction to its applicability. This paper may be regarded as the extension and the completion of that previous work.

In the next section we calculate the wave functions of an electron moving in a spherically symmetric potential. We need not specify the shape of the electron potential inside the nucleus. Terms representing the screening correction are added formally. In § 3 it is shown how to calculate the correction factor. Special emphasis is laid on the correspondence of our formulas to those in the $K-U$ approximation, which will make our formulas familiar. In this respect the idea of "the effective nuclear radii", which should replace the usual nuclear radius, plays an important role. The 1st forbidden transition in the mixed interaction of scalar, tensor and pseudoscalar types (called STP interaction hereafter) is treated as an application of our formulas. The formulas for the allowed, 2nd forbidden and 3rd forbidden transitions in the same mixed interaction are given in Appendix 1. § 4 is prepared particularly for persons who are not interested in the derivation of the formulas but in the applications only. One of the interesting characteristics of our theory is that our formulas do not depend explicitly either on the nuclear radius or on the nuclear charge distribution. This indicates that we cannot get any quantitative information about the nuclear charge distributions only from the research on β-spectra. Detailed argument on this problem is given in Appendix 2. In § 3 and Appendix 1 we take into account only the states in which the sum of the magnitudes of the orbital angular momenta of electron and neutrino takes the minimum value permitted by the selection rule. This approximation is somewhat insufficient for the allowed and unique forbidden transitions. We give the supplementary formulas for these transitions in Appendix 3.

§ 2. Wave function of the electron

As far as we treat nuclear matrix elements as adjustable parameters, we need not know the wave function of the nucleus. All we have to calculate are the wave functions of the emitted electron and neutrino. The neutrino causes no problem because of its very weak interaction with other particles. It is the wave function of the electron that comes into question. The motion of the electron will be influenced mainly by the Coulomb potential due to the nuclear charge and the atomic electrons. In this paper we assume the spherically symmetric charge distribution for the nucleus and its atomic electrons. Other effects neglected here (e. g. radiative correction, effect of nuclear moments) do not seem to be very important, but we intend to investigate them in future. The recoil of the nucleus is surely unimportant.

The above assumption makes it possible to use the partial wave analysis. A convenient quantum number in the relativistic motion of an electron is denoted by κ. $\kappa=\mp(j+1/2)$ according as $j=l\pm 1/2$, where j is the magnitude of the total angular momentum and l is that of the orbital angular momentum of the large component. We adopt the representation employed by Rose[13] which has been used in most articles on the β-decay. To each value of κ there belong two radial wave functions called the large component $g_\kappa(r)$ and the small component $f_\kappa(r)$. These names come from the behaviors of the electron in the non-relativistic motion. $f_\kappa(r)$ and $g_\kappa(r)$ for the finite nuclear

charge distribution satisfy the following coupled integral equations[14]:

$$\begin{aligned} r f_\kappa(r) &= r^\kappa \left\{ d_1 + \int_0^r r_1^{1-\kappa} [V(r_1) - (W-1)] g_\kappa(r_1)\, dr_1 \right\}, \\ r g_\kappa(r) &= r^{-\kappa} \left\{ d_2 - \int_0^r r_1^{1+\kappa} [V(r_1) - (W+1)] f_\kappa(r_1)\, dr_1 \right\}, \end{aligned} \tag{1}$$

where r is the distance between the electron and the nuclear center, $V(r)$ the (central) potential for the electron, and W the electron energy including its rest mass energy. We use the natural unit ($\hbar = c = 1$) and also take the electron mass as unity ($m = 1$). d_1 and d_2 are constants.

If we neglect both the finite nuclear size effect and the screening effect, i. e. if we assume $V(r) = -\alpha Z/r$ (α : fine structure constant, Z : atomic number of the daughter nucleus with a positive sign for β^--decay and a negative sign for β^+-decay), the solution of the Dirac equation from which eqs. (1) come are simply expressed by the confluent hypergeometric functions (It should be mentioned that eqs. (1) are unsuitable for the point charge nucleus). The above two effects change the solution into a more complicated form. The properties of these corrections are very different from each other. The finite nuclear size effect changes the behavior of the electron motion drastically but only in the neighborhood of the nucleus. On the other hand, the screening effect does not appreciably alter the electron motion near the nucleus, but it does produce a slowly varying change in the electron wave function in a region large compared with the nuclear charge. Since the β-decay occurs inside the nucleus, we have to know the electron wave function only in the nucleus, where the finiteness of the nucleus has a direct effect. The screening effect occurs through the normalization of the electron eigenfunction, because it must be normalized suitably at large distances.

We take the following steps to get the normalized electron wave functions in the nucleus. In the first sub-section we calculate the unnormalized wave functions in the nucleus. In the second sub-section the regular and irregular wave functions for a point nuclear charge will be given as those for an unscreened nucleus plus screening correction terms. Then in the third sub-section the inner wave functions are multiplied by normalization factors to be smoothly connected to the linear combinations of the regular and irregular outside wave functions. In the last sub-section the behavior of our wave functions are examined more explicitly by assuming a few nuclear charge distributions.

2A. *Unnormalized electron wave function inside the nucleus*

Eqs. (1) have suitable forms for obtainning the inner wave functions. The unnormalized (regular) solution of eqs. (1) is obtained by putting $d_1 = 0$, $d_2 = 1$ (these values are chosen only for convenience) for $\kappa < 0$, and $d_1 = 1$, $d_2 = 0$ for $\kappa > 0$. The nuclear radius ρ is 0.01 to 0.02 in our unit for most nuclei, though the definition is somewhat ambiguous. The electron energy W is usually less than 10. $V(r)$ is of the order of $\alpha Z/\rho$, and αZ is less than unity. If we regard W and αZ as of the order of unity, and r as of the order of ρ, we can solve eqs. (1) in power series in ρ. The third term

in this expansion is about $(W\rho)^2$ times as large as the first term, and does not seem important in most cases. Inclusion of this term will increase the number of nuclear matrix elements so tremendously that we neglect it. Then we get the unnormalized (regular) solution of eqs. (1) as follows:

$$
\begin{aligned}
f^{(1)}_{-k}(r) &= r^k \left\{ r^{-2k-1} \int_0^r dr_1 r_1^{2k} V(r_1) J_k(r_1) \right. \\
&\quad \left. - \frac{W-1}{2k+1} J_k(r) - \frac{2(2kW+1)}{(2k+1)(2k-1)} K_k^{(1)}(r) \right\}, \\
g^{(1)}_{-k}(r) &= r^{k-1} \left\{ J_k(r) - \frac{W+1}{2k-1} r^{-2k-1} \int_0^r dr_1 r_1^{2k} V(r_1) J_k(r_1) \right. \\
&\quad \left. + \frac{2(2kW+1)}{(2k+1)(2k-1)} K_k^{(2)}(r) \right\}, \\
f^{(1)}_{k}(r) &= r^{k-1} \left\{ J_k(r) - \frac{W-1}{2k-1} r^{-2k+1} \int_0^r dr_1 r_1^{2k} V(r_1) J_k(r_1) \right. \\
&\quad \left. + \frac{2(2kW-1)}{(2k+1)(2k-1)} K_k^{(2)}(r) \right\}, \\
g^{(1)}_{k}(r) &= -r^k \left\{ r^{-2k-1} \int_0^r dr_1 r_1^{2k} V(r_1) J_k(r_1) \right. \\
&\quad \left. - \frac{W+1}{2k+1} J_k(r) - \frac{2(2kW-1)}{(2k+1)(2k-1)} K_k^{(1)}(r) \right\},
\end{aligned}
\tag{2}
$$

where $k=|\kappa|$, and $J_k(r)$, $K_k^{(1)}(r)$ and $K_k^{(2)}(r)$ satisfy the following equations:

$$
\begin{aligned}
J_k(r) &= 1 - \int_0^r dr_1 r_1^{-2k} V(r_1) \int_0^{r_1} dr_2 r_2^{2k} V(r_2) J_k(r_2), \\
K_k^{(1)}(r) &= r^{2k-1} \left\{ \int_0^r dr_1 r_1 V(r_1) \int_0^{r_1} dr_2 r_2^{2k} V(r_2) J_k(r_2) \right. \\
&\quad \left. - \int_0^r dr_1 r_1^{2k} V(r_1) K_k^{(2)}(r_1) \right\}, \\
K_k^{(2)}(r) &= \int_0^r dr_1 r_1 V(r_1) \{ J_k(r_1) + K_k^{(1)}(r_1) \}.
\end{aligned}
\tag{3}
$$

It should be noted that these functions are independent of W. Eqs. (3) can be easily solved for the finite nucleus by the iteration method, and the convergency is good. In expression (2) the first terms are the largest terms and the second and third terms are one order higher in ρ. Here we should make a comment that our equality signs are

used not only for the absolute equality but also for the approximate equality within our approximation.

The solution (2) is still not simple. In the case of the allowed decay or the unique forbidden decay these formulas might be handled but with rather troublesome effort. However, in the case of the non-unique forbidden transition, we can never control many adjustable parameters coming from all the terms in (2). Therefore, we make the additional approximation in which $\alpha Z W\rho$ is neglected in comparison with unity. This means we neglect the second terms of $g_{-k}(r)$ and $f_k(r)$ and all the third terms in (2). The ratios of the second terms of $f_{-k}(r)$ and $g_k(r)$ to their first terms are of the order of $W\rho/(\alpha Z)$, so that they are retained. In order to understand the nature of this approximation these neglected terms are shown as percentages of the retained terms in Figs. 1a—1d and 2a—2d for $Z=\pm 90$ and $W=5$, by assuming the uniform and surface nuclear charge distributions. As they are roughly proportional to $\alpha Z W\rho$, we can estimate these quantities for other Z and W values.

In should be mentioned here that the above errors are not those in the transition matrix elements but only those in the eigenfunctions. The former errors will usually be

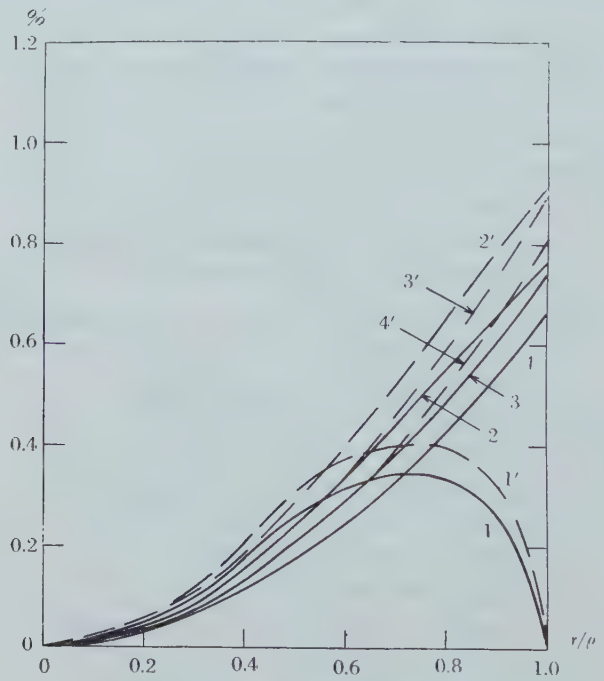


Fig. 1a. In $f^{\text{in}}_{-k}(r)$

Figs. 1a—1d. Neglected terms in the inner electron wave functions as percentages of retained terms, for $Z=\pm 90$, ($A=230$) and $W=5$. Uniform nuclear charge distributions is assumed with (minimum) $\rho=1.2A^{1/3}\times 10^{-13}$ cm. The numbers on the graphs are k's and primes are for positrons. For details see eqs. (2) and the following explanations.

of the same order of magnitude of the latter one, although the detailed value will differ in each particular case because the radial dependence of the neglected term is different from that of the retained term. In particular, the error may be considerably magnified when the main matrix element becomes extraordinarily small, or when a large cancellation among several partial matrix elements occurs. An unusually large *ft* value is an indication of such a case. One must be careful in interpreting the result of an analysis in such a case. It is dangerous to deduce a definite conclusion from it, although a plausible conclusion might be drawn.

2*B*. *Electron wave functions outside the nucleus*

Outside the nucleus the potential is independent of the nuclear charge distribution. Therefore, the actual wave function of the electron is a linear combination of the regular and irregular wave functions for the point nuclear charge. If we neglect the screening effect and also neglect $(W_{k'})^2$ in comparison with unity, the regular radial wave functions are given by[13]

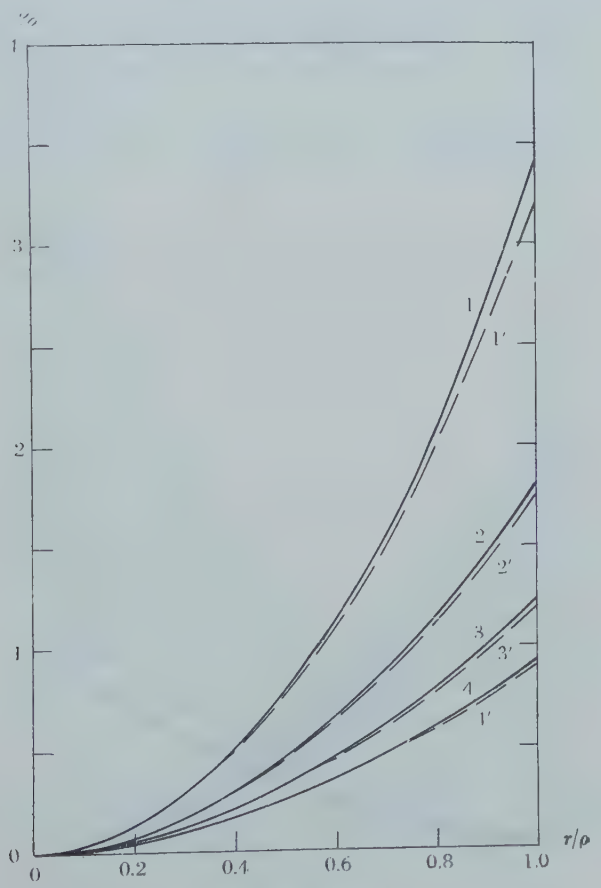


Fig. 1b. In $g^{m}_{-k}(r)$

$$f_{-k}^{\mathrm{reg}}(r)=-i\sqrt{\frac{W-1}{W}}\sqrt{F_{k-1}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k+\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k+\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1+2\gamma_k}\,ipr\right)-c.\,c.\right\},$$

$$f_{-k}^{\mathrm{reg}}(r)=-\sqrt{\frac{W+1}{W}}\,\nu'\sqrt{F_{k-1}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k+\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k+\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1+2\gamma_k}\,ipr\right)+c.\,c.\right\},\qquad(4)$$

$$f_k^{\mathrm{reg}}(r)=\sqrt{\frac{W-1}{W}}\,\nu'\sqrt{F_{k-1}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k-\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k+\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1+2\gamma_k}\,ipr\right)+c.\,c.\right\},$$

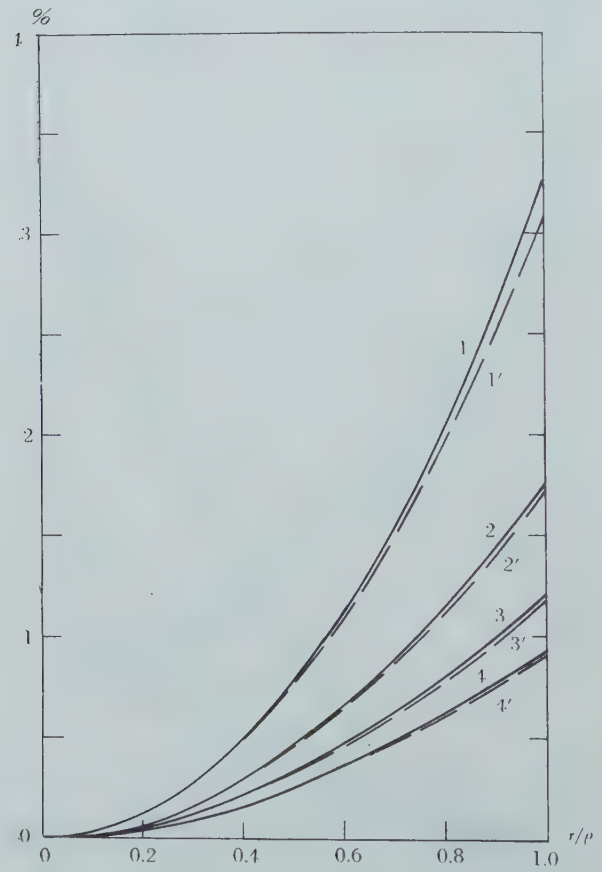


Fig. 1c. In $f_k^{\mathrm{in}}(r)$

$$g_k^{\mathrm{reg}}(r) = -i\sqrt{\frac{W+1}{W}}\sqrt{F_{k-1}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k-\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k-\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1+2\gamma_k}\,ipr\right)-c.\,c.\right).$$

Similarly, the irregular wave functions are[7),13)]

$$f_{-k}^{\mathrm{irr}}(r) = \sqrt{\frac{W-1}{W}}\sqrt{F_{k-1}^{\mathrm{irr}}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k+\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k-\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1-2\gamma_k}\,ipr\right)+c.\,c.\right\},$$
$$g_{-k}^{\mathrm{irr}}(r) = -i\sqrt{\frac{W+1}{W}}\sqrt{F_{k-1}^{\mathrm{irr}}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k+\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k-\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1-2\gamma_k}\,ipr\right)-c.\,c.\right\}, \qquad (5)$$
$$f_k^{\mathrm{irr}}(r) = i\sqrt{\frac{W-1}{W}}\sqrt{F_{k-1}^{\mathrm{irr}}(r)}\,\frac{(2p)^k(k-1)!}{2(2k)!}\,r^{k-1}$$
$$\times\left\{\left(k-\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k-\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1-2\gamma_k}\,ipr\right)-c.\,c.\right\},$$

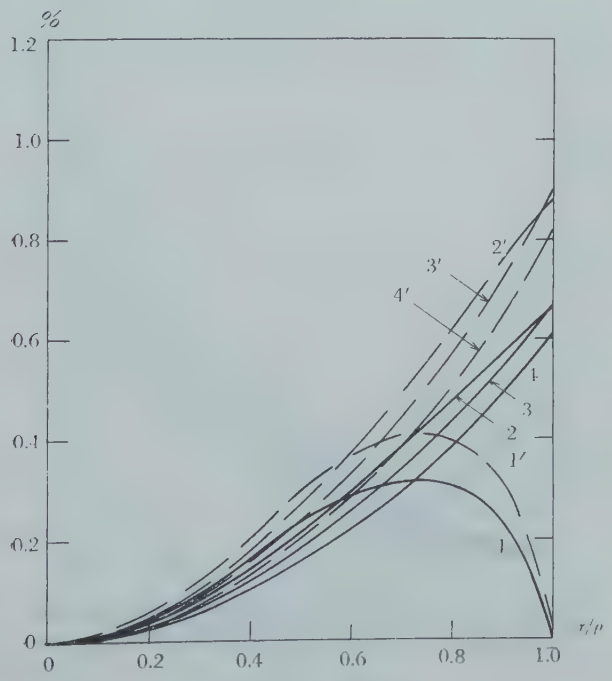


Fig. 1d. In $g_k^{\mathrm{in}}(r)$

$$g_k^{\mathrm{irr}}(r)=\sqrt{\frac{W}{W}+1}\ \sqrt{F_{k-1}^{\mathrm{irr}}(r)}\ \frac{(2p)^k(k-1)!}{2(2k)!}\ r^{k-1}$$

$$\times\left\{\left(k-\frac{i\alpha Z}{p}\right)^{1/2}\left(\gamma_k-\frac{i\alpha ZW}{p}\right)^{1/2}\left(1+\frac{1+2i\alpha ZW/p}{1-2\gamma_k}ipr\right)+c.c.\right\}.$$

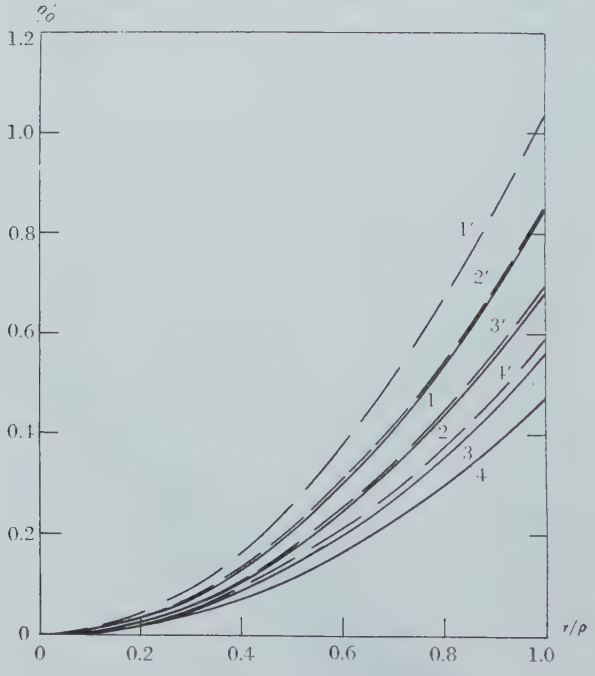
In (4) and (5) p is the magnitude of the electron momentum, $\gamma_k=(k^2-\alpha^2Z^2)^{1/2}$, and *c. c.* means complex conjugate. The Fermi functions are defined as

$$\sqrt{F_{k-1}(r)}=\frac{(2k)!}{(k-1)!\,\Gamma(1+2\gamma_k)}\ (2pr)^{\gamma_k-k}e^{\pi\alpha ZW/(2p)}\ \Gamma\left(\gamma_k+\frac{i\alpha ZW}{p}\right),$$

$$\sqrt{F_{k-1}^{\mathrm{irr}}(r)}=\frac{(2k)!}{(k-1)!\,|\Gamma(1-2\gamma_k)|}\ (2pr)^{-\gamma_k-k}e^{\pi\alpha ZW/(2p)}\left|\Gamma\left(-\gamma_k+\frac{i\alpha ZW}{p}\right)\right|. \tag{6}$$

(6)

The absolute signs of the wave functions are not important, and we may take the value of the two-valued functions $(k\pm i\alpha Z/p)^{1/2}$ and $(\gamma_k\pm i\alpha ZW/p)^{1/2}$ on the branches where they become positive for $Z\to 0$. The above wave functions are normalized so that there are R electrons in a large sphere of radius R. We note here that the factor $k-\gamma_k$ in the denominator of the higher terms in p_l' in the expansion of the irregular function



Figs. 2a—2d. The same percentages as Figs. 1a—1d except that here the nuclear charge is assumed to be entirely at the surface.

Fig. 2a. In $f_{-k}^{\mathrm{irr}}(r)$

causes no trouble for $Z \to 0$, but the approximate expressions (5) certainly fails for $\gamma_k \to 1/2$.

If we take into account the screening effect, the functions (4) and (5) are no longer the exact wave function even for the point nuclear charge. The shift of the electron potential near the nucleus due to this effect was evaluated by Rose[3], using the *WKB* method as $V_0 \simeq 1.13\,\alpha^2 Z^{4/3}$. Even if the *WKB* method is crude in this case, it certainly gives the order of magnitude correctly. The above V_0 is quite small in comparison with the magnitude of the total potential, so that the unnormalized wave function near the nucleus is practically independent of the screening effect. The screening effect is expressed as a change of the normalization in this case too. From the above consideration we may write the normalized regular and irregular wave functions of the electron for the point nuclear charge as

$$[1+\Delta_\kappa(r)]f_\kappa^{\mathrm{reg}}(r), \quad [1+\Delta_\kappa(r)]g_\kappa^{\mathrm{reg}}(r),$$
$$[1+\Delta_\kappa^{\mathrm{irr}}(r)]f_\kappa^{\mathrm{irr}}(r), \quad [1+\Delta_\kappa^{\mathrm{irr}}(r)]g_\kappa^{\mathrm{irr}}(r). \tag{7}$$

$\Delta_\kappa(r)$ and $\Delta_\kappa^{\mathrm{irr}}(r)$ are correction terms for the screening effect, and also depend on Z and W. They are almost independent of r which is as large as ρ. It seems difficult to express them in an analytical form. At present we do not have the numerical values for them, but we can expect from the experimental data for allowed transitions[5] that their effect is appreciable only for β^--decay of relatively low energies, where K electron capture dominates. Even if we do not specify $\Delta_\kappa(r)$ and $\Delta_\kappa^{\mathrm{irr}}(r)$, we shall see in the following that fairly convenient formulas can be obtained and the screening effect can be

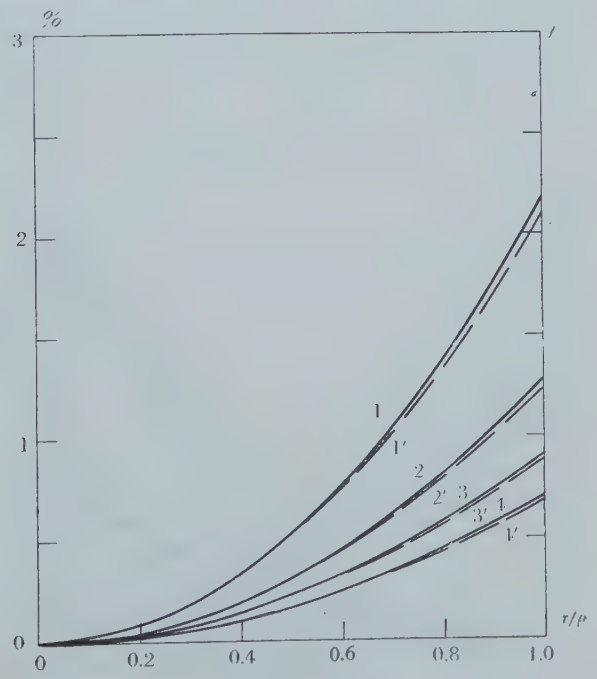


Fig. 2b. In $g_{-k}^{\mathrm{in}}(r)$

taken into account without difficulty once we know the numerical values of $\Delta_\kappa(\rho)$.

2C. *Normalization of the inner wave functions*

The actual electron wave functions outside the nucleus normalized in the same way as (5) or (6) are written as their linear combinations,

$$\left.\begin{aligned} f_\kappa(r) &= a_\kappa[1+\Delta_\kappa(r)]f_\kappa^{\mathrm{reg}}(r)+b_\kappa[1+\Delta_\kappa^{\mathrm{irr}}(r)]f_\kappa^{\mathrm{irr}}(r), \\ g_\kappa(r) &= a_\kappa[1+\Delta_\kappa(r)]g_\kappa^{\mathrm{reg}}(r)+b_\kappa[1+\Delta_\kappa^{\mathrm{irr}}(r)]g_\kappa^{\mathrm{irr}}(r). \end{aligned}\right\} : r>\rho \tag{8}$$

We are assuming that there is no nuclear charge at $r>\rho$ (This condition gives the definition of ρ as used in this paper). The normalized inner wave functions in which we are most interested can be represented as follows:

$$\left.\begin{aligned} f_\kappa(r) &= c_\kappa f_\kappa^{\mathrm{in}}(r), \\ g_\kappa(r) &= c_\kappa g_\kappa^{\mathrm{in}}(r). \end{aligned}\right\} : r<\rho \tag{9}$$

Here, c_κ is obtained by connecting (8) and (9) smoothly, and is solved formally as

$$c_\kappa = \frac{f_\kappa^{\mathrm{irr}}(\rho)g_\kappa^{\mathrm{reg}}(\rho)-g_\kappa^{\mathrm{irr}}(\rho)f_\kappa^{\mathrm{reg}}(\rho)}{f_\kappa^{\mathrm{irr}}(\rho)g_\kappa^{\mathrm{in}}(\rho)-g_\kappa^{\mathrm{irr}}(\rho)f_\kappa^{\mathrm{in}}(\rho)}[1+\Delta_\kappa]a_\kappa. \tag{10}$$

Here $\Delta_\kappa(\rho)$ is written simply as Δ_κ because it is independent of ρ in our approximation. We shall see in § 3 that this normalization factor is independent of the so-called large cancellation, and the error in it is not magnified contrary to the error in the $f_\kappa^{\mathrm{in}}(r)$ and $g_\kappa^{\mathrm{in}}(r)$. Therefore, we may calculate c_κ's somewhat more roughly than $f_\kappa^{\mathrm{in}}(r)$ or $g_\kappa^{\mathrm{in}}(r)$.

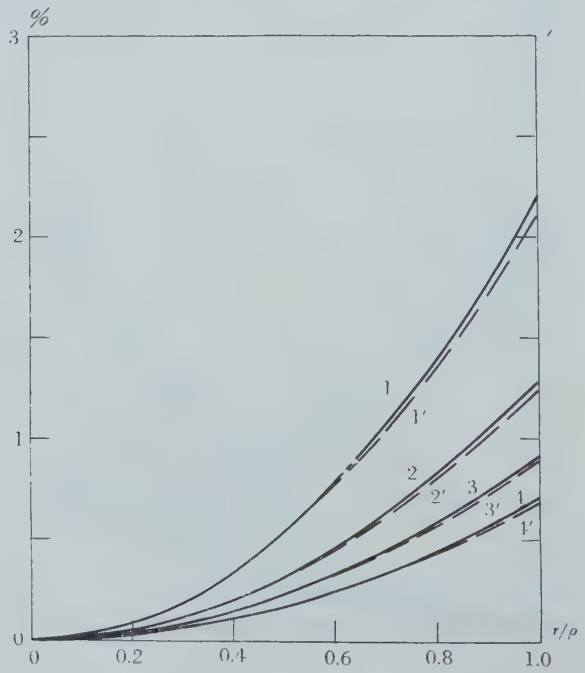


Fig. 2c. In $f_k^{\mathrm{in}}(r)$

First we can take $a_\varkappa=1$ in (10) as a suitable approximation. The first terms in the right-hand side of (8) are of the same order of magnitude as the second terms at $r=\rho$, which may be seen from considerations of self-consistency. Since the irregular function is much larger than the regular function near the nucleus, $b_\varkappa$ is much smaller than $a_\varkappa$. A simple order estimation shows that $b_\varkappa$ is about $(p\rho)^{2\gamma_k}$ times as large as $a_\varkappa$. $\gamma_k\rangle 1/2$ for $Z<118$. This suggests that we can practically neglect the second terms in (8) when we discuss the properties of (8) at $r>\infty$, because the irregular functions are as large as the regular functions at $r\to\infty$ according to the normalization condition. Then $a_\varkappa=1$ follows from the normalization of $f_\varkappa(r)$ and $g_\varkappa(r)$. (For the detailed discussion see reference 7.) It should be clear from the above argument that $a_\varkappa=1$ in our approximation is not equivalent to $b_\varkappa=0$. The numerical estimate has been performed for $Z=90$, $W=5$, $k=1$ by assuming the uniform and the surface nuclear charge distributions, and has confirmed that $a_\varkappa=1$ within 1% error. With this approximation we get

$$c_{-k}=-\sqrt{\frac{W+\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_{-k}]\frac{(2p)^k\gamma_k k!}{k(2k)!\,D_k^-},$$

$$c_k=\sqrt{\frac{W-\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_k]\frac{(2p)^k\gamma_k k!}{k(2k)!\,D_k^+}, \tag{11}$$

with

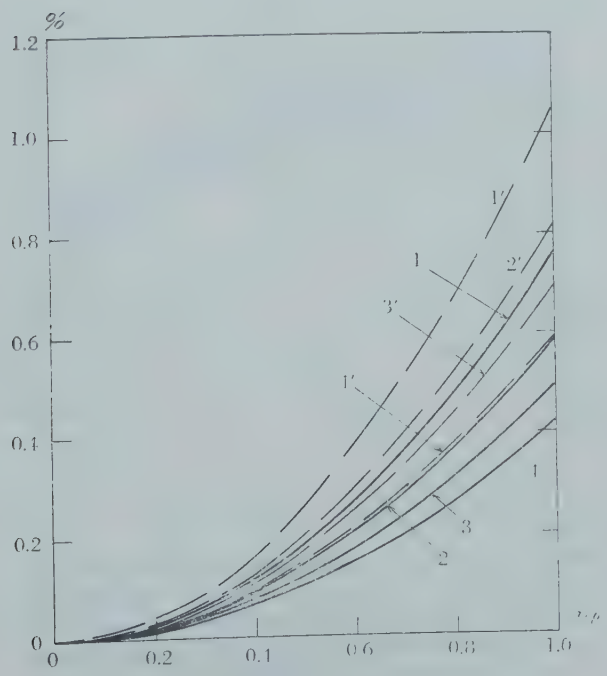


Fig. 2d. In $g_k^{\text{in}}(r)$

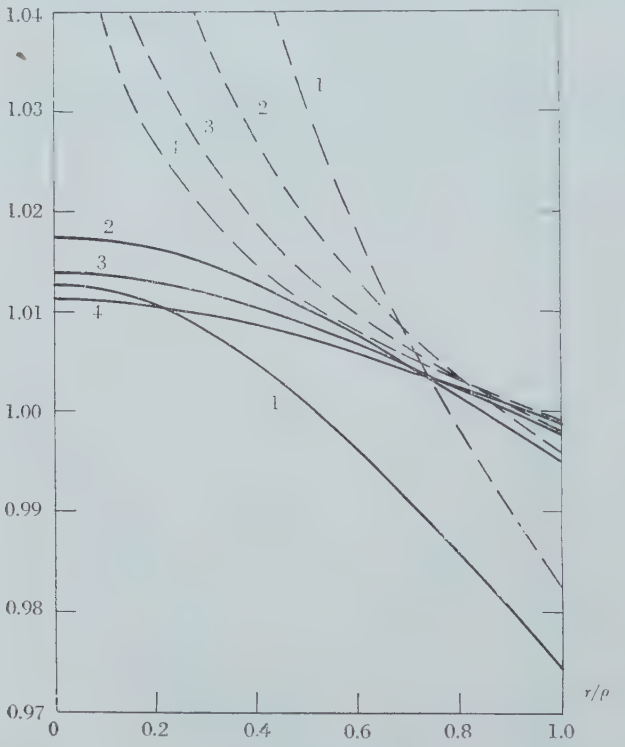
$$D_k^{\mp}=\sqrt{\frac{k+\gamma_k}{2k}}J_k(\rho)+\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}\rho^{-2k}\int_0^{\rho}dr\,r^{2k}V(r)J_k(r)$$

$$+\frac{2(2kW\pm1)\rho}{(2k+1)(2k-1)}\left\{\frac{\alpha Z}{2\gamma_k-1}\left[(2k-1)\sqrt{\frac{k+\gamma_k}{2k}}J_k(\rho)\right.\right.$$

$$\left.+(2k+1)\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}\rho^{-2k}\int_0^{\rho}dr\,r^{2k}V(r)J_k(r)\right]$$

$$\left.-\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}K_k^{(1)}(\rho)+\sqrt{\frac{k+\gamma_k}{2k}}\rho^{-1}K_k^{(2)}(\rho)\right\}.$$

In (11) $W^2\rho^2$ is neglected in comparison with unity. It is not impossible to handle all the terms in the denominators $D_k^{\mp}$ of c_{-k} and c_k. However, if we consider that we have to make many other approximations of the same order, it seems not only more



Figs. 3a—3b. Functions appearing in the electron wave functions (14) for $Z=\pm50$ and $A=120$. The solid curves are for uniform charge distributions with (minimum) $\rho=1.2A^{1/3}\times10^{-13}$ cm, and the dashed curves are for a point charge nucleus. The numbers on the figures are k's.

Fig. 3a. $I_k(r)$

convenient but also more consistent to neglect the third terms. Their orders of magnitudes are $\alpha ZW\rho/(2\gamma_k-1)$, but they are not magnified by the cancellation. Moreover, if one wants to include these terms, one may attribute their effects to J_κ. J_κ has been supposed to represent the screening correction, but it may be considered to be more

Table I. Neglected terms in the normalization factors $c_{\mp k}$ as percentages of retained terms. Uniform and surface nuclear charge distributions are assumed with (minimum) $\rho=1.2A^{1/3}\times10^{-13}$ cm. For details see eqs. (11) and the following explanations.

	k	1	2	3	4
$Z=\pm50$	uniform	0.31 $(W\pm1/2)$	0.11 $(W\pm1/4)$	0.07 $(W\pm1/6)$	0.05 $(W\pm1/8)$
$(A=120)$	surface	0.44 $(W\pm1/2)$	0.15 $(W\pm1/4)$	0.09 $(W\pm1/6)$	0.07 $(W\pm1/8)$
$Z=\pm90$	uniform	1.21 $(W\pm1/2)$	0.25 $(W\pm1/4)$	0.15 $(W\pm1/6)$	0.11 $(W\pm1/8)$
$(A=230)$	surface	1.75 $(W\pm1/2)$	0.36 $(W\pm1/4)$	0.22 $(W\pm1/8)$	0.16 $(W\pm1/8)$

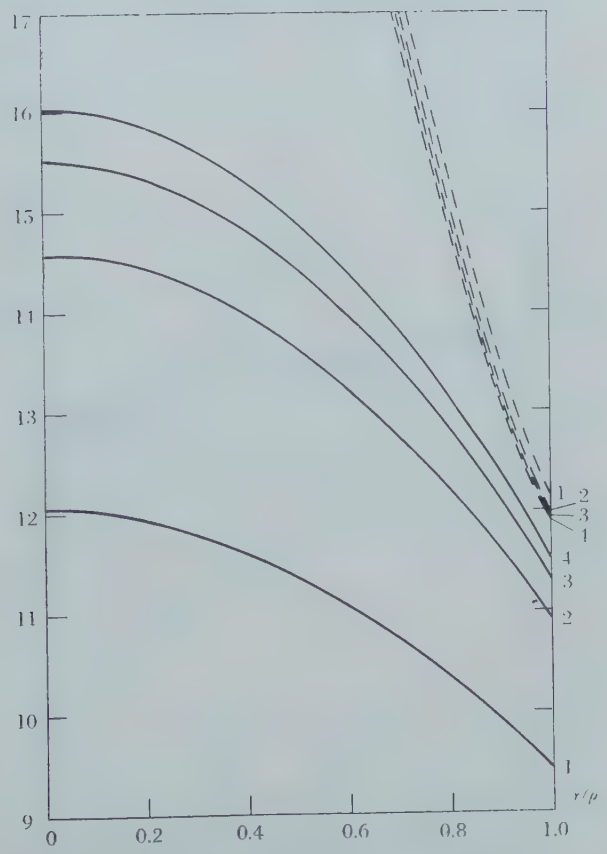
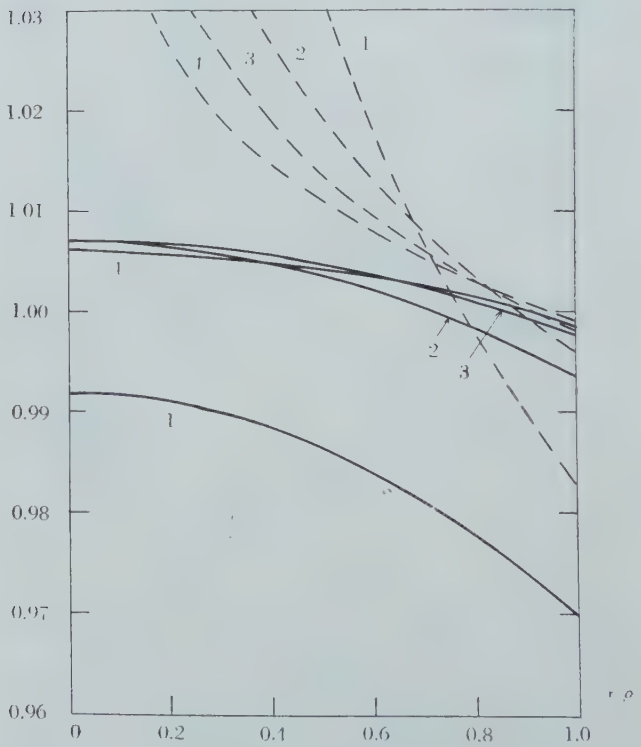


Fig. 3b. $\left|kr^{-2k-1}\int_0^r dr_1\cdot r_1^{2k}V(r_1)I_k(r_1)\right|$

flexible. The third terms in $D_k^{\mp}$ are listed in Table 1 as percentages of the main terms for a few particular cases.

After all we have obtained the following normalized wave functions:

$$f_{-k}(r) = -\sqrt{\frac{W+\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_{-k}]\frac{(2p)^k\gamma_k k!}{k(2k)!}r^k$$
$$\times\left\{r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)J_k(r_1) - \frac{W-1}{2k+1}J_k(r)\right\}D_{ok}^{-1},$$
$$g_{-k}(r) = -\sqrt{\frac{W+\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_{-k}]\frac{(2p)^k\gamma_k k!}{k(2k)!}r^{k-1}J_k(r)D_{ok}^{-1},$$
$$f_k(r) = \sqrt{\frac{W-\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_k]\frac{(2p)^k\gamma_k k!}{k(2k)!}r^{k-1}J_k(r)D_{ok}^{-1}, \qquad (12)$$
$$g_k(r) = -\sqrt{\frac{W-\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_k]\frac{(2p)^k\gamma_k k!}{k(2k)!}r^k$$
$$\times\left\{r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)J_k(r_1) - \frac{W+1}{2k+1}J_k(r)\right\}D_{ok}^{-1},$$



Figs. 4a—4b. The same functions as Figs. 3a—3b except that the solid curves are for a surface charge distributions.

Fig. 4a. $I_k(r)$

with

$$D_{ok}=\sqrt{\frac{k+\gamma_k}{2k}}J_k(\rho)+\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}\rho^{-2k}\int_0^\rho dr r^{2k}V(r)J_k(r).$$

If we define $I_k(r)$ as

$$I_k(r)=(\gamma_k/k)J_k(r)D_{ok}^{-1}, \tag{13}$$

(12) can also be written as

$$f_{-k}(r)=-\sqrt{\frac{W+\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_{-k}]\frac{(2p)^k k!}{(2k)!}r^k$$

$$\times\left\{r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)I_k(r_1)-\frac{W-1}{2k+1}I_k(r)\right\},$$

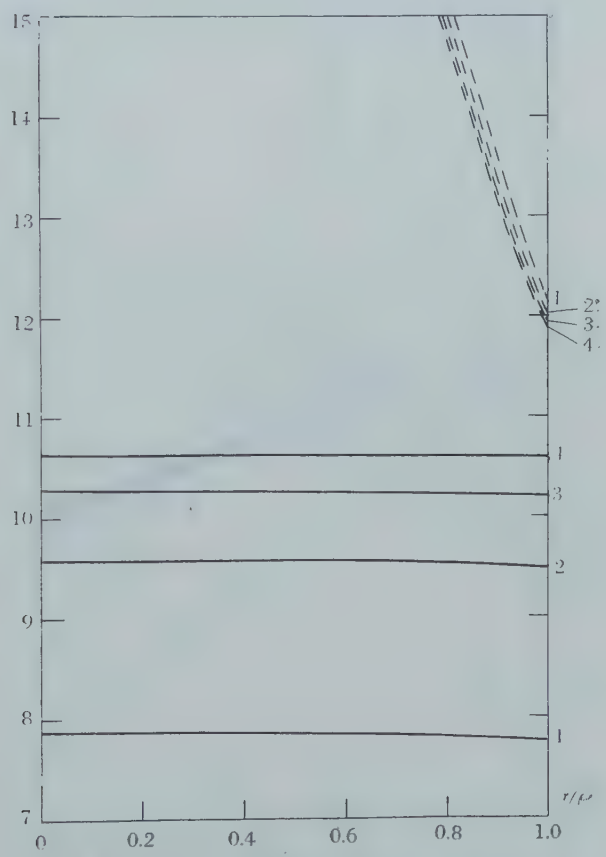


Fig. 4b. $kr^{-2k-1}\int_0^r dr_1 \cdot r_1^{2k}V(r_1)I(r_1)$

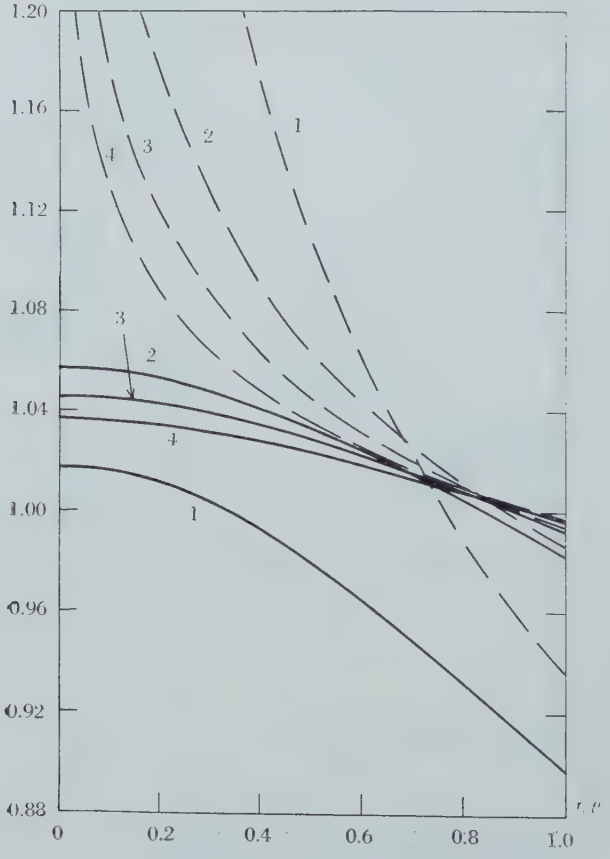
$$g_{-k}(r) = -\sqrt{\frac{W+\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_{-k}]\frac{(2p)^k k!}{(2k)!}r^{k-1}I_k(r),$$

$$f_k(r) = \sqrt{\frac{W-\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_k]\frac{(2p)^k k!}{(2k)!}r^{k-1}I_k(r), \tag{14}$$

$$g_k(r) = -\sqrt{\frac{W-\gamma_k/k}{W}}\sqrt{F_{k-1}(\rho)}[1+\Delta_k]\frac{(2p)^k k!}{(2k)!}r^k$$

$$\times\left\{r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)I_k(r_1) - \frac{W+1}{2k+1}I_k(r)\right\}.$$

Expressions (12) and (14) are our final electron wave functions. They include the effects of both the screening and the finite nucleus in a consistent way. Among the latter effects, which consist of the finite nuclear size effect in the narrow sense[7] the



Figs. 5a—5b. The same functions as Figs. 3a—b except that $Z=\pm 90$ and $A=230$.

Fig. 5a. $I_k(r)$

effect of the β-decay at the inside of the nucleus[9] and the finite de Broglie wave length effect,[8] one may suspect the omission of the last effect. As far as an unscreened point charge nucleus is concerned, the $K-U$ approximation might look better than ours because the former includes some higher terms in p_1^0 other than ours. But, their approximation was performed after the transition matrix element was squared, while our approximation is made before squaring. Suppose the neglected term in the transition matrix element is of the order of 1% compared with the largest terms. If a cancellation between the largest terms reduces the matrix element to $1/10$, which corresponds to a cancellation to $1/100$ in the squared matrix element, our error will be of the order of 10%. However, if the neglected term in the squared matrix element is of the order of 1%, the error after the same cancellation may be 100%. Actually this may happen in the case of the $K-U$ non-low-Z approximation, but their low-Z approximation fortunately gets rid of this tremendous increase of error accompanying a cancellation.[10] From this consideration it can be seen that our theory includes the main part of the finite de Broglie wave length

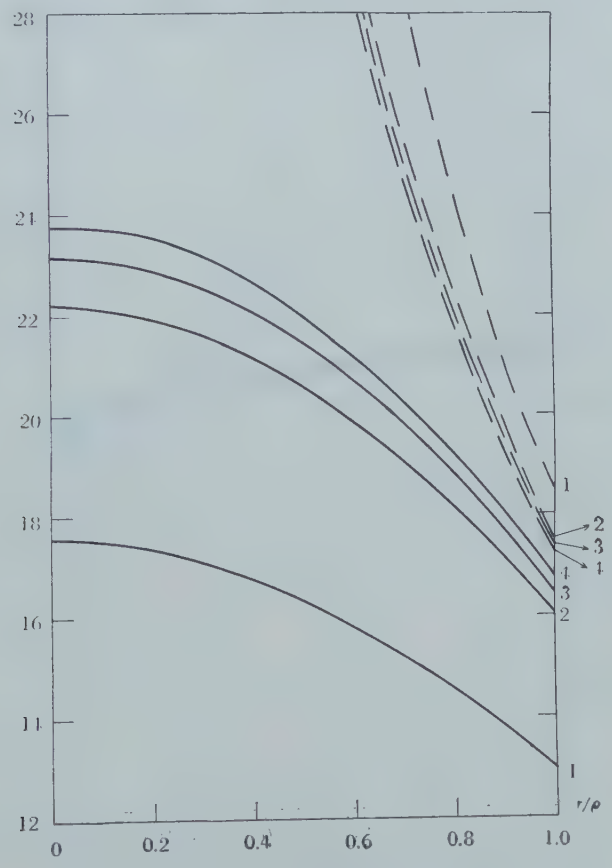


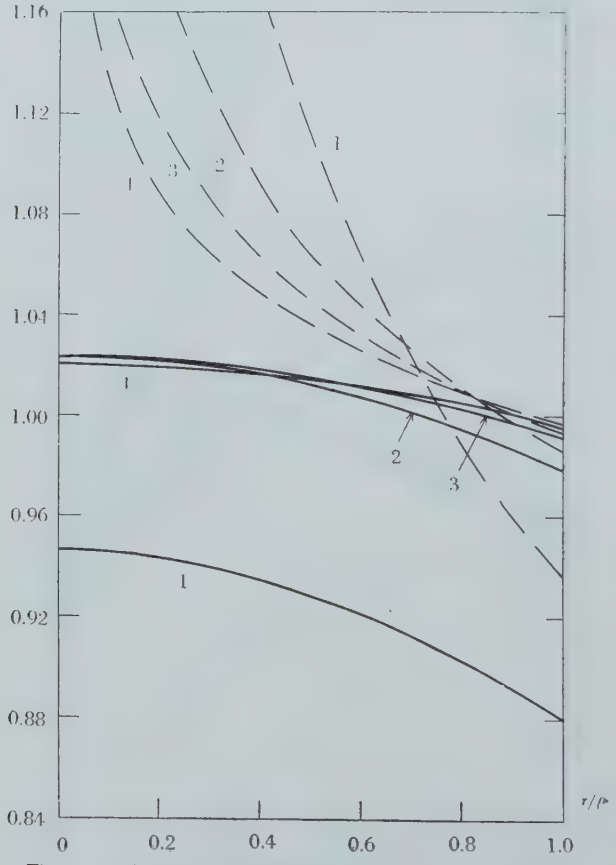
Fig. 5b. $\left| kr^{-2k-1} \int_0^r dr_1 \cdot r_1^{2k} V(r_1) I_k(r_1) \right|$

effect. Higher terms in $p\rho$ considered by Rose et al.[8], which are neglected in our calculation, are understood to be only a possibility if we notice that many terms of the same order of magnitude are neglected in their theory. We avoid uneven approximations and keep the order of the error in our mind.

The normalized wave functions should be independent of the choice of the hypothetical nuclear radius ρ. Each wave function in (12) and (14) has two ρ-dependent terms. Note that $I_k(r)$ depends on ρ. It is proved in Appendix 2 that every function in (12) and (14) is actually ρ-independent by the cancellation of the ρ-dependences of two terms. Although ρ can be chosen somewhat arbitrarily in our theory within the restriction of absence of the nuclear charge at $r > \rho$, the smallest value is favourable to estimate the error in our approximation, which is followed in this paper unless stated explicitly.

2D. *Behavior of our wave functions*

Our final forms of the electron wave functions (14) include $I_k(r)$ which depends on both k and the nuclear charge distributions. It is desirable to get an idea about that



Figs. 6a—6b. The same functions as Figs. 4a—4b except that $Z=\pm 90$ and $A=230$.

Fig. 6a. $I_k(r)$

behavior of $I_k(r)$ for later purposes. In Figs. 3a—6b $I_k(r)$ and $kr^{-2k-1}\int_0^{r} dr_1 r_1^{2k} V(r_1) \times I_k(r_1)$ are shown for $Z=\pm 50$ and ± 90 (independent of the sign) and for the uniform, surface and point nuclear charge distributions. For the point charge nucleus the $I_k(r)$ cannot be obtained by the procedure given above because the boundary condition of the first equation in (3) is unsuitable in this case. However, we can get it directly by comparing (4) with (14) as

$$I_k(r) = \sqrt{\frac{k+\gamma_k}{2k}}\left(\frac{r}{\rho}\right)^{\gamma_k - k} \quad : \text{for point charge}$$

This function satisfies the differential equation derived from the integral equation (3). It should be noted that the function for the point nuclear charge does not correspond to the $K-U$ approximation. The $K-U$ approximation rather corresponds to

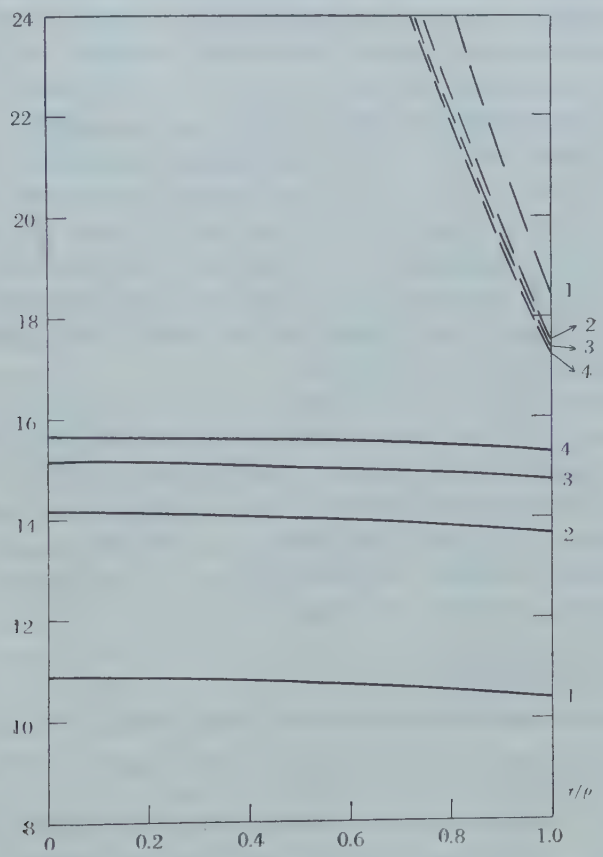


Fig. 6b. $\left| kr^{-2k-1}\int_0^{r} dr_1 \cdot r_1^{2k} V(r_1) I_k(r_1) \right|$

$$I_k(r)=\sqrt{\frac{k+\gamma_k}{2k}},$$

$$-r^{-2k-1}\int_0^r dr_1 r_1^{2k} V(r_1) I_k(r_1)=\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}\frac{1}{\rho},$$: for $K-U$ approximation,

although this correspondence is not perfect because of the different method in their approximation. Thus we have obtained the electron wave functions in quite a general form. In the next section we calculate the correction factors for β-spectra by using these wave functions.

§ 3. Correction factor for the beta-spectrum with effective nuclear radii

In this section we show how to derive and arrange the correction factors for β-spectra using the electron wave functions obtained in the previons section. As an example we treat the 1st forbidden transition assuming the *STP* interaction. See Appendix 1 for the allowed transition and the 2nd and 3rd forbidden transitions with the same interaction. In the first sub-section a primitive form of the correction factor is presented, which includes both the finite nuclear size effect and the screening effect. This form is still inconvenient for practical purposes. In the second sub-section we introduce the "effective nuclear radii" to make the form of the new correction factor quite familiar. In the third sub-section we discuss the domain of the possible values of the nuclear matrix elements and the effective nuclear radii. In the final sub-section further parametrization characteristic of the *STP* interaction is performed, which will not only help the actual calculation but also be useful for understanding the nature of our approximation.

3A. *Primitive form of the correction factor*

The correction factor for the β-spectrum is such a function of the electron energy that gives the actual β-spectrum after multiplication with the ideal spectrum (allowed-shape spectrum), $(1/2\pi^3)F_0(\rho)pWq^2dw$ (q : neutrino energy). We shall normalize our correction factor so that the above mentioned (actual) spectrum also gives the absolute transition probability. The β-decay interaction is so weak that we may use perturbation theory. In this paper we assume the Fermi theory for the β-decay interaction. The interaction Hamiltonian density in this theory is

$$\begin{aligned}
H_\beta(\boldsymbol{r}) =& G_S\psi_p^*(\boldsymbol{r})\beta\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\psi_\nu(\boldsymbol{r})\\
&+G_V[\psi_p^*(\boldsymbol{r})\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\psi_\nu(\boldsymbol{r})-\psi_p^*(\boldsymbol{r})\alpha\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\alpha\psi_\nu(\boldsymbol{r})]\\
&+G_T[\psi_p^*(\boldsymbol{r})\beta\sigma\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\sigma\psi_\nu(\boldsymbol{r})+\psi_p^*(\boldsymbol{r})\beta\alpha\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\alpha\psi_\nu(\boldsymbol{r})]\\
&+G_A[\psi_p^*(\boldsymbol{r})\sigma\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\sigma\psi_\nu(\boldsymbol{r})-\psi_p^*(\boldsymbol{r})\gamma_5\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\gamma_5\psi_\nu(\boldsymbol{r})]\\
&+G_P\psi_p^*(\boldsymbol{r})\beta\gamma_5\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\gamma_5\psi_\nu(\boldsymbol{r})\\
&+(\text{Hermite conjugate}).
\end{aligned}\tag{15}$$

Here $\psi_p(\boldsymbol{r})$, $\psi_n(\boldsymbol{r})$, $\psi_e(\boldsymbol{r})$ and $\psi_\nu(\boldsymbol{r})$ are the field operators for the proton, neutron, electron and neutrino respectively. β, α, σ and $\gamma_5 = -i\alpha_x\alpha_y\alpha_z$ are the Dirac matrices, and G_S, G_V, G_T, G_A and G_P are the coupling constants. The recent experiment by Wu et al.[2] showed that the interaction (15) is not the true one, but the interaction may be revised as

$$H_\beta^{(1)}(\boldsymbol{r}) = H_\beta(\boldsymbol{r}) + H_\beta'(\boldsymbol{r}) \tag{15'}$$

with

$$\begin{aligned}
H_\beta'(\boldsymbol{r}) = {} & G_S'\psi_p^*(\boldsymbol{r})\beta\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\gamma_5\psi_\nu(\boldsymbol{r}) \\
& + G_V'[\psi_p^*(\boldsymbol{r})\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\gamma_5\psi_\nu(\boldsymbol{r}) - \psi_p^*(\boldsymbol{r})\boldsymbol{\alpha}\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\boldsymbol{\sigma}\psi_\nu(\boldsymbol{r})] \\
& + G_T'[\psi_p^*(\boldsymbol{r})\beta\boldsymbol{\sigma}\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\boldsymbol{\alpha}\psi_\nu(\boldsymbol{r}) + \psi_p^*(\boldsymbol{r})\beta\boldsymbol{\alpha}\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\boldsymbol{\sigma}\psi_\nu(\boldsymbol{r})] \\
& + G_A'[\psi_p^*(\boldsymbol{r})\boldsymbol{\sigma}\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\boldsymbol{\alpha}\psi_\nu(\boldsymbol{r}) - \psi_p^*(\boldsymbol{r})\gamma_5\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\psi_\nu(\boldsymbol{r})] \\
& + G_P'\psi_p^*(\boldsymbol{r})\beta\gamma_5\psi_n(\boldsymbol{r})\cdot\psi_e^*(\boldsymbol{r})\beta\psi_\nu(\boldsymbol{r}) \\
& + (\text{Hermite conjugate}).
\end{aligned}$$

This interaction $H_\beta^{(1)}(\boldsymbol{r})$ includes two different kinds of terms, those which have primes attached to the coupling constants and those which do not. It is well known that the interaction (15') necessarily violates the parity conservation. The interference term of these two kinds of interactions causes left-right asymmetry. However, there occurs no such interference term in the β-spectrum, because the spectrum is a property integrated over all directions and there arises no difference between right-handed and left-handed coordinates. We can not distinguish ψ_ν and $\gamma_5\psi_\nu$ by themselves since we assume zero rest mass for the neutrino. Therefore, we get precisely the same correction factor for each kind of interactions in (15') if the prime signs are ignored. Accordingly the results of this paper are easily extended to the interaction (15'). To do so we have only to add the correction factor with primed coupling constants to that with non-primed coupling constants. We do not assume invariance of the theory under the time reversal in this paper, so that the coupling constants may take any complex numbers.[15]

The β-decay interaction has become fairly well-known now.[16] One of the clearest facts is the existence of both the Fermi type (scalar and vector type) and the Gamow-Teller type (tensor and axial vector type) interactions; i. e. allowed transitions occur for nuclear transitions of spin change 0 and 1 and parity change no. If we assume the Fermi theory (not to be confused with the Fermi type interaction or the Fermi selection rule), the 1st forbidden transitions occur whenever the nuclear spin change is 0, 1 (non-unique transition), or 2 (unique transition) and the parity change is yes. The n-th forbidden transitions for $n \geqq 2$ have the selection rule of spin change n (non-unique transition) or $n+1$ (unique transition) and parity change $(-1)^n$. In this paper we define the order of forbiddenness according to this selection rule. Then for the n-th forbidden transition, the electron and neutrino partial waves concerned can be found from the following rules: When the Dirac matrix for the nucleon in the cofactor is 1, β, σ

or $\beta\sigma$, the sum of the magnitudes of the orbital angular momenta of the electron and the neutrino is not less than n. When the Dirac matrix for the nucleon is γ_5, $\beta\gamma_5$, α or $\beta\alpha$, the sum of the magnitudes of the orbital angular momenta of the electron and the neutrino is not less than $n-1$. In the text (eq. (16) and § 3D) and Appendix 1 we take into account only the partial waves leading to the smallest sums of orbital angular momenta permitted by the above rules. For non-unique forbidden transitions this approximation means the neglect of $\alpha ZW_{l'}$ in comparison with unity, and is consistent with the approximation made in § 2. For the allowed and unique forbidden transitions, however, it means the neglect of $(\alpha Z)^2$ in comparison with unity. This approximation is usually cruder than that in § 2, so that we supplement the formulas for these transitions in Appendix 3. For more explicit arguments, see references 10 and 17. The radial wave functions of the neutrino are the Bessel functions. We use only their first terms in the expansion in $(qr)^2$.

The above two approximations (without the refinement for the allowed and unique forbidden transitions) are the same as those which were made by $K-U$ and many others. But an important difference is that we do not regard the radial wave function of the electron as constant inside the nucleus, even if it is divided by $r^{l'}$ (l' : appropriate orbital angular momenta for the large and small components). The radial wave function of the neutrino divided by $r^{l'}$ is surely constant in our approximation, but the effect of the Coulomb field is so large that we cannot make the same approximation for the electron wave function.

In the following example of a calculation by our method we assume $G_V=G_A=0$ in (15), which seems to be probable from the experiments.[16] We also assume that G_P is not much bigger than G_S or G_T because it does not seem likely[16] and furthermore an extraordinary large G_P ($|G_P/G_T|>10$) requires quite a different procedure.[15] We also assume that G_S and G_T are of the same order of magnitude. In the text we treat only the 1st forbidden transitions because the long formulas required for the others may be annoying. In Appendix 1 the allowed, the 2nd forbidden and the 3rd forbidden transitions are treated assuming the same interaction. It is not difficult to get the formulas for other interactions by the aid of reference 9.

With the above mentioned approximations we get the following primitive form of the correction factor for the 1st forbidden transitions

$$
\begin{aligned}
C_{1STP}=&+|G_S|^2\{(q^2/3)\,L_0^+(\beta\boldsymbol{r},\ \beta\boldsymbol{r})+2L_1^+(\beta\boldsymbol{r},\ \beta\boldsymbol{r})+M_0^+(\beta\boldsymbol{r},\ \beta\boldsymbol{r})\\
&+(q/3)[N_0^+(\beta\boldsymbol{r},\ \beta\boldsymbol{r})+c.\,c.]\}\\
&+|G_T|^2\{(q^2/9)\,L_0^+(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r},\ \beta\boldsymbol{\sigma}\cdot\boldsymbol{r})+M_0^+(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r},\ \beta\boldsymbol{\sigma}\cdot\boldsymbol{r})\\
&+(q/3)[N_0^+(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r},\ \beta\boldsymbol{\sigma}\cdot\boldsymbol{r})+c.\,c.]\\
&+L_0^+(\beta\alpha,\ \beta\alpha)+(q^2/6)\,L_0^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\boldsymbol{\sigma}\times\boldsymbol{r})+(1/2)\,L_1^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\boldsymbol{\sigma}\times\boldsymbol{r})\\
&+M_0^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\boldsymbol{\sigma}\times\boldsymbol{r})-(q/3)[N_0^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\boldsymbol{\sigma}\times\boldsymbol{r})+c.\,c.]\\
&-(q/3)\lfloor L_0^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\alpha)+c.\,c.]+[N_0^+(\beta\boldsymbol{\sigma}\times\boldsymbol{r},\ \beta\alpha)+c.\,c.]
\end{aligned}
$$

$$+(q^2/12)L_0^+(B_{ij}^\beta,\ B_{ij}^\beta)+(3/4)L_1^+(B_{ij}^\beta,\ B_{ij}^\beta)\}$$
$$+|G_P|^2 L_0^+(\beta\gamma_5,\ \beta\gamma_5)$$
$$+\{iG_S{}^*G_T[L_1^+(\beta\mathbf{r},\ \beta\sigma\times\mathbf{r})-M_0^+(\beta\mathbf{r},\ \beta\sigma\times\mathbf{r})+(q/3)N_0^+(\beta\mathbf{r},\ \beta\sigma\times\mathbf{r})$$
$$-(q/3)N_0^{+*}(\beta\sigma\times\mathbf{r},\ \beta\mathbf{r})-(q/3)L_0^+(\beta\mathbf{r},\ \beta\alpha)-N_0^+(\beta\mathbf{r},\ \beta\alpha)]+c.c.\}$$
$$+\{iG_T{}^*G_P[-(q/3)L_0^+(\beta\sigma\cdot\mathbf{r},\ \beta\gamma_5)-N_0^+(\beta\sigma\cdot\mathbf{r},\ \beta\gamma_5)]+c.c.\}. \qquad (16)$$

Here *c. c.* means complex conjugate, and

$$L_{k-1}^{\pm}(X,\ Y)=(2F_0(\rho)p^2)^{-1}$$
$$\times\left[\left(\int X\frac{g_{-k}(r)}{r^{k-1}}\right)^*\left(\int Y\frac{g_{-k}(r)}{r^{k-1}}\right)\pm\left(\int X\frac{f_k(r)}{r^{k-1}}\right)^*\left(\int Y\frac{f_k(r)}{r^{k-1}}\right)\right],$$
$$M_{k-1}^{\pm}(X,\ Y)=(2F_0(\rho)p^2)^{-1}$$
$$\times\left[\left(\int X\frac{f_{-k}(r)}{r^k}\right)^*\left(\int Y\frac{f_{-k}(r)}{r^k}\right)\pm\left(\int X\frac{g_k(r)}{r^k}\right)^*\left(\int Y\frac{g_k(r)}{r^k}\right)\right], \qquad (17)$$
$$N_{k-1}^{\pm}(X,\ Y)=(2F_0(\rho)p^2)^{-1}$$
$$\times\left[\left(\int X\frac{f_{-k}(r)}{r^k}\right)^*\left(\int Y\frac{g_{-k}(r)}{r^{k-1}}\right)\mp\left(\int X\frac{g_k(r)}{r^k}\right)^*\left(\int Y\frac{f_k(r)}{r^{k-1}}\right)\right].$$

X and Y are used for arbitrary nucleon operators operating on any one of all the nucleons. If they are vectors or tensors the scalar product should be taken in (17). $\int$ means not only the matrix element between the initial and the final nuclear states, but also, when two $\int$'s are multiplied, the average over the directions of the initial nuclear spins which are assumed to be random and also the sum over the directions of the final nuclear spins. Instead of using the above definition, we may define $\int X$ as "the reduced nuclear matrix element", a quantity similar to that denoted by Racah's $(\ ,\ ||\ )$ symbol or Yamada-Morita's $\mathfrak{M}$ symbol.[19] According to this definition $\int X$ is free from all the directions of X, the initial nuclear spin $\boldsymbol{J}$, and the final nuclear spin $\boldsymbol{J}'$. In order to match the normalization to the former definition, $\int X$ should be $(2J'+1)^{1/2}/(2J+1)^{1/2}$ times Yamada-Morita's $\mathfrak{M}(X)$.[19] This latter definition of $\int X$ becomes inevitable for the calculation of angular distributions. B_{ij}^β in this paper is a tensor operator whose matrix element is the "B_{ij} of $K-U$"[3]. The electron wave functions always appear in such combinations as (17) in the Fermi theory, and only those with $+$ indices occur in the *STP* interaction. In the $K-U$ approximation which replaces $f_{-k}(r)/r^k$, $g_{-k}(r)/r^{k-1}$, $f_k(r)/r^{k-1}$, $g_k(r)/r^k$ by their values at $r=\rho$, (17) tends to

$$L_{k-1}^{+}(X,\ Y)\to\left(\int X\right)^*\left(\int Y\right)L_{k-1},\quad L_{k-1}^{-}(X,\ Y)\to\left(\int X\right)^*\left(\int Y\right)L_{k-1}^{-},$$
$$M_{k-1}^{+}(X,\ Y)\to\left(\int X\right)^*\left(\int Y\right)M_{k-1},\quad M_{k-1}^{-}(X,\ Y)\to\left(\int X\right)^*\left(\int Y\right)M_{k-1}^{-},$$

$$N_{k-1}^{+}(X,\ Y)\to\left(\int X\right)^{*}\left(\int Y\right)N_{k-1},\qquad N_{k-1}^{-}(Y,\ Y)\to\left(\int X\right)^{*}\left(\int Y\right)N_{k-1}^{-}.$$

(17 $K-U$)

In the above expressions L_{k-1}^{-}, M_{k-1}^{-} and N_{k-1}^{-} are the symbols used by Smith.[4]

3B. *Effective nuclear radii*

The above form of the correction factor is inconvenient for practical purposes because the functions $L_{k-1}^{\pm}(X,\ Y)$, etc., have energy dependences in the integrands. These energy dependences are separated by substituting (14) in (17),

$$L_{k-1}^{\pm}(X,\ Y)=\left(\int XI_k(r)\right)^{*}\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$

$$\times\frac{1}{2}\left\{\frac{W+\gamma_k/k}{W}[1+\Delta_{-k}]^2\pm\frac{W-\gamma_k/k}{W}[1+\Delta_k]^2\right\},$$

$$M_{k-1}^{\pm}(X,\ Y)=\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$

$$\times\left[\left(\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)\right)^{*}\left(\int Yr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)\right)\right.$$

$$\times\frac{1}{2}\left\{\frac{W+\gamma_k/k}{W}[1+\Delta_{-k}]^2\pm\frac{W-\gamma_k/k}{W}[1+\Delta_k]^2\right\}$$

$$-\left\{\left(\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)\right)^{*}\left(\int YI_k(r)\right)\right.$$

$$\left.+\left(\int XI_k(r)\right)^{*}\left(\int Yr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)\right)\right\}$$

$$\times\frac{1}{2(2k+1)}\left\{\frac{(W+\gamma_k/k)\,(W-1)}{W}[1+\Delta_{-k}]^2\pm\frac{(W-\gamma_k/k)\,(W+1)}{W}[1+\Delta_k]^2\right\}$$

$$+\left(\int XI_k(r)\right)^{*}\left(\int YI_k(r)\right)\frac{1}{2(2k+1)^2}$$

$$\left.\times\left\{\frac{(W+\gamma_k/k)\,(W-1)^2}{W}[1+\Delta_{-k}]^2\pm\frac{(W-\gamma_k/k)\,(W+1)^2}{W}[1+\Delta_k]^2\right\}\right],$$

$$N_{k-1}^{\pm}(X,\ Y)=\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$

$$\times\left[\left(\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)\right)^{*}\left(\int YI_k(r)\right)\right.$$

$$\left.\times\frac{1}{2}\left\{\frac{W+\gamma_k/k}{W}[1+\Delta_{-k}]^2\pm\frac{W-\gamma_k/k}{W}[1+\Delta_k]^2\right\}\right.$$

$$-\left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{1}{2(2k+1)}$$

$$\times\left\{\frac{(W+\gamma_k/k)(W-1)}{W}[1+\Delta_{-k}]^2\pm\frac{(W-\gamma_k/k)(W+1)}{W}[1+\Delta_k]^2\right\}\Bigg]. \quad (18)$$

In order to write (18) more concisely we use the following notation :

$$\lambda_k^{\pm}=\frac{1}{2}\left\{\frac{W+\gamma_k/k}{W}[1+\Delta_{-k}]^2+\frac{W-\gamma_k/k}{W}[1+\Delta_k]^2\right\},$$

$$\lambda_k^{-}=\frac{W}{2}\left\{\frac{W+\gamma_k/k}{W}[1+\Delta_{-k}]^2-\frac{W-\gamma_k/k}{W}[1+\Delta_k]^2\right\},$$

$$\mu_k^{+}=\frac{1}{2p^2}\left\{\frac{(W+\gamma_k/k)(W-1)^2}{W}[1+\Delta_{-k}]^2+\frac{(W-\gamma_k/k)(W+1)^2}{W}[1+\Delta_k]^2\right\},$$

$$\mu_k^{-}=-\frac{W}{2p^2}\left\{\frac{(W+\gamma_k\ k)(W-1)^2}{W}[1+\Delta_{-k}]^2-\frac{(W-\gamma_k/k)(W+1)^2}{W}[1+\Delta_k]^2\right\},$$

$$\nu_k^{+}=\frac{W}{2p^2}\left\{\frac{(W+\gamma_k/k)(W-1)}{W}[1+\Delta_{-k}]^2+\frac{(W-\gamma_k/k)(W+1)}{W}[1+\Delta_k]^2\right\},$$

$$\nu_k^{-}=\frac{W^2}{2p^2}\left\{\frac{(W+\gamma_k/k)(W-1)}{W}[1+\Delta_{-k}]^2-\frac{(W-\gamma_k/k)(W+1)}{W}[1+\Delta_k]^2\right\}. \quad (19)$$

They are not independent, but

$$\mu_k^{+}=\lambda_k^{+}+(2/p^2)(\lambda_k^{+}-\lambda_k^{-}), \qquad \mu_k^{-}=\lambda_k^{+}+(W^2+1)/p^2\cdot(\lambda_k^{+}-\lambda_k^{-}),$$

$$\nu_k^{-}=\lambda_k^{-}+(1/p^2)(\lambda_k^{-}-\lambda_k^{-}), \qquad \nu_k^{-}=-W^2/p^2\cdot(\lambda_k^{+}-\lambda_k^{-}).$$

If the screening effect is neglected, i. e. if we put $\Delta_k=\Delta_{-k}=0$, (19) becomes

$$\lambda_k^{+}\to 1, \qquad \lambda_k^{-}\to 1-(1-\gamma_k/k),$$

$$\mu_k^{+}\to 1+(2/p^2)(1-\gamma_k/k), \qquad \mu_k^{-}\to 1+(W^2+1)/p^2\cdot(1-\gamma_k/k),$$

$$\nu_k^{+}\to 1+(1/p^2)(1-\gamma_k/k), \qquad \nu_k^{-}\to -W^2/p^2\cdot(1-\gamma_k/k). \quad (20)$$

$\lambda_k^{\pm}$, $\mu_k^{\pm}$ and $\nu_k^{\pm}$ can be regarded as the factors which make it possible to take into account the screening correction. $1-\gamma_k/k$ is considerably smaller than unity and can be neglected in most cases. p^2 in the denominators in $\mu_k^{\pm}$ and $\nu_k^{\pm}$ causes no trouble because these quantities are always multiplied by p^2 in the correction factor.

Now we define the " effective nuclear radius " $\rho_k(X)$ as

$$\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)I_k(r_1)=-\frac{1}{k}\frac{\alpha Z}{2\rho_k(X)}\int XI_k(r),$$

or

$$\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k} V(r_1) J_k(r_1) = -\frac{1}{k}\frac{\alpha Z}{2\rho_k(X)}\int XJ_k(r). \tag{21}$$

Since the effective nuclear radius is defined by the ratio of two nuclear matrix elements, it depends on both k and X, but is independent of the electron energy. It should be noted that our effective nuclear radius is a well-defined quantity contrary to the hypothetical nuclear radius ρ. When we know the nuclear wave function thoroughly there is nothing arbitrary about the effective radius as can be easily seen from the latter form of its definition (21).

Substituting (19) and (21) in (18) we get

$$L_{k-1}^{+}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2\lambda_k^{+},$$

$$L_{k-1}^{-}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2\frac{\lambda_k^{-}}{W},$$

$$M_{k-1}^{+}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$
$$\times\left[\frac{1}{k^2}\left(\frac{\alpha Z}{2\rho_k^*(X)}\right)\left(\frac{\alpha Z}{2\rho_k(Y)}\right)\lambda_k^{-} + \frac{p^2}{k(2k+1)W}\left(\frac{\alpha Z}{2\rho_k^*(X)} + \frac{\alpha Z}{2\rho_k(Y)}\right)\nu_k^{-}\right.$$
$$\left. + \frac{p^2}{(2k+1)^2}\mu_k^{+}\right],$$

$$M_{k-1}^{-}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$
$$\times\frac{1}{W}\left[\frac{1}{k^2}\left(\frac{\alpha Z}{2\rho_k^*(X)}\right)\left(\frac{\alpha Z}{2\rho_k(Y)}\right)\lambda_k^{-} + \frac{p^2}{k(2k+1)W}\left(\frac{\alpha Z}{2\rho_k^*(X)} + \frac{\alpha Z}{2\rho_k(Y)}\right)\nu_k^{-}\right.$$
$$\left. - \frac{p^2}{(2k+1)^2}\mu_k^{-}\right],$$

$$N_{k-1}^{+}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$
$$\times\left[-\frac{1}{k}\frac{\alpha Z}{2\rho_k^*(X)}\lambda_k^{+} - \frac{p^2}{(2k+1)W}\nu_k^{+}\right],$$

$$N_{k-1}^{-}(X, Y) = \left(\int XI_k(r)\right)^*\left(\int YI_k(r)\right)\frac{F_{k-1}(\rho)}{F_0(\rho)}\left[\frac{2^k k!}{(2k)!}p^{k-1}\right]^2$$
$$\times\frac{1}{W}\left[-\frac{1}{k}\frac{\alpha Z}{2\rho_k^*(X)}\lambda_k^{-} - \frac{p^2}{(2k+1)W}\nu_k^{-}\right]. \tag{22}$$

If we compare the above formulas with those in the $K-U$ low-Z approximation, we find a great similarity between them. If we put $I_k(r)=1$, $F_{k-1}(\rho)/F_0(\rho)=1$, $\lambda_k^{\pm}=\mu_k^{\pm}=\nu_k^{+}=1$, $\nu_k^{-}=0$ and $\rho_k^*(X)=\rho_k(Y)=\rho$ in our formulas, they reduce to the

formulas in the $K-U$ low-Z approximation. Actually $I_k(r)$ is not much different from 1, especially for small Z, as can be seen from Figs. 3a—6a. $F_{k-1}(\rho)/F_0(\rho)$ differs from 1 somewhat more than $I_k(r)$ does[20], but $F_{k-1}(\rho)/F_0(\rho)=1$ does not seem to be an essential part of the $K-U$ low-Z approximation. As the $K-U$ approximation does not take the screening effect into account, it is natural that its formulas should correspond to ours with $\lambda_k^{\pm}=\mu_k^{\pm}=\nu_k^{+}=1$, $\nu_k^{-}=0$ which are fairly good approximations except for very heavy nuclei. The fact that in the $K-U$ approximation $\rho_k^*(X)=\rho_k(Y)=\rho$ seems to be the most essential difference between that approximation and ours. However, the $K-U$ low-Z approximation can be regarded as a special case of ours because we may assume $\rho_k(X)$ or $\rho_k(Y)$ to be either equal to or different from ρ, as far as we can regard nuclear matrix elements as adjustable parameters (see § 3 C). On the other hand the $K-U$ non-low-Z approximation is very different from ours, and cannot be regarded as a special case of our approximation.

3C. *Interrelations among the nuclear matrix elements and the effective nuclear radii*

From the argument of the last sub-section we see that in our formulation there appear two kinds of nuclear matrix elements $\int XI_k(r)$ and $\int Xr^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)I_k(r_1)$. The effective nuclear radius $\rho_k(X)$ can be obtained from the ratio of the above two nuclear matrix elements. In most cases we cannot calculate the above quantities with reliable accuracy. Therefore, they are treated as adjustable parameters, and may in principle be determined from the experimental data. However, they cannot be regarded as independent parameters since there are interrelations among them.

In a given transition there are generally a few X's and k values. It seems difficult to get any reliable relation between the nuclear matrix elements of different X's in the present situation, so that they may be considered to be independent of each other. However, $I_k(r)$ does not vary significantly for different k values, so that $\int XI_k(r)$ will not depend strongly on k. This means that we cannot regard such quantities as $\int XI_j(r)$ and $\int XI_{j+1}(r)$ as independent adjustable parameters. It is not easy to say generally how much difference is allowed between these two nuclear matrix elements, because this difference depends not only on Z and nuclear charge distributions but also on other factors such as nuclear models when applicable. However, the following method will give a fairly reasonable estimate of the ratio of two nuclear matrix elements.

Let us consider $\int XI_2(r)/\int XI_1(r)$, which can be written

$$\int XI_2(r)/\int XI_1(r)=I_2(\bar{r})/I_1(\bar{r})+\int X[I_1(\bar{r})I_2(r)-I_1(r)I_2(\bar{r})]/I_1(\bar{r})\int XI_1(r). \tag{23}$$

Here $\bar{r}$ is a certain average of r and is taken so that the radial dependence of the function $I_1(\bar{r})I_2(r)-I_1(r)I_2(\bar{r})$ differs from that of $I_1(r)$ as far as possible. $\bar{r}=0.8\rho$ may be a good choice if the smallest ρ value (see § 2C) is chosen. $I_2(\bar{r})/I_1(\bar{r})$ will serve

as a middle point of the range of $\int XI_2(r)/\int XI_1(r)$. On the other hand the second term in the right-hand side of (23) will take a positive sign as often as a negative sign because the radial dependences of the integrands of the numerator and denominator are so different. The integrand of the numerator vanishes at $r=\bar{r}$. If we choose $\bar{r}=0.8\rho$ as stated before the largest contribution to the numerator will come from the region around $r=\bar{r}_1\simeq 0.6\rho$ and $r=\bar{r}_2\simeq 0.9\rho$. Then $n\sum_{i=1,2}|I_1(\bar{r})I_2(\bar{r}_i)-I_1(\bar{r}_i)I_2(\bar{r})|/[2I_1^2(\bar{r})]$ will serve as a measure of the range of the ratio $\int XI_2(r)/\int XI_1(r)$, or as a measure of the radius of the range on the complex plane if we do not assume invariance under the time reversal even for the nuclear interaction[21]:

$$\left|\int XI_2(r)/\int XI_1(r)-I_2(\bar{r})/I_1(\bar{r})\right| \lesssim n\sum_{i=1,2}|I_1(\bar{r})I_2(\bar{r}_i)-I_1(\bar{r})I_2(\bar{r})|/[2I_1^2(\bar{r})], \tag{24a}$$

with $$\bar{r}\simeq 0.8\rho,\quad \bar{r}_1\simeq 0.6\rho,\quad \bar{r}_2\simeq 0.9\rho.$$

Here the "ratio of risk" of (24a) depends on the value of n. If we choose n as 1 the ratio of risk is expected to be of the order of 1/2, because the "probability" that

$$\left|\int X[I_1(\bar{r})I_2(r)-I_1(r)I_2(\bar{r})]/\sum_{i=1,2}[I_1(\bar{r})I_2(\bar{r}_i)-I_1(\bar{r}_i)I_2(\bar{r})]\right|$$

is larger than $\left|\int XI_1(r)/[2I_1(r)]\right|$ seems to be about 1/2 when the nuclear wave function is completely unknown. For $n\gg 1$ failure of (24a) occurs when $\left|\int XI_1(r)\right|$ is very small due to a cancellation in the radial integral. It is natural to assume that the *a priori* "probability" for $\left|\int XI_1(r)\right|$ to take a very small value is independent of that value. This means that the "ratio of risk" of (24a) for $n\gg 1$ is inversely proportional to n. From the above argument we may regard the "ratio of risk" as of the order of $1/(2n)$,

$$\text{"Ratio of risk" of (24a)}=O(1/(2n)). \tag{24b}$$

The authors feel that $n=5$ and 0.1 risk are suitable for many cases.

Although the above argument is very crude, (24a) and (24b) will be helpful for practical calculations. The numerical results of the above method depend on nuclear charge distributions. We assume uniform charge distributions which seem to be pretty close to the actual one, and calculate the allowable ranges of some important ratios of nuclear matrix elements in the same way as (24a):

$$\int XI_i(r)/\int XI_j(r)=1, \qquad : Z=0 \tag{25a}$$

$$\left.\begin{aligned}\left|\int XI_2(r)/\int XI_1(r)-1.01_6\right|&<0.003_5n,\\ \left|\int XI_3(r)/\int XI_1(r)-1.01_7\right|&<0.004_8n,\end{aligned}\right|$$

$$\left.\begin{aligned}
&\left|\int XI_4(r)/\int XI_1(r) - 1.01_7\right| < 0.005_6 n,\\
&\left|\int XI_3(r)/\int XI_2(r) - 1.00_1\right| < 0.001_4 n,\\
&\left|\int XI_4(r)/\int XI_2(r) - 1.00_1\right| < 0.002_2 n,\\
&\left|\int XI_4(r)/\int XI_3(r) - 1.00_0\right| < 0.000_8 n,
\end{aligned}\right\} : Z = \pm 50 \qquad (25b)$$

$$\left.\begin{aligned}
&\left|\int XI_2(r)/\int XI_1(r) - 1.07_8\right| < 0.012_3 n,\\
&\left|\int XI_3(r)/\int XI_1(r) - 1.08_2\right| < 0.017_5 n,\\
&\left|\int XI_4(r)/\int XI_1(r) - 1.08_1\right| < 0.020_2 n,\\
&\left|\int XI_3(r)/\int XI_2(r) - 1.00_4\right| < 0.004_7 n,\\
&\left|\int XI_4(r)/\int XI_2(r) - 1.00_3\right| < 0.007_2 n,\\
&\left|\int XI_4(r)/\int XI_3(r) - 1.00_0\right| < 0.002_5 n,
\end{aligned}\right\} : Z = \pm 90 \qquad (25c)$$

$$\left.\begin{aligned}
&|\rho/\rho_1(X) - 0.87_2| < 0.045_0 n,\\
&|\rho/\rho_2(X) - 1.01_7| < 0.064_2 n,\\
&|\rho/\rho_3(X) - 1.07_2| < 0.075_0 n,
\end{aligned}\right\} : Z \approx 0 \ (\neq 0) \qquad (26a)$$

$$\left.\begin{aligned}
&|\rho/\rho_1(X) - 0.88_2| < 0.042_9 n,\\
&|\rho/\rho_2(X) - 1.02_4| < 0.063_6 n,\\
&|\rho/\rho_3(X) - 1.07_4| < 0.074_6 n,
\end{aligned}\right\} : Z = \pm 50 \qquad (26b)$$

$$\left.\begin{aligned}
&|\rho/\rho_1(X) - 0.90_2| < 0.037_7 n,\\
&|\rho/\rho_2(X) - 1.04_5| < 0.062_4 n,\\
&|\rho/\rho_3(X) - 1.07_9| < 0.073_7 n,
\end{aligned}\right\} : Z = \pm 90 \qquad (26c)$$

$$\left.\begin{aligned}
&\left|\frac{2\int Xr^{-5}\int_0^r dr_1 r_1^4 V(r_1) I_2(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^2 V(r_1) I_1(r_1)} - 1.16_6\right| \lesssim 0.013_4 n,\\
&\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^6 V(r_1) I_3(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^2 V(r_1) I_1(r_1)} - 1.23_0\right| \lesssim 0.022_7 n,
\end{aligned}\right\} : Z \approx 0 \ (\neq 0) \qquad (27a)$$

$$\left.\begin{aligned}
&\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^{6} V(r_1) I_3(r_1)}{2\int Xr^{-5}\int_0^r dr_1 r_1^{4} V(r_1) I_2(r_1)} - 1.05_4\right| \lesssim 0.007_1 n\,,\\
&\left|\frac{2\int Xr^{-5}\int_0^r dr_1 r_1^{4} V(r_1) I_2(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^{2} V(r_1) I_1(r_1)} - 1.18_2\right| \lesssim 0.012_8 n\,,\\
&\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^{6} V(r_1) I_3(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^{2} V(r_1) I_1(r_1)} - 1.24_1\right| \lesssim 0.021_1 n\,,
\end{aligned}\right\} : Z=\pm 50 \qquad (27b)$$

$$\left.\begin{aligned}
&\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^{6} V(r_1) I_3(r_1)}{2\int Xr^{-5}\int_0^r dr_1 r_1^{4} V(r_1) I_2(r_1)} - 1.04_9\right| \lesssim 0.006_1 n\,,\\
&\left|\frac{2\int Xr^{-5}\int_0^r dr_1 r_1^{4} V(r_1) I_2(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^{2} V(r_1) I_1(r_1)} - 1.24_9\right| \lesssim 0.008_0 n\,,\\
&\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^{6} V(r_1) I_3(r_1)}{\int Xr^{-3}\int_0^r dr_1 r_1^{2} V(r_1) I_1(r_1)} - 1.29_5\right| \lesssim 0.013_3 n\,,
\end{aligned}\right\} : Z=\pm 90 \qquad (27c)$$

$$\left|\frac{3\int Xr^{-7}\int_0^r dr_1 r_1^{6} V(r_1) I_3(r_1)}{2\int Xr^{-5}\int_0^r dr_1 r_1^{4} V(r_1) I_2(r_1)} - 1.03_6\right| \lesssim 0.004_3 n\,,$$

with

$$\text{“Ratios of risk of eqs. (25a) — (27c)”} = O(1/(2n))\,.$$

The application of (25a) — (27c) are given in the next sub-section and Appendix 1. In concluding this sub-section, it should be mentioned again that the above argument is based on our ignorance of nuclear wave functions.

3D. Further parametrization in the 1st forbidden transition in the STP interaction

The argument untill now is quite general and not restricted to a special β-decay interaction. However, the procedure given in this sub-section is rather characteristic of the assumed *STP* interaction.

After substituting (22) in (16), we can arrange (16) into

$$C_{1STP} = \lambda_1^{+}\left|\frac{\alpha Z}{2\rho_1(\beta\sigma\cdot\boldsymbol{r})} G_T\left(\int \beta\sigma\cdot\boldsymbol{r} I_1(r)\right) + iG_I\left(\int \beta\gamma_5 I_1(r)\right)\right|^2$$

$$+\frac{1}{3}\left(q\lambda_1^+-\frac{p^2}{W}\nu_1^+\right)\left\{\left(G_T\int\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}\,I_1(r)\right)^*\right.$$

$$\left.\times\left[\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r})}G_T\left(\int\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}\,I_1(r)\right)+iG_P\left(\int\beta\gamma_5 I_1(r)\right)\right]+c.\,c.\right\}$$

$$+\frac{1}{9}\left(q^2\lambda_1^+ + p^2\mu_1^+ - \frac{2qp^2}{W}\nu_1^+\right)\left|G_T\int\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}\,I_1(r)\right|^2$$

$$+\lambda_1^+\left|G_T\left[\left(\int\beta\boldsymbol{\alpha}\,I_1(r)\right)-\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{\sigma}\times\boldsymbol{r})}\left(\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_1(r)\right)\right]\right.$$

$$\left.-i\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{r})}G_S\left(\int\beta\boldsymbol{r}\,I(r)\right)\right|^2$$

$$+\frac{1}{3}\left(q\lambda_1^+-\frac{p^2}{W}\nu_1^+\right)\left[\left(iG_S\int\beta\boldsymbol{r}\,I_1(r)\right)^*\left\{G_T\left[\left(\int\beta\boldsymbol{\alpha}\,I_1(r)\right)\right.\right.\right.$$

$$\left.\left.\left.-\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{\sigma}\times\boldsymbol{r})}\left(\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_1(r)\right)\right]-i\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{r})}G_S\left(\int\beta\boldsymbol{r}\,I_1(r)\right)\right\}+c.\,c.\right]$$

$$-\frac{1}{3}\left(q\lambda_1^+-\frac{p^2}{W}\nu_1^+\right)\left(G_T\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_1(r)\right)^*\left\{G_T\left[\left(\int\beta\boldsymbol{\alpha}\,I_1(r)\right)\right.\right.$$

$$\left.\left.-\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r})}\left(\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_1(r)\right)\right]-i\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{r})}G_S\left(\int\beta\boldsymbol{r}\,I_1(r)\right)\right\}+c.\,c.\Big]$$

$$+\frac{1}{9}\left(3q^2\lambda_1^+ + p^2\mu_1^+ - \frac{2qp^2}{W}\nu_1^+\right)\left|G_S\int\beta\boldsymbol{r}\,I_1(r)\right|^2$$

$$+\frac{2p^2}{9}\frac{F_1(\rho)}{F_0(\rho)}\lambda_2^+\left|G_S\int\beta\boldsymbol{r}\,I_2(r)\right|^2$$

$$+\frac{1}{18}\left(3q^2\lambda_1^+ + 2p^2\mu_1^+ + \frac{4qp^2}{W}\nu_1^+\right)\left|G_T\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_2(r)\right|^2$$

$$+\frac{p^2}{18}\frac{F_1(\rho)}{F_0(\rho)}\lambda_2^+\left|G_T\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_2(r)\right|^2$$

$$+\frac{p^2}{9}\mu_1^+\left[\left(iG_S\int\beta\boldsymbol{r}\,I_1(r)\right)^*G_T\left(\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_1(r)\right)+c.\,c.\right]$$

$$-\frac{p^2}{9}\frac{F_1(\rho)}{F_0(\rho)}\lambda_2^+\left[\left(iG_S\int\beta\boldsymbol{r}\,I_2(r)\right)^*G_T\left(\int\beta\boldsymbol{\sigma}\times\boldsymbol{r}\,I_2(r)\right)+c.\,c.\right]$$

$$+\frac{q^2}{12}\lambda_1^+\sum_{i,j}\left|G_T\int B_{ij}^3 I_1(r)\right|^2+\frac{p^2}{12}\frac{F_1(\rho)}{F_0(\rho)}\lambda_2^+\sum_{i,j}\left|G_T\int B_{ij}^3 I_2(r)\right|^2. \qquad (28)$$

Now we define the (adjustable) parameters independent of W and ρ as follows;

$$x_1^{(0)}=\frac{\alpha Z}{2\rho_1(\beta\sigma\cdot\boldsymbol{r})}+\frac{iG_P\int\beta\gamma_5 I_1(r)}{G_T\int\beta\sigma\cdot\boldsymbol{r}\,I_1(r)},$$

$$x_1^{(1)}=\Big[G_T\Big(\int\beta\alpha\, I_1(r)\Big)-\frac{\alpha Z}{2\rho_1(\beta\sigma\times\boldsymbol{r})}G_T\Big(\int\beta\sigma\times\boldsymbol{r}\,I_1(r)\Big)$$
$$-i\frac{\alpha Z}{2\rho_1(\beta\boldsymbol{r})}G_S\Big(\int\beta\boldsymbol{r}\,I_1(r)\Big)\Big]\Big[G_T\int\beta\sigma\times\boldsymbol{r}\,I_1(r)\Big]^{-1},$$

$$y_1^{(1)}=\frac{iG_S\int\beta\boldsymbol{r}\,I_1(r)}{G_T\int\beta\sigma\times\boldsymbol{r}\,I_1(r)},\quad y_2^{(1)}=\frac{iG_S\int\beta\boldsymbol{r}\,I_2(r)}{G_T\int\beta\sigma\times\boldsymbol{r}\,I_1(r)},\tag{29}$$

$$z_2^{(1)}=\frac{\int\beta\sigma\times\boldsymbol{r}\,I_2(r)}{\int\beta\sigma\times\boldsymbol{r}\,I_1(r)},\quad u_2^{(2)}=\frac{\int B_{ij}^{\beta}I_2(r)}{\int B_{ij}^{\beta}I_1(r)}.$$

Each index of the parameter in (29) indicates a magnitude of the (vector) sum of the emitted electron and neutrino momenta. Substituting (29) in (28) we get a fairly concise expression for the correction factor,

$$C_{1STP}=\Big|G_T\int\beta\sigma\cdot\boldsymbol{r}\,I_1(r)\Big|^2\Big\{\lambda_1|x_1^{(0)}|^2-\frac{2}{3}\Big(q\lambda_1^{+}-\frac{p^2}{W}\nu_1^{+}\Big)\mathfrak{Re}\,(x_1^{(0)})$$
$$+\frac{q^2}{9}\lambda_1^{+}+\frac{p^2}{9}\mu_1^{+}-\frac{2qp^2}{9W}\nu_1^{+}\Big\}$$
$$+\Big|G_T\int\beta\sigma\cdot\boldsymbol{r}\,I_1(r)\Big|^2\Big\{\lambda_1^{+}|x_1^{(1)}|^2+\frac{2}{3}\Big(q\lambda_1^{+}-\frac{p^2}{W}\nu_1^{+}\Big)\mathfrak{Re}\,(x_1^{(1)*}y_1^{(1)})$$
$$-\frac{2}{3}\Big(q\lambda_1^{+}+\frac{p^2}{W}\nu_1^{+}\Big)\mathfrak{Re}\,(x_1^{(1)})+\frac{1}{3}\Big(q^2\lambda_1^{+}+\frac{p^2}{3}\mu_1^{+}-\frac{2qp^2}{3W}\nu_1^{+}\Big)|y_1^{(1)}|^2$$
$$+\frac{2p^2}{9}\mu_1^{+}\mathfrak{Re}\,(y_1^{(1)})+\frac{q^2}{6}\lambda_1^{+}+\frac{p^2}{9}\mu_1^{+}+\frac{2qp^2}{9W}\nu_1^{+}$$
$$+\frac{F_1(\rho)}{F_0(\rho)}\frac{2p^2}{9}\lambda_2^{+}\Big[|y_2^{(1)}|^2-\mathfrak{Re}\,(y_2^{(1)*}z_2^{(1)})+\frac{1}{4}|z_2^{(1)}|^2\Big]\Big\}$$
$$+\sum_{i,j}\Big|G_T\int B_{ij}^{\beta}I(r)\Big|^2\cdot\frac{1}{12}\Big\{q^2\lambda_1^{+}+\frac{F_1(\rho)}{F_0(\rho)}p^2\lambda_2^{+}|u_2^{(2)}|^2\Big\}.\tag{30}$$

(30) is our final form for the first forbidden correction factor in the *STP* interrc-tion. $\mathfrak{Re}$ and $\mathfrak{Im}$ mean the real and imaginary parts respectively. Among the nuclear matrix elements appearing above $\int\beta\boldsymbol{r}\,I_k(r)$, $\int\beta\sigma\cdot\boldsymbol{r}\,I_k(r)$, $\int\beta\sigma\times\boldsymbol{r}\,I_k(r)$ and $\int B_{ij}^{\beta}I_k(r)$ are considered to be of the same order of magnitude, and $\int\beta\alpha\,I_k(r)$ to be of the order of

$\alpha Z/2\rho$ times larger.[22] The magnitude of $\int \beta\gamma_5 I_k(r)$ is still very ambiguous. It may be much smaller than $\int \beta\boldsymbol{r} I_k(r)$, but may be as large as $\int \beta\alpha I_k(r)$. If we regard Z, p, q and W as of the order of unity, we can classify the terms in (28) according to the power of ρ except for the $\int \beta\gamma_5 I_k(r)$ terms. Usually the lowest terms in ρ are the largest. In (29) $x_1^{(0)}$ and $x_1^{(1)}$ are one order lower in ρ, and $|x_1^{(0)}|^2$ and $|x_1^{(1)}|^2$ terms are the largest in (30) if there is no large cancellation among the terms in the numerators of $x_1^{(0)}$ and $x_1^{(1)}$ in (29). However, it is possible that some large cancellation makes $x_1^{(0)}$ and $x_1^{(1)}$ of the order of other parameters in (29). In such cases (30) is very convenient for the actual calculation because the large terms do not appear explicitly in (30). The large cancellation which requires a very troublesome calculation in the primitive form of the correction factor can be taken into account merely by taking small values for $x_1^{(0)}$ and $x_1^{(1)}$.

From the above argument and (30) it is clear that a large cancellation can occur even if the screening effect is taken into account. On the other hand the correction factor is a homogenious linear function in the screening factors λ_k^{τ}, μ_k^{τ} and ν_k^{τ}, so that the order of the magnitude of the screening effect does not change appreciably with the degree of cancellation. The cancellation which depends only on (unnormalized) inner electron wave function $f_\varkappa^{\mathrm{in}}(r)$ and $g_\varkappa^{\mathrm{in}}(r)$ has quite a different nature from the screening effect or the effect of normalization which is determined by the region outside the nucleus.

As we stated in the last sub-section it is important to notice that the parameters (29) are not always independently adjustable parameters. We can easily see from (25a), (25b) and (25c) that $z_2^{(1)}$, $u_2^{(2)}$ and $y_2^{(1)}/y_1^{(1)}$ are almost unity or slightly larger than unity. $y_1^{(1)}$ itself is an independent parameter in the following sense: 1) If time reversal does not hold in the nuclear interaction, $y_1^{(1)}$ can be any complex number. 2) In the other case, $y_1^{(1)}$ can be any number whose phase is that of $\pm G_S/G_T$. $x_1^{(1)}$ is also a fairly independent parameter: 1) If time reversal does not hold in the nuclear interaction, $x_1^{(1)}$ can be any complex number. 2) If the time reversal holds in both the nuclear interaction and the β-decay interaction, $x_1^{(1)}$ can be any real number. 3) If time reversal holds in the nuclear interaction but not in the β-decay interaction, $\Re\mathfrak{e}(x_1^{(1)})$ can take any number but $\Im\mathfrak{m}(x_1^{(1)})$ is equal to $-\alpha Z/(2\rho_1(\beta\boldsymbol{r}))\cdot\Im\mathfrak{m}(y_1^{(1)})$. In this case, therefore, the allowed range for $\Im\mathfrak{m}(x_1^{(1)})/\Im\mathfrak{m}(y_1^{(1)}) = -\alpha Z/(2\rho_1(\beta\boldsymbol{r}))$ is restricted as can be obtained from (26a) – (26c). It is interesting to note that only in this case 3) a large cancellation does not occur unless we make $\Im\mathfrak{m}(y_1^{(1)})$ very small. The restriction for $x_1^{(0)}$ is most complicated. 1) If the contribution from the pseudoscalar interaction is negligible, $x_1^{(0)} \simeq \alpha Z/(2\rho_1(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}))$ whose allowed region is known from (26a) – (26c). 2) If the contribution from the pseudoscalar interaction is comparable to that from the tensor interaction and time reversal does not hold for the nuclear interaction, $x_1^{(0)}$ can be any complex number. 3) If the contribution from the pseudoscalar interaction is comparable to that from the tensor interaction and time reversal holds for the nuclear interaction, $x_1^{(0)} - \alpha Z/(2\rho_1(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r}))$ must have the same phase as $\pm G_P/G_T$. The three typical nuclear matrix

elements $\int \beta\sigma \cdot \boldsymbol{r} I_1(r)$, $\int \beta\sigma \times \boldsymbol{r} I_1(r)$ and $\int B_{ij}^3 I_1(r)$ can surely be regarded as independent parameters.

As was mentioned in § 3 B, the main difference between our theory and the $K-U$ low-Z theory is that our effective nuclear radii are more flexible than the usual nuclear radius ρ. In the above mentioned restrictions to the range of parameters, the effective nuclear radii appeared only in two cases, namely those in which: 1) time reversal holds for the nuclear interaction but not for the β-decay interaction, 2) the pseudoscalar interaction has no effect (for no-spin-change transitions only). In the other cases the difference of our theory from the K-U low-Z theory is insignificant in the 1st forbidden transition. It is a remarkable characteristic of the STP interaction that the correction factor is almost independent of Z and charge distributions in the 1st forbidden transitions if we neglect the relations between the adjustable parameters and the nuclear wave functions.

Our correction factor (30) is ρ-dependent which comes from the ρ-dependence of the normalization of the ideal allowed spectrum to which the correction should be applied. However, this ρ-dependence is completely independent of the electron energy, so that it does not affect the spectrum shape. It is necessary to pay attention to the ρ-dependence only in connection with the ft-value.

Thus the mathematical manipulation has ended. The next section includes nothing new, and may be omitted without loss of any theoretical points.

§ 4. Summary

This section is prepared especially for persons not interested in the derivation of the formulas but only in their application.

In § 2 we obtained the radial wave functions of the electron as (14) which are normalized so that there are R electrons in a large sphere of radius R. (14) is obtained by taking into account the finite nuclear size[7),8)] and the screening by atomic electrons.[5)] The conventional notation and units in the theory of β-decay are used as far as possible. $V(r)$ is the electron potential which depends on nuclear charge distributions. Δ_k is the screening correction term ($\Delta_k=0$ without the correction). We do not have simple formulas or tables for Δ_k yet, but we can get some idea about it from the published papers on the screening correction.[5)] $I_k(r)$ is an energy-independent function shown in Figs. 3a–6a for a few particular cases. $I_k(r)$ is not much different from unity and $I_k(r)\to 1$ for $Z\to 0$.

Of course, (14) is not completely exact. The errors in (14) are composed of two parts: one is in the normalization, and another is the error which would remain after normalizing (14) exactly. The former is roughly proportional to $\alpha Z W\rho/(2\gamma_k-1)$, and the latter is roughly proportional to $\alpha Z W\rho$. They are shown in Table I and Figs. 1a–2d in a few particular cases. These errors are given only for the purpose of indicating the limit of our theory. It is extremely difficult to go beyond our approximation. The error in our correction factor is mostly of the order of the error in (14), but it may sometimes be cancelled out, or may sometimes be magnified by piling up of errors or

especially by cancellation of the main terms as is indicated by a large ft value. One must be careful in interpreting the results of analyses when the ft value is unusually large.

In the Fermi theory of β-decay, the correction factor for the spectrum can be written by using combinations of electron wave functions as (17). For example the correction factor for the non-unique first forbidden transition in the *STP* interaction is expressed in (16). For the other interactions see reference 9. X and Y in (17) represent arbitrary nucleon operators. (17) can be written as (22) using (14). It should be noted that no approximation is used in this procedure. The electron wave function is not replaced by its surface value divided by $r^{l'}$ (l': magnitude of orbital angular momentum) as is usually done. Therefore, our theory fully takes into account the effect of β-decay from inside of the nucleus. The symbols $\lambda_k^{\pm}$, $\mu_k^{\pm}$ and $\nu_k^{\pm}$ in (22), which are introduced to take into account the screening effect, are defined in (19). If we neglect the screening effect we may use (20). $\rho_k(X)$ defined in (21) is the "effective nuclear radius" which should replace the ordinary nuclear radius appearing in the $K-U$ (Konopinski-Uhlenbeck) low-Z approximation[3] and is more flexible than ρ. Substituting (22) in (16) or in similar formulas for other interactions we can get a fairly simple form of the correction factor. Our formulas are similar to those of the $K-U$ low-Z approximation. The main difference is the flexibility of our effective nuclear radii which unfortunately increases the number of adjustable parameters. However, our adjustable parameters are subject to some restrictions owing to the similarity of the operators whose matrix elements are related with the parameters under discussion. Important restrictions are listed in (25a)–(27c) which are based on the assumption of our ignorance of nuclear wave functions. The "Probabilities" that (25a)–(27c) fail ("ratios of risk" of (25a)–(27c)) are estimated to be about $1/(2n)$. (25a)–(27c) will be sufficient to interpolate for other Z values.

For the non-unique transitions in the *STP* interaction we arranged the terms more concisely as (30) (1st forbidden), (A3) (2nd forbidden) and (A4) (3rd forbidden). The x's, y's, z's, u's and the nuclear matrix elements explicitly written in these formulas are adjustable parameters, but they cannot be adjusted independently of each other. The important restrictions to the adjustabilities which are based on (25a)–(27c) are stated next to the formulas. Since the accuracies of (30), (A3) and (A4) are somewhat insufficient for unique forbidden transitions we supplement the formulas for these transitions in Appendix 3.

§ 5. Additional remarks

We have obtained a fairly satisfactory formulation of expression for the β-spectrum in the Fermi theory. It includes all the corrections which are considered to be important at present, yet it is not very complicated.

The ft-value has a quantitative meaning only in the allowed and the unique forbidden transitions. We need not change the interpretation of the ft-value because our theory does not present these transitions in a new aspect, especially for low-Z nuclei where accurate ft-values are sometimes meaningful.

In the next paper we shall treat the angular distribution of electrons and neutrinos by our method. The radiative effect and the effect of nuclear moments which are quite ambiguous in the present situation will also become our future problem.

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Appendix 1. Allowed, 2nd forbidden and 3rd forbidden transitions in the $ST(P)$ interaction

Under the assumption made in the text the pseudoscalar interaction has no effect except for the 1st forbidden, so that we omit the suffix P in the following correction factors C_{nSTP}. For the allowed transition the correction factor is

$$C_{0ST}=|G_S\int\beta I_1(r)|^2\lambda_1^+ + |G_T\int\beta\sigma I_1(r)|^2\lambda_1^+. \tag{A1}$$

There is little difference from the old theory in the allowed transition. If we neglect the screening correction, (A1) has no energy dependence.

In order to write the formulas for the 2nd and 3rd forbidden transitions concisely we put

$$x_k^{(L)}=\Big[G_T\Big(\int Q_L(i\beta\alpha I_k(r)\Big)-\frac{1}{k}\frac{\alpha Z}{2i\nu_k[Q_L(i\beta\sigma\times\boldsymbol{r})]}G_T\Big(\int Q_L(i\beta\sigma\times\boldsymbol{r})I_k(r)\Big)$$
$$-i\frac{1}{k}\frac{\alpha Z}{2i\nu_k[Q_L(\beta\boldsymbol{r})]}G_S\Big(\int Q_L(\beta\boldsymbol{r})I_k(r)\Big)\Big]\Big[G_T\int Q_L(\beta\sigma\times\boldsymbol{r})I_1(r)\Big]^{-1},$$

$$y_k^{(L)}=\frac{iG_S\int Q_L(\beta\boldsymbol{r})I_k(r)}{G_T\int Q_L(\beta\sigma\times\boldsymbol{r})I_1(r)},$$

$$z_k^{(L)}=\frac{\int Q_L(i\beta\sigma\times\boldsymbol{r})I_k(r)}{\int Q_L(\beta\sigma\times\boldsymbol{r})I_1(r)},\quad v_k^{(L)}=\frac{\int Q_L(i\beta\sigma)I_k(r)}{\int Q_L(\beta\sigma)I_1(r)}, \tag{A2}$$

where $Q_L(X)$'s are tensor operators whose matrix elements are "$Q_L(X)$'s of Pursey[4]". Then the correction factor for the 2nd forbidden transition in the $ST(P)$ interaction is

$$C_{2ST}=\sum_{i,j}|G_T\int T^{\beta}_{ij}I_1(r)|^2\Big\{\frac{q^2}{12}\lambda_1^+|x_1^{(2)}|^2+\frac{q^2}{6}\Big(\frac{q}{5}\lambda_1^+-\frac{p^2}{3W}\nu_1^+\Big)\mathrm{Re}\,(x_1^{(2)*}y_1^{(2)})$$
$$-\frac{q^2}{6}\Big(\frac{q}{5}\lambda_1^+ + \frac{p^2}{3W}\nu_1^+\Big)\mathrm{Re}\,(x_1^{(2)})+\frac{q^2}{12}\Big(\frac{q^2}{10}\lambda_1^+ + \frac{p^2}{9}\mu_1^+ - \frac{2qp^2}{15W}\nu_1^+\Big)|y_1^{(2)}|^2$$
$$+\frac{q^2p^2}{54}\mu_1^+\mathrm{Re}\,(y_1^{(2)})+\frac{q^2}{18}\Big(\frac{q^2}{10}\lambda_1^+ + \frac{p^2}{6}\mu_1^+ + \frac{qp^2}{5W}\nu_1^+\Big)$$

$$+\frac{F_1(\rho)}{F_0(\rho)}\frac{p^2}{6}\left[\frac{1}{2}\lambda_2^+|x_2^{(2)}|^2+\left(\frac{q}{3}\lambda_2^+-\frac{p^2}{5W}\nu_2^+\right)\mathrm{Re}\,(x_2^{(2)*}y_2^{(2)})\right.$$

$$-\left(\frac{q^2}{3}\lambda_2^++\frac{p^2}{5W}\nu_2^+\right)\mathrm{Re}\,(x_2^{(2)*}z_2^{(2)})+\left(\frac{q^2}{9}\lambda_2^++\frac{p^2}{50}\mu_2^+-\frac{qp^2}{15W}\nu_2^+\right)|y_2^{(2)}|^2$$

$$-\left(\frac{q^2}{9}\lambda_2^+-\frac{p^2}{25}\mu_2^+\right)\mathrm{Re}\,(y_2^{(2)*}z_2^{(2)})$$

$$\left.+\left(\frac{q^2}{18}\lambda_2^++\frac{p^2}{50}\mu_2^++\frac{qp^2}{15W}\nu_2^+\right)|z_2^{(2)}|^2\right]$$

$$+\frac{F_2(\rho)}{F_0(\rho)}\frac{p^4}{50}\lambda_3^+\left[\frac{1}{4}|y_3^{(2)}|^2-\frac{1}{3}\mathrm{Re}\,(y_3^{(2)*}z_3^{(2)})+\frac{1}{9}|z_3^{(2)}|^2\right]\Bigg\}$$

$$+\sum_{i,j,k}|G_T\int S_{ijk}^{\beta}\,I_1(r)|^2\cdot\frac{1}{108}\left\{\frac{q^4}{10}\lambda_1^++\frac{F_1(\rho)}{F_0(\rho)}\frac{p^2q^2}{3}\lambda_2^+|u_2^{(3)}|^2+\frac{F_2(\rho)}{F_0(\rho)}\frac{p^4}{10}\lambda_3^+|u_3^{(3)}|^2\right\}. \tag{A3}$$

In (A3) we used the $K-U$'s notation T_{ij}^{β} and S_{ijk}^{β} instead of of $Q_L(X)$'s. The 3rd forbidden correction factor is

$$G_{3ST}=|G_T\int Q_3(\beta\sigma\times\mathbf{r})\,I_1(r)|^2\left\{\frac{q^4}{30}\lambda_1^+|x_1^{(3)}|^2\right.$$

$$+\frac{q^4}{15}\left(\frac{q}{7}\lambda_1^+-\frac{p^2}{3W}\nu_1^+\right)\mathrm{Re}\,(x_1^{(3)*}y_1^{(3)})-\frac{q^4}{15}\left(\frac{q^2}{7}\lambda_1^++\frac{p^2}{3W}\nu_1^+\right)\mathrm{Re}\,(x_1^{(3)})$$

$$+\frac{q^4}{45}\left(\frac{q^2}{14}\lambda_1^++\frac{p^2}{6}\mu_1^+-\frac{qp^2}{7W}\nu_1^+\right)|y_1^{(3)}|^2+\frac{q^4p^2}{135}\mu_1^+\mathrm{Re}\,(y_1^{(3)})$$

$$+\frac{q^4}{15}\left(\frac{q^2}{56}\lambda_1^++\frac{p^2}{18}\mu_1^++\frac{qp^2}{21W}\nu_1^+\right)+\frac{F_1(\rho)}{F_0(\rho)}\frac{q^2p^2}{9}\left[\lambda_2^+|x_2^{(3)}|^2\right.$$

$$+\frac{2}{5}\left(q\lambda_2^+-\frac{p^2}{W}\nu_2^+\right)\mathrm{Re}\,(x_2^{(3)*}y_2^{(3)})-\frac{2}{5}\left(q\lambda_2^++\frac{p^2}{W}\nu_2^+\right)\mathrm{Re}\,(x_2^{(3)*}z_2^{(3)})$$

$$+\frac{1}{5}\left(\frac{q^2}{3}\lambda_2^++\frac{p^2}{5}\mu_2^+-\frac{2qp^2}{5W}\nu_2^+\right)|y_2^{(3)}|^2$$

$$-\frac{2}{5}\left(\frac{q^2}{6}\lambda_2^+-\frac{p^2}{5}\mu_2^+\right)\mathrm{Re}\,(y_2^{(3)*}z_2^{(3)})$$

$$\left.+\left(\frac{q^2}{24}\lambda_2^++\frac{p^2}{25}\mu_2^++\frac{2qp^2}{25W}\nu_2^+\right)|z_2^{(3)}|^2\right]$$

$$+\frac{F_2(\rho)}{F_0(\rho)}\frac{p^4}{15}\left[\frac{1}{2}\lambda_3^+|x_3^{(3)}|^2+\left(\frac{q}{3}\lambda_3^+-\frac{p^2}{7W}\nu_3^+\right)\mathrm{Re}\,(x_3^{(3)*}y_3^{(3)})\right.$$

$$\left.-\left(\frac{q}{3}\lambda_3^++\frac{p^2}{7W}\nu_3^+\right)\mathrm{Re}\,(x_3^{(3)*}z_3^{(3)})\right.$$

$$+\left(\frac{q^2}{10}\lambda_3^{+}+\frac{p^2}{98}\mu_3^{-}-\frac{qp^2}{21W}\nu_3^{+}\right)|y_3^{(3)}|^2$$

$$-2\left(\frac{q^2}{15}\lambda_3^{+}-\frac{p^2}{98}\mu_3^{+}\right)\mathfrak{Re}\,(y_3^{(3)}*z_3^{(3)})$$

$$+\left(\frac{7q^2}{120}\lambda_3^{+}+\frac{p^2}{98}\mu_3^{+}+\frac{qp^2}{21W}\nu_3^{+}\right)|z_3^{(3)}|^2\Bigg]$$

$$+\frac{F_3(\rho)}{F_0(\rho)}\frac{p^6}{245}\lambda_4\cdot\left[\frac{1}{8}|z_4^{(3)}|^2+\frac{2}{9}|y_4^{(3)}|^2-\frac{1}{3}\mathfrak{Re}\,(y_4^{(3)}*z_4^{(3)})\right]\Bigg\}$$

$$+|G_T\int Q_4(\beta\sigma)\,I_1(r)\,|^2\cdot\frac{1}{90}\cdot\left\{\frac{q^6}{7}\lambda_1^{+}+\frac{F_1(\rho)}{F_0(\rho)}q^4p^2\lambda_2^{+}|u_2^{(1)}|^2\right.$$

$$\left.+\frac{F_2(\rho)}{F_0(\rho)}q^2p^4\lambda_3^{+}|u_3^{(1)}|^2+\frac{F_3(\rho)}{F_0(\rho)}\frac{p^6}{7}\lambda_4^{+}|u_4^{(4)}|^2\right\}. \tag{A4}$$

As in the case of 1st forbidden transitions the parameters in (A3) and (A4) are not independent. From (25) and (26) we can easily see that $z_k^{(L)}$, $u_k^{(L)}$ and $y_i^{(L)}/y_1^{(L)}$ $(i\geq 2)$ are almost unity or slightly larger than 1. There is practically no difference between $z_k^{(L)}$'s, between $u_k^{(L)}$'s or between $y_k^{(L)}$'s. $y_1^{(L)}$ itself can be regarded as an independent parameter in the same sense as $y_1^{(1)}$ in the 1st forbidden transition. $x_k^{(L)}$ obeys a restriction similar to that for $x_1^{(1)}$ in the first forbidden transition except that in case 3) $\mathfrak{Im}\,(x_k^{(L)})/\mathfrak{Im}\,(y_k^{(L)})=-\alpha Z/[2k\rho_k(Q_L(\beta r))]$. The main difference from the case of the 1st forbidden transitions is that the $x_k^{(L)}$'s are not quite independent mutually. There exist the following relations among them:

$$x_j^{(L)}-x_k^{(L)}\cdot\frac{\int Q_L(\beta\alpha)\,I_j(r)}{\int Q_L(\beta\alpha)\,I_k(r)}=-\frac{1}{j}\frac{\alpha Z}{2\rho_j(Q_L(\beta\sigma\times\boldsymbol{r}))}\frac{\int Q_L(\beta\sigma\times\boldsymbol{r})\,I_j(r)}{\int Q_L(\beta\sigma\times\boldsymbol{r})\,I_1(r)}$$

$$\times\left[1-\frac{\int Q_L(\beta\sigma\times\boldsymbol{r})\,r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)}{\int Q_L(\beta\sigma\times\boldsymbol{r})\,r^{-2j-1}\int_0^r dr_1 r_1^{2j}V(r_1)\,I_j(r_1)}\cdot\frac{\int Q_L(\beta\alpha)\,I_j(r)}{\int Q_L(\beta\alpha)\,I_k(r)}\right]$$

$$-\frac{1}{j}\frac{\alpha Z}{2\rho_j(Q_L(\beta\boldsymbol{r}))}y_j^{(L)}$$

$$\times\left[1-\frac{\int Q_L(\beta\boldsymbol{r})\,r^{-2k-1}\int_0^r dr_1 r_1^{2k}V(r_1)\,I_k(r_1)}{\int Q_L(\beta\boldsymbol{r})\,r^{-2j-1}\int_0^r dr_1 r_1^{2j}V(r_1)\,I_j(r_1)}\frac{\int Q_L(\beta\alpha)\,I_j(r)}{\int Q_L(\beta\alpha)\,I_k(r)}\right]. \tag{A5}$$

The $K-U$ low-Z approximation corresponds to $I_j(r)=I_k(r)=1$, $\rho_j(X)=\rho_k(X)=\rho$ and $y_j^{(L)}=y_k^{(L)}(=y^{(L)})$, so that only one of the $x_k^{(L)}$'s is independent. The quantities in (A5) are more flexible in our approximation. Among these flexibilities those of the effective

nuclear radii are much larger than others. Therefore, we first fix the ratios of the nuclear matrix elements in (A5) to their "mean values". Then we get, using (25a)–(25c) and (27a)–(27c),

$$\left.\begin{aligned}
&x_1^{(L)}-x_2^{(L)}\simeq -0.41_7\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_1^{(L)}-x_3^{(L)}\simeq -0.59_0\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_2^{(L)}-x_3^{(L)}\simeq -0.14_9\left[\frac{\alpha Z}{2\rho_2(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_2(Q_L(\beta\boldsymbol{r}))}y_2^{(L)}\right],
\end{aligned}\right\}\ :\ Z\approx 0\ \ (\neq 0)$$

$$\left.\begin{aligned}
&x_1^{(L)}-0.98_4x_2^{(L)}\simeq -0.41_8\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_1^{(L)}-0.98_3x_3^{(L)}\simeq -0.59_3\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_2^{(L)}-0.99_9x_3^{(L)}\simeq -0.15_1\left[\frac{1.01_6\alpha Z}{2\rho_2(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_2(Q_L(\beta\boldsymbol{r}))}y_2^{(L)}\right],
\end{aligned}\right\}\ :\ Z=\pm 50$$

(A6)

$$\left.\begin{aligned}
&x_1^{(L)}-0.92_2x_2^{(L)}\simeq -0.42_1\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_1^{(L)}-0.91_8x_3^{(L)}\simeq -0.60_1\left[\frac{\alpha Z}{2\rho_1(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_1(Q_L(\beta\boldsymbol{r}))}y_1^{(L)}\right],\\
&x_2^{(L)}-0.99_6x_3^{(L)}\simeq -0.15_6\left[\frac{1.07_8\alpha Z}{2\rho_2(Q_L(\beta\sigma\times\boldsymbol{r}))}+\frac{\alpha Z}{2\rho_2(Q_L(\beta\boldsymbol{r}))}y_2^{(L)}\right].
\end{aligned}\right\}\ :\ Z=\pm 90$$

From (A6) we can see that the large cancellation over all electron energies is less probable in the higher forbidden transitions than in the 1st forbidden transition.

Appendix 2. ρ-Independence of our theory

It is one of the advantages of our theory that the results do not explicitly depend on ρ. The condition to ρ is only that there is no nuclear charge at $r>\rho$. Therefore, there is much arbitrariness in choosing ρ, although the smallest value is favourable for estimating the error of our theory. This situation is similar to that in the formal theory of nuclear reaction.[23]

It is easy to prove the ρ-independence of our theory. First we prove that our electron wave functions (14), or more suitably, the expressions (12), are ρ-independent. There are two ρ-dependent terms $\sqrt{F_{k-1}}(\rho)$ and D_{0k}^{-1} in (12). In order to prove the ρ-independence of (12) we have only to show that the logarithmic derivatives of $\sqrt{F_{k-1}}(\rho)$ and D_{0k} are equal to each other. From (6)

$$\frac{d}{d\rho}[\log\sqrt{F_{k-1}(\rho)}]=-\frac{k-\gamma_k}{\rho}. \tag{A7}$$

On the other hand we get from (12)

$$\frac{dD_{0k}}{d\rho}=\sqrt{\frac{k+\gamma_k}{2k}}\,\frac{dJ_k(\rho)}{d\rho}+\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}(-2k)\rho^{-2k-1}\int_0^\rho dr\,r^{2k}V(r)J_k(r)$$

$$+\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}V(\rho)J_k(\rho). \tag{A8}$$

As $r=\rho$ lies outside the nucleus and the screening correction for $V(\rho)$ is small (see § 2),

$$V(\rho)=-\alpha Z/\rho. \tag{A9}$$

Using (A9) the third term of (A8) becomes

$$-\sqrt{\frac{k+\gamma_k}{2k}}\,\frac{k-\gamma_k}{\rho}J_k(\rho). \tag{A10}$$

From (3) and (A9)

$$\frac{dJ_k(\rho)}{d\rho}=\alpha Z\rho^{-2k-1}\int_0^\rho dr\,r^{2k}V(r)J_k(r). \tag{A11}$$

Using (A11) the sum of the first and the second terms in (A8) we have

$$-\frac{\alpha Z}{\sqrt{2k(k+\gamma_k)}}\cdot\frac{k-\gamma_k}{\rho}\rho^{-2k}\int_0^\rho dr\,r^{2k}V(r)J_k(r). \tag{A12}$$

Comparing the sum of (A10) and (A12) with D_{0k} we get

$$\frac{d}{d\rho}[\log D_{0k}]=-\frac{k-\gamma_k}{\rho}. \tag{A13}$$

Thus the proof has been completed.

From the above proof it is clear that $I_k(r)$ has a ρ-dependence $\rho^{k-\gamma_k}$. Since the ideal allowed spectrum has a ρ-dependence of $\rho^{2\gamma_1-2}$, the correction factor must have a ρ-dependence of $\rho^{2-2\gamma_1}$ so as to make the whole theory ρ-independent. All the ρ-dependence of the correction factor is in $L^{\pm}_{k-1}(X, Y)$, $M^{\pm}_{k-1}(X, Y)$ and $N^{\pm}_{k-1}(X, Y)$ which surely have a ρ-dependence of $\rho^{2-2\gamma_1}$ as can be easily seen from (22). Note that neither the effective nuclear radius nor the screening correction factor has ρ-dependence.

Although the ρ-independence of our theory is quite obvious in the stage of § 3 B, it becomes somewhat obscure in (30), (A3) and (A4). In these formulas the adjustable parameters, x's, y's, z's and u's, have a ρ-dependence to make the ρ-dependence of the correction factor $\rho^{2-2\gamma_1}$. This situation is not desirable but seems inevitable as we usually have to use a certain coordinate system for the practical calculation even though the

theory is independent of the coordinate system.

Instead of the dependence on ρ, our formulas depend on the effective nuclear radii. They depend not only on charge distributions but on the details of the nuclear wave function. Generally the latter is much more unknown. Moreover, the effective radii are only numbers. Thus we can understand that we cannot generally get any information about nuclear charge distributions (this is a function of r) from analyses of β-spectra. This is a similar situation to that we cannot determine the radial form of the nucleon-nucleon potential from analyses of low energy data.[21]

Appendix 3. Supplement to the formulas for allowed and unique forbidden transitions

As was mentioned in the text our formulas in § 3 D, Appendix 1 and eq. (16) are not sufficiently accurate for allowed and unique forbidden transitions. In these transitions α^2Z^2 has been neglected in comparison with 1, while any power of αZ is retained in the non-unique forbidden transition. Our general principle for approximation is to neglect $\alpha ZW\rho$ or $\alpha ZW\rho/(2\gamma_k-1)$, not α^2Z^2, in comparison with 1. This appendix supplements the insufficient formulas for allowed and unique forbidden transitions in the *STP* interaction.

The terms which are of the order of α^2Z^2 times as large as the main term come from the velocity type nuclear matrix elements whose operators include γ_5 or α. For example, the ratio of the contributions of the $\int\beta\alpha\times\boldsymbol{r}$ (in the $K-U$ approximation) type term to that of the $\int\beta\sigma$ type term is of the order of α^2Z^2. One αZ factor comes from the ratio of two nuclear matrix elements, the other from the ratio of the electron wave functions concerned. For the correction terms only the leading terms of $f_{-k}(r)$ and $g_k(r)$ in (14) are necessary. The contributions of the other parts are $\alpha ZW\rho$ times as large as the main term, and consistently neglected.

We need not calculate the correction from the beginning if we note that the results are invariant under the transformation of the neutrino field, $\psi_\nu\to\gamma_5\psi_\nu$ since the neutrino rest mass is taken as zero. For the *STP* interaction, the correction factors for allowed and unique forbidden transitions can be obtained as follows: First, pick up the largest order terms from the correction factor for the transitions whose order of forbiddenness is higher by one than that of the transition under consideration. Make the following replacements of nucleon operators and coupling constants; $G_S\beta\to G_P\beta\gamma_5$, $G_P\beta\gamma_5\to G_S\beta$, $G_T\beta\sigma\to G_T\beta\alpha$, $G_T\beta\alpha\to G_T\beta\sigma$. Then we get the final answer. The results are given for allowed, 1st forbidden unique, and 2nd forbidden unique transitions as follows:

$$C_{0STP}=\left|iG_S\left(\int\beta I_1(r)\right)+\frac{\alpha Z}{2\rho_1(\beta\alpha\cdot\boldsymbol{r})}G_T\left(\int\beta\alpha\cdot\boldsymbol{r}\,I_1(r)\right)\right|^2\lambda_1^+$$

$$+\left|G_T\left(\int\beta\sigma I_1(r)\right)-\frac{\alpha Z}{2\rho_1(\beta\alpha\times\boldsymbol{r})}G_T\left(\int\beta\alpha\times\boldsymbol{r}\,I_1(r)\right)\right.$$

$$-i\frac{\alpha Z}{2\rho_1(\beta\gamma_5\boldsymbol{r})}G_P\Big(\int\beta\gamma_5\boldsymbol{r}\,I_1(r)\Big)\Big|^2\lambda_1^+, \tag{A14}$$

$$C_{1STP\ \text{unique}}=\Big|G_T\Big(\int B_{ij}^{\beta}\,I_1(r)\Big)-\frac{\alpha Z}{2\rho_1(T_{ij}^{\beta\gamma_5})}G_T\Big(\int T_{ij}^{\beta\gamma_5}\,I_1(r)\Big)$$

$$-i\frac{\alpha Z}{2\rho_1(R_{ij}^{\beta\gamma_5})}G_P\Big(2\int R_{ij}^{\beta\gamma_5}\,I_1(r)\Big)\Big|^2\frac{q^2}{12}\lambda_1^+$$

$$+\Big|G_T\Big(\int B_{ij}^{\beta}\,I_2(r)\Big)-\frac{1}{2}\cdot\frac{\alpha Z}{2\rho_2(T_{ij}^{\beta\gamma_5})}G_T\Big(\int T_{ij}^{\beta\gamma_5}\,I_2(r)\Big)$$

$$-\frac{i}{2}\cdot\frac{\alpha Z}{2\rho_1(R_{ij}^{\beta\gamma_5})}G_P\Big(2\int R_{ij}^{\beta\gamma_5}\,I_2(r)\Big)\Big|^2\frac{F_1(\rho)}{F_0(\rho)}\frac{p^2}{12}\lambda_2^+,$$

$$C_{2STP\ \text{unique}}=\Big|G_T\Big(\int Q_3(\beta\sigma)\,I_1(r)\Big)-\frac{\alpha Z}{2\rho_1[Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})]}G_T\Big(\int Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})\,I_1(r)\Big)$$

$$-i\frac{\alpha Z}{2\rho_1[Q_3(\beta\gamma_5\boldsymbol{r})]}G_P\Big(\int Q_3(\beta\gamma_5\boldsymbol{r})\,I_1(r)\Big)\Big|^2\frac{q^4}{30}\lambda_1^+$$

$$+\Big|G_T\Big(\int Q_3(\beta\sigma)\,I_2(r)\Big)-\frac{1}{2}\cdot\frac{\alpha Z}{2\rho_2[Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})]}G_T\Big(\int(Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})\,I_2(r)\Big)$$

$$-\frac{i}{2}\frac{\alpha Z}{2\rho_2[Q_3(\beta\gamma_5\boldsymbol{r})]}G_P\Big(\int Q_3(\beta\gamma_5\boldsymbol{r})\,I_2(r)\Big)\Big|^2\frac{F_1(\rho)}{F_0(\rho)}\frac{p^2q^2}{9}\lambda_2^+$$

$$+\Big|G_T\Big(\int Q_3(\beta\sigma)\,I_3(r)\Big)-\frac{1}{3}\cdot\frac{\alpha Z}{2\rho_3[Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})]}G_T\Big(\int Q_3(\beta\boldsymbol{\alpha}\times\boldsymbol{r})\,I_3(r)\Big)$$

$$-\frac{i}{3}\frac{\alpha Z}{2\rho_3[Q_3(\beta\gamma_5\boldsymbol{r})]}G_P\Big(\int(Q_3(\beta\gamma_5\boldsymbol{r})\,I_3(r)\Big)\Big|^2\frac{F_2(\rho)}{F_0(\rho)}\frac{p^4}{30}\lambda_3^+. \tag{A16}$$

In (A15), the indices $\beta\gamma_5$ of $T_{ij}^{\beta\gamma_5}$ and $R_{ij}^{\beta\gamma_5}$ mean that nucleon operators include extra factors of $\beta\gamma_5$. Note that $B_{ij}^{\beta\gamma_5}=A_{ij}^{\beta}$ and $A_{ij}^{\beta\gamma_5}=B_{ij}^{\beta}$. We see from (A14) that the spectrum for allowed transition does not change even in this approximation. On the other hand, the spectra for unique forbidden transitions are not so "unique" when Z is very large or when the nuclear matrix elements of the main terms are unusually small for some reason.

Note added in proof (To § 3, Appendix 1 and 3). After submitting this paper the β-decay interaction has become known to be more likely VA than $ST(P)$. Formulas for VA are easily obtained from those for STP by the following substitutions: $G_S\beta\to G_V$, $G_T\beta\sigma\to G_A\boldsymbol{\sigma}$, $G_T\beta\boldsymbol{\alpha}\to G_V\boldsymbol{\alpha}$, $G_P\beta\gamma_5\to G_A\gamma_5$, and $q\to -q$. The above Dirac matrices are those for nucleons. Relations similar, but not quite the same, to those given for STP hold between nuclear matrix elements and parameters for VA.

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On the Static Approximation of the γ_5 Meson Theory

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The $1/m$ correction to the static approximation assumed by Sugawara has been estimated within the non-recoil framework in the case of the low energy pion-nucleon scattering. It is here assumed that the Foldy transformation generates a valid static Hamiltonian in the relativistic γ_5-theory. It is shown that two extra parameters appear besides three renormalized coupling constants known thus far in describing S- and P-wave pion-nucleon scattering. However, a crude numerical estimate shows that they may be quite negligible. Other effect of the $1/m$ correction is to modify the numerical values of three coupling constants, which would, according to the present rough numerical estimate, not be so great as invalidating the static approximation assumed by Sugawara.

§ 1. Introduction

The low energy P-wave pion-nucleon scattering has been successfully described in terms of the static model by Chew.[1] Some models have been proposed also as regards the low energy S-wave interaction, among which the static model by Drell et al.[2] is the only reasonable one, though less satisfactory than the P-wave model. It is, however, not yet quite clear whether these models would really follow from the covariant γ_5 theory as the static limit. According to a recent method due to Sugawara,[3] it is possible in principle to investigate this problem. It was assumed there that the static approximation is permissible after a certain canonical transformation is applied to the relativistic γ_5 Hamiltonian. It is the purpose of this paper to examine to what extent this static approximation assumed by Sugawara[3] is valid.

In the present work we assume the Foldy transformation[4] as the canonical transformation generating a valid static Hamiltonian. The motivation is that the expansion in powers of the inverse of nucleon mass m would be reasonable in the low energy region; it is known that the Foldy transformation produces such an expansion. It is thus straightforward to retain the terms of order $1/m$ in the transformed Hamiltonian (Section 2).

We derive in Section 3 the Low-type integral equation for the pion-nucleon scattering amplitude, taking into account the nucleon velocity up to order $1/m$. In this integral equation the nucleon coordinates have already been eliminated and only the physical nucleon momenta (*c-number*) appear explicitly.

It is seen that the effect of the terms of order $1/m$ consists of two parts, one of which depends explicitly on the physical nucleon momenta and the other contains only meson field quantities. The former is the leading term of the essentially recoil effect,

while the latter is the $1/m$ correction to the static Hamiltonian within the non-recoil framework.

If we neglect the terms containing physical nucleon momenta, according to a recent investigation,[3] the integral equation then assumes a form which permits the same technique as that employed by Sugawara. Thus the $1/m$ correction to the static approximation is shown firstly to modify the numerical values of three renormalized coupling constants characterizing the low energy S- and P-wave pion-nucleon scattering and secondly to give rise to two extra parameters which did not exists in the simple static approximation. All these five renormalized coupling constants are given as the physical nucleon expectation values of certain complicated meson field operators, thus permitting us to calculate them in terms of the unrenormalized *ps-ps* coupling constant and the cut-off momentum.

A very rough numerical estimate of these parameters is given in Section 4. Though the numerical values should not be taken too seriously, the $1/m$ correction to the static Hamiltonian does not seem to be so great, though not quite negligible; on the other hand, it seems almost certain that two extra parameters which did not appear in Sugawara's approximation are quite negligible. Conclusions are summarized at the end of Section 4.

§ 2. Static approximation

As was mentioned in Introduction, we assume the Foldy transformation as generating a valid static approximation. The meson-nucleon system is described in terms of the Hamiltonian, dropping the renormalization terms,

$$H=\int\psi^*\left\{-i\boldsymbol{\alpha}\frac{\partial}{\partial\boldsymbol{x}}+m\beta\right\}\psi d\boldsymbol{x}+(1/2)\int\{\pi^2+\mathrm{grad}^2\phi+\mu^2\phi^2\}\,d\boldsymbol{x}$$
$$+if\int\psi^*\beta\gamma_5\tau\phi\psi d\boldsymbol{x}, \tag{1}$$

where the notations of the operators are the usual ones.

According to Foldy,[4] we apply to (1) a unitary transformation U;

$$U=\exp(iS),\quad S=(f/2m)\int\psi^*\gamma_5\tau\phi\,w(\chi)\psi d\boldsymbol{x},$$
$$w(\chi)=(1/2\chi)\tan^{-1}2\chi,\quad \chi=(f/2m)\sqrt{\boldsymbol{\phi}^2}\,. \tag{2}$$

The result is

$$H'=UHU^{-1}=A+\int\psi^*(\mathfrak{S}+\mathfrak{O})\psi d\boldsymbol{x}+\int\psi^*\beta m\psi d\boldsymbol{x}+(1/2)\int|\psi^*\boldsymbol{D}\psi|^2 d\boldsymbol{x} \tag{3}$$

with

$$A=H_\pi=(1/2)\int\{\pi^2+\mathrm{grad}^2\boldsymbol{\phi}+\mu^2\boldsymbol{\phi}^2\}\,d\boldsymbol{x},$$
$$\mathfrak{S}=(g/\mu)(\sigma\boldsymbol{\nabla}\tau\boldsymbol{\phi})f(\chi)+(g/\mu)^3(\tau\boldsymbol{\phi})(\sigma\boldsymbol{\nabla}\tau\boldsymbol{\phi})(\tau\boldsymbol{\phi})g(\chi)$$
$$+2m\beta h(\chi)+(g/\mu)^2\left\{\frac{1}{2}\tau\cdot(\boldsymbol{\phi}\times\boldsymbol{\pi})k(\chi)+h.c.\right\},$$

$$\mathcal{O}=\rho_1\{(1/2)(g/\mu)(n(\chi)\tau\pi+h.c.)+(g/\mu)^3(g(\chi)(\tau\phi)(\phi\pi)+h.c.)$$
$$-(g/\mu)^2k(\chi)\tau\cdot(\sigma\nabla\phi\times\phi)+\sigma\boldsymbol{p}\},$$
$$\boldsymbol{D}=(g/\mu)\{\rho_1 n(\chi)\tau+2\rho_1(g/\mu)^2 g(\chi)(\tau\phi)\phi-(g/\mu)k(\chi)\phi\times\tau\},$$

where

$$f(\chi)=(1/2)\left(\frac{1}{\zeta}+\frac{1}{\zeta^2}\right),\ g(\chi)=(1/2\chi^2)\left(\frac{1}{\zeta^2}-\frac{1}{\zeta}\right),\ h(\chi)=(1/2)(\zeta-1),$$
$$k(\chi)=-\zeta g(\chi),\ n(\chi)=1/\zeta,\ \zeta=(1+4\chi^2)^{1/2},$$

and $g=(\mu/2m)f$ is the equivalent *ps-pv* coupling constant.

The term $+(1/2)\int|\psi^*\boldsymbol{D}\psi|^2d\boldsymbol{x}$ in H' will be neglected in the following, since most of its effect concerns the renormalization which is neglected in this paper.

The static approximation assumed by Sugawara is simply to adopt the static limit in H' given by (3), dropping the odd term $\int\psi^*\mathcal{O}\psi d\boldsymbol{x}$. It is the purpose of this paper to see the effect of this term. We, therefore, apply furthermore a canonical transformation

$$U'=\exp(iS'),\ S'=-(i/2m)\int\psi^*\beta\mathcal{O}\psi d\boldsymbol{x} \tag{4}$$

to H', which is transformed into

$$H''=U'H'U'^{-1}=H_\pi+\int\psi^*\mathcal{E}\psi d\boldsymbol{x}$$
$$+(1/2m)\int\psi^*\beta\mathcal{O}^2\psi d\boldsymbol{x}+O(1/m^2), \tag{5}$$

where the third term is the $1/m$ correction mainly interested in this paper.

We now neglect the last term in (5) and restrict us within the nonrelativistic framework for a nucleon, which leads to a Hamiltonian

$$H''=H_0+H_1+H_2+H_3, \tag{6}$$

with

$$H_0=H_\pi+(\boldsymbol{p}^2/2m),$$
$$H_1=(g/\mu)\sigma\nabla(\tau\phi)f(\chi)+(g/\mu)^3(\tau\phi)(\sigma\nabla\tau\phi)(\tau\phi)g(\chi)$$
$$+2mh(\chi)+(g/\mu)^2(1/2)\{\tau\cdot(\phi\times\pi)k(\chi)+h.c.\},$$
$$H_2=(1/2m)[(g/\mu)^4(dk/d\chi)(1/\chi)\sigma((\phi\nabla\phi)\times(\nabla\phi\times\phi)\cdot\tau)$$
$$+(1/4)(g/\mu)^2\{(n\pi+\pi n)^2+i\tau\cdot((n\pi+\pi n)\times(n\pi+\pi n))\}$$
$$-(1/2)(g/\mu)^4\{(n\pi+\pi n)(g\phi(\phi\pi)+h.c.)$$
$$+i\tau\cdot[(n\pi+\pi n)\times(g\phi(\phi\pi)+h.c.)]+h.c.\}$$
$$+(1/2)(g/\mu)^3\{(n\pi+\pi n)(\phi\times\sigma\nabla\phi)k+i\tau[(n\pi+\pi n)\times(\phi\times\sigma\nabla\phi)]k+h.c.\}$$

$$+(g/\mu)^6\{(g\phi(\phi\pi)+h.c.)^2+i\tau\cdot[(g\phi(\phi\pi)+h.c.)\times(g\phi(\phi\pi)+h.c.)]\}$$
$$-(g/\mu)^5\{(g\phi(\phi\pi)+h.c.)(\phi\times\sigma\nabla\phi)k$$
$$+i\tau\cdot[(g\phi(\phi\pi)+h.c.)\times(\phi\times\sigma\nabla\phi)k]+h.c.\}$$
$$+(g/\mu)^4k^2[\nabla\phi\times\phi]^2],$$
$$H_3=(1/2m)[(1/2)(g/\mu)(n\pi\tau+\tau\pi n)\sigma p$$
$$-(g/\mu)^3(g(\tau\phi)(\phi\pi)+h.c.)\sigma p+(g/\mu)^2k\tau[\phi\times\nabla\phi]\cdot p+h.c.],$$

where p is the bare nucleon momentum $-i\partial/\partial r$, and the nucleon coordinate r is to be substituted in all the unintegrated meson field quantities in the Hamiltonian.

Sugawara's investigation is based solely on the interaction Hamiltonian H_1 in (6). The $1/m$ correction terms are contained in H_2 and H_3, the former of which does not contain p and the latter contains p.

§ 3. Integral equation for the scattering amplitude

The Schrödinger equation can be written as

$$\begin{aligned} H\Psi^{(+)}(p,q)&=(p^2/2m+\omega_q)\Psi^{(+)}(p,q),\\ \Psi^{(+)}(p,q)&=a^*(q)\psi(p)+\bar{\chi}^{(+)}(p,q),\\ H\psi(p)&=(p^2/2m)\psi(p), \end{aligned} \tag{7}$$

where $\Psi^{(+)}(p,q)$ is a scattering state in which a meson of momentum q collides with a nucleon of momentum p. It is here presumed that p and q specify also spin and charge variables. $\psi(p)$ describes a physical nucleon with momentum p, and $a^*(q)$ is a creation operator of a meson with momentum q.

It is well known that the bare nucleon coordinate r can be eliminated by putting the total momentum eigenstate $\Psi(p')$ of the system as

$$\Psi(p')=\exp\{i(p'-G)r\}\,\Phi(p'),$$
$$-i(\partial/\partial r)\,\Phi(p')=0,$$

where G is the meson field momentum, p' being the total momentum eigenvalue. Thus, by putting

$$\begin{aligned} \Psi^{(+)}(p,q)&=\exp\{i(p+q-G)r\}\cdot\Phi^{(+)}(p,q),\\ \bar{\chi}^{(+)}(p,q)&=\exp\{i(p+q-G)r\}\cdot\chi^{(+)}(p,q),\\ \psi(p)&=\exp\{i(p-G)r\}\cdot\phi(p), \end{aligned} \tag{8}$$

we can eliminate r from Eq. (7), which become

$$\begin{aligned} H(p+q)\,\Phi^{(+)}(p,q)&=(p^2/2m+\omega_q)\,\Phi^{(+)}(p,q),\\ \Phi^{(+)}(p,q)&=a^*(q)\phi(p)+\chi^{(+)}(p,q),\\ H(p)\phi(p)&=(p^2/2m)\phi(p), \end{aligned} \tag{9}$$

where $H(\boldsymbol{p})$ is defined as

$$H(\boldsymbol{p}) = \exp\{-i(\boldsymbol{p}-\boldsymbol{G})\boldsymbol{r}\}\, H(-i(\partial/\partial\boldsymbol{r}),\ \phi(\boldsymbol{r}),\ \pi(\boldsymbol{r}))\exp\{i(\boldsymbol{p}-\boldsymbol{G})\boldsymbol{r}\}$$
$$= H(\boldsymbol{p}-\boldsymbol{G},\ \phi(0),\ \pi(0)).$$

It is to be remarked that $\boldsymbol{p}$ in (9) represents the physical nucleon momentum.

The scattering amplitude T_{fi} is defined in terms of the relevant S-matrix element $\tilde{S}_{fi}$ by

$$\tilde{S}_{fi} = (\Psi^{(-)}(\boldsymbol{p}_f,\ \boldsymbol{q}_f)\,|\,\Psi^{(+)}(\boldsymbol{p}_i,\ \boldsymbol{q}_i)) = \delta_{fi} - 2\pi i\delta(E_f - E_i)\,T_{fi}. \tag{10}$$

Using the relation

$$\Psi^{(\pm)}(\boldsymbol{p},\ \boldsymbol{q}) = a^*(\boldsymbol{q})\,\phi(\boldsymbol{p})$$
$$-[H(\boldsymbol{p}+\boldsymbol{q}) - (\omega_q + \boldsymbol{p}^2/2m) \pm i\varepsilon]^{-1}\cdot[H'(\boldsymbol{p}+\boldsymbol{q}/2),\ a^*(\boldsymbol{q})]\,\phi(\boldsymbol{p}), \tag{11}$$

which is derived from (9), we get

$$T_{fi} = (\Psi^{(-)}(\boldsymbol{p}_f,\ \boldsymbol{q}_f)\,|[H'(\boldsymbol{p}_i+\boldsymbol{q}_i/2),\ a^*(\boldsymbol{q}_i)].|\phi(\boldsymbol{p}_i)), \tag{12}$$

where H' is the total interaction Hamiltonian and the dot besides the commutator implies that $\boldsymbol{G}$ in H' is regarded as a c-number (See Appendix). An integral equation for T_{fi} can now be derived as usual from (11) and (12):

$$T_{fi} = (\phi(\boldsymbol{p}_f)\,|[a(\boldsymbol{q}_f),\ [H'(\boldsymbol{p}_i/2),\ a^*(\boldsymbol{q}_i)].]|\phi(\boldsymbol{p}_i))$$
$$+\sum_{\boldsymbol{p}_n=\boldsymbol{p}_i+\boldsymbol{q}_i=0} \frac{(\phi(\boldsymbol{p}_f)\,|[H'(\boldsymbol{p}_f/2),\ a(\boldsymbol{q}_f)].|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|[H'(\boldsymbol{p}_i/2),\ a^*(\boldsymbol{q}_i)].|\phi(\boldsymbol{p}_i))}{E_n - (\boldsymbol{p}_f^2/2m) - \omega_f - i\varepsilon}$$
$$+\sum_{\boldsymbol{p}_n=\boldsymbol{p}_i-\boldsymbol{q}_f} \frac{(\phi(\boldsymbol{p}_f)\,|[H'(\boldsymbol{p}_f/2),\ a^*(\boldsymbol{q}_i)].|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|[H'(\boldsymbol{p}_i+\boldsymbol{p}_f/2),\ a(\boldsymbol{q}_f)].|\phi(\boldsymbol{p}_i))}{E_n + \omega_f - (\boldsymbol{p}_i^2/2m)} \tag{13}$$

which is written in the centre of mass system, where $\boldsymbol{p}_i+\boldsymbol{q}_i=\boldsymbol{p}_f+\boldsymbol{q}_f=0$.

We shall in the following consider only those parts of the $1/m$ correction to Sugawara's static approximation, which do not depend on the physical nucleon velocity $\boldsymbol{p}_i/m$ and $\boldsymbol{p}_f/m$. This is because those corrections which depend explicitly on $\boldsymbol{p}_i/m$ and $\boldsymbol{p}_f/m$ are the essentially recoil corrections, which are, however, shown to be small according to a recent work by Chew et al.[5] We thus put $\boldsymbol{p}_i/m$ and $\boldsymbol{p}_f/m$ in (13) equal to zero and obtain

$$T_{fi} = (\Psi^{(-)}(\boldsymbol{q}_f)\,|[H',\ a^*(\boldsymbol{q}_i)].|\phi_0)$$
$$= (\phi_0|[a(\boldsymbol{q}_f),\ [H',\ a^*(\boldsymbol{q}_i)].]|\phi_0)$$
$$+\sum_{n\neq 0}(1/\omega_f)[(\phi_0|[a(\boldsymbol{q}_f),\ H'].|\phi_{0n})\,(\phi_{0n}|[H',\ a^*(\boldsymbol{q}_i)].|\phi_0)$$
$$+(\phi_0|[H',\ a^*(\boldsymbol{q}_i)].|\phi_{0n})\,(\phi_{0n}|[H',\ a(\boldsymbol{q}_f)].|\phi_0)]$$
$$+\sum_{n\geq 1}\left[\frac{(\phi_0|[H',\ a(\boldsymbol{q}_f)].|\Psi_n^{(-)})\,(\Psi_n^{(-)}|[H',\ a^*(\boldsymbol{q}_i)].|\phi_0)}{E_n - \omega_f - i\varepsilon}\right.$$

$$+\frac{(\phi_0|[H',\ a^*(\boldsymbol{q}_f)].|\Psi_n^{(-)})\,(\Psi_n^{(-)}|[H',\ a^*(\boldsymbol{q}_f)].|\phi_0)}{E_n+\omega_f}\Big]. \tag{14}$$

A corresponding equation follows immediately for $S_{fi}=(\Psi^{(-)}(\boldsymbol{q}_f)|[H',\ a(\boldsymbol{q}_i)].|\phi_0)$. This, together with (14), gives a coupled integral equation for T_{fi} and S_{fi}, if we assume the one meson approximation.

§ 4. Estimation of the $1/m$ correction and conclusion

The next problem is to examine the inhomogeneous terms in the integral equation (14). For this purpose we have only to follow the procedure adopted by Sugawara.[3] Therefore, only the outline is given below.

We first expand the meson fields as usual:

$$\boldsymbol{\phi}_\alpha=\sum_k \frac{v_k}{\sqrt{\omega_k}}A_{k\alpha},\quad \boldsymbol{\pi}_\alpha=\sum_k v_k\sqrt{\omega_k}\,B_{k\alpha} \tag{15}$$

with

$$A_{k\alpha}=(1/\sqrt{2})\,(a_\alpha^*(\boldsymbol{k})+a_\alpha(\boldsymbol{k})),\quad B_{k\alpha}=(i/\sqrt{2})\,(a_\alpha^*(\boldsymbol{k})-a_\alpha(\boldsymbol{k})),$$
$$[A_{k\alpha},\ B_{k'\alpha'}]=i\delta_{kk'}\delta_{\alpha\alpha'}, \tag{16}$$

where the cut-off function v_k is introduced explicitly. Then we can show the relations

$$(\phi_0|[H',\ A_i].|\phi_0)=iG_P\frac{v_i}{\sqrt{\omega_i}}(\boldsymbol{k}_i\boldsymbol{\sigma})\tau_i,$$

$$(\phi_0|[A_f,\ [H',\ A_i].]|\phi_0)=G_P'\frac{v_iv_f}{\sqrt{\omega_i\omega_f}}(\boldsymbol{k}_i\boldsymbol{k}_f)+G_S''v_iv_f\sqrt{\omega_i\omega_f}\delta_{if},$$

$$(\phi_0|[A_f,\ [H',\ B_i].]|\phi_0)=-(\phi_0|[B_f,\ [H',\ A_i].]|\phi_0)=G_S'\frac{v_iv_f}{\sqrt{\omega_i\omega_f}}\omega_f\varepsilon_{i\alpha f}\tau_\alpha,$$

$$(\phi_0|[B_f,\ [H',\ B_i].]|\phi_0)=2G_S\frac{v_iv_f}{\sqrt{\omega_i\omega_f}}\delta_{if},\quad (\phi_0|[H',\ B_i].|\phi_0)=0, \tag{17}$$

where $\varepsilon_{i\alpha f}$ is equal to $+1$ or -1 for even or odd permutations of its indices. G's in (17) are the renormalized coupling constants effective in the low energy pion-nucleon scattering. It is to be added that (17) follows under the assumption that no D and higher waves appear in the physical nucleon ϕ_0. More exactly, G's in (17) are only nearly constant, since the meson field momentum G appears in H'. This would be permitted, however, as far as the low energy pion-nucleon scattering is concerned.

It is readily noticed that G_P' and G_S'' are two additional parameters which did not appear in the simple static approximation.[2),3)] The meaning of these coupling constants would be clarified if we give explicitly the effective Hamiltonian

$$H_{eff}=G_{10}\sigma\nabla\tau\boldsymbol{\phi}+G_{10}'(\nabla\boldsymbol{\phi})^2+G_{S0}\boldsymbol{\phi}^2$$
$$+G_{S0}'\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}+G_{S0}''\boldsymbol{\pi}^2, \tag{18}$$

which gives rise to exactly the same integral equation as (14) under the one-meson

approximation. We put the subscript 0 to the coupling constants since those in (18) are unrenormalized yet, which are related to those in (17) as follows:

$$G_P = G_{P0}(\phi_0|\tau\sigma|\phi_0) + 2G'_{P0}(\phi_0|\nabla\phi|\phi_0), \quad G_P' = G'_{P0}$$
$$G_S = G_{S0}, \quad G_S' = G_{S0}(\phi_0|\tau|\phi_0), \quad G_S'' = G''_{S0}. \tag{19}$$

We thus see that the effect of the $1/m$ correction is two-fold: one is to modify the numerical values of the renormalized coupling constants defined by Sugawara and the other is to produce two extra parameters. We write explicitly as

$$G_P = G_P^0 + \Delta G_P, \quad G_S = (G_S^0 + \Delta G_S)/\mu, \tag{20}$$
$$G_S' = (G_S'^0 + \Delta G_S')/\mu^2, \quad G_P' = \Delta G_P'/\mu^3, \quad G_S'' = \Delta G_S''/\mu^3,$$

where ΔG's are the $1/m$ corrections.

To estimate these corrections we assume the same approximation as was done by Sugawara; namely we approximate $|\phi_0)$, according to the intermediate coupling theory,[6] as

$$|\phi_0) = C_0|u\rangle + (C_1/3)\sigma_i\tau_\alpha b^*_{\alpha i}|u\rangle, \tag{21}$$

with

$$b^*_{\alpha i} = i\int g(k)\,(3/4\pi)^{1/2}\cdot \boldsymbol{k}_i/k\cdot a_\alpha^*(\boldsymbol{k})\,d^3\boldsymbol{k},$$
$$g(k) = Nk/\omega(\omega+\mu), \quad \int g^2(k)\,k^2dk = 1,$$

Table 1. The numerical values of the renormalized coupling constants. The definitions are given in (20) in the text.

Case A: $k_{\max,P}=5.0\mu$, $k_{\max,S}=5.0\mu$, $(g^2/4\pi)=0.26$.

C_1	G_P^0	ΔG_P	G_S^0	ΔG_S	$G_S'^0$	$\Delta G_S'$	$\Delta G_P'$	$\Delta G_S''$
−0.3	0.50	0.01	11	−2.1	0.34	0.14	0.006	0.033
−0.5	0.43	0.08	11	−1.6	0.34	0.13	0.006	0.033

Case B: $k_{\max,P}=5.0\mu$, $k_{\max,S}=4.0\mu$, $(g^2/4\pi)=0.44$.

C_1	G_P^0	ΔG_P	G_S^0	ΔG_S	$G_S'^0$	$\Delta G_S'$	$\Delta G_P'$	$\Delta G_S''$
−0.3	0.65	0.06	20	−0.50	0.56	0.20	0.010	0.055
−0.5	0.55	0.15	20	−1.6	0.56	0.17	0.010	0.055

Chew-Low and Drell et al.'s values.

G_P	$G_S\cdot\mu$	$G_S'\cdot\mu^2$	$G_P'\cdot\mu^3$	$G_S''\cdot\mu^3$
1.0	0.4	0.4	0.0	0.0

where $|u\rangle$ is the bare nucleon spinor and $g(k)$ specifies the P-wave configuration inside the pion cloud.

Results are given in Table 1 for two sets of parameters, both of which correspond to $\chi_0^2=(1/3)\langle(f/2m)^2\phi^2\rangle_0=1.0$, using the notation in reference 3. It is not attempted here to reproduce the experimental values, since the approximation (21) is certainly not sufficient. It is here aimed to see how large the $1/m$ correction is. For this purpose, even a crude approximation (21) may give us some information.

We can now summarize some important conclusions deduced from Table 1:

a) If we cut the virtual meson momentum below the nucleon rest mass, the $1/m$ correction would not be so great as invalidating the static approximation assumed by Sugawara.

b) It is to be noted that two additional parameters seem quite negligible, though their effects in the integral equation (14) have not yet been studied in detail. In this meaning the static model containing three coupling oonstants can have a strong validity.

The author would like to thank Prof. Sugawara for helpful discussions and his careful reading of the draft of this paper. He also thanks Dr. H. Tanaka and many other members of the Institute for valuable discussions.

Appendix

Let us derive the relation (11) from (9). $H(\boldsymbol{p})$ defined in Section 3 is written more explicitly as

$$H(\boldsymbol{p})=H_\pi+(\boldsymbol{p}-\boldsymbol{G})^2/2m+H_1+H_2+H_3(\boldsymbol{p})+H_3(-\boldsymbol{G}), \tag{A1}$$

where $H_3(-\boldsymbol{G})$ has a form

$$H_3(-\boldsymbol{G})=\boldsymbol{A}(\phi,\pi,\sigma,\tau)\cdot\boldsymbol{G}+\boldsymbol{G}\cdot\boldsymbol{A}.$$

Using the relations

$$\begin{aligned}
&[(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{G})^2,\ a^*(\boldsymbol{q})]=a^*(\boldsymbol{q})\{(\boldsymbol{p}-\boldsymbol{G})^2-(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{G})^2\},\\
&[H_3(-\boldsymbol{G}),\ a^*(\boldsymbol{q})]=[H_3(-\boldsymbol{G}),\ a^*(\boldsymbol{q})]\cdot-a^*(\boldsymbol{q})H_3(\boldsymbol{q})\\
&\qquad\qquad -(1/2)[H_3(\boldsymbol{q}),\ a^*(\boldsymbol{q})],\\
&H_3(\boldsymbol{p}+\boldsymbol{q})=H_3(\boldsymbol{p})+H_3(\boldsymbol{q}),
\end{aligned} \tag{A2}$$

it is easy to justify the following manipulation

$$\begin{aligned}
H(\boldsymbol{p}+\boldsymbol{q})\Psi^{(+)}(\boldsymbol{p},\boldsymbol{q})&=H(\boldsymbol{p}+\boldsymbol{q})\{a^*(\boldsymbol{q})\phi(\boldsymbol{p})+\chi^{(+)}(\boldsymbol{p},\boldsymbol{q})\}\\
&=a^*(\boldsymbol{q})H(\boldsymbol{p}+\boldsymbol{q})\phi(\boldsymbol{p})+[H(\boldsymbol{p}+\boldsymbol{q}),\ a^*(\boldsymbol{q})]\phi(\boldsymbol{p})+H(\boldsymbol{p}+\boldsymbol{q})\chi^{(+)}(\boldsymbol{p},\boldsymbol{q})\\
&=\{a^*(\boldsymbol{q})H(\boldsymbol{p}+\boldsymbol{q})+[H_\pi,\ a^*(\boldsymbol{q})]+[(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{G})^2/2m,\ a^*(\boldsymbol{q})]\\
&\quad+[H_1+H_2,\ a^*(\boldsymbol{q})]+[H_3(\boldsymbol{p}+\boldsymbol{q}),\ a^*(\boldsymbol{q})]+[H_3(-\boldsymbol{G}),\ a^*(\boldsymbol{q})]\}\phi(\boldsymbol{p})\\
&\quad+H(\boldsymbol{p}+\boldsymbol{q})\chi^{(+)}(\boldsymbol{p},\boldsymbol{q})
\end{aligned}$$

$$
\begin{aligned}
&= \{a^*(\boldsymbol{q})\,(H(\boldsymbol{p}+\boldsymbol{q}) + \omega_q + (\boldsymbol{p}-\boldsymbol{G})^2/2m - (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{G})^2/2m - H_3(\boldsymbol{q}))\\
&\quad + [H_1 + H_2 + H_3(\boldsymbol{p}+\boldsymbol{q}) - (1/2)\,H_3(\boldsymbol{q}) + H_3(-\boldsymbol{G}),\ a^*(\boldsymbol{q})\,].\}\,\phi(\boldsymbol{p})\\
&\quad + H(\boldsymbol{p}+\boldsymbol{q})\,\chi^{(-)}(\boldsymbol{p},\ \boldsymbol{q})\\
&= a^*(\boldsymbol{q})\,(H(\boldsymbol{p}) + \omega_q)\,\phi(\boldsymbol{p}) + [H'(\boldsymbol{p}+\boldsymbol{q}/2),\ a^*(\boldsymbol{q})\,].\,\phi(\boldsymbol{p})\\
&\quad + H(\boldsymbol{p}+\boldsymbol{q})\,\chi^{(-)}(\boldsymbol{p},\ \boldsymbol{q})\\
&= (\boldsymbol{p}^2/2m + \omega_q)\,\Psi^{(+)}(\boldsymbol{p},\ \boldsymbol{q}). \qquad \text{(A3)}
\end{aligned}
$$

The relation (11) follows directly from (A3).

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Letters to the Editor

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On the Recombination Processes by the Auger Effect

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December 23, 1957

It is the aim of this note to show how much the Auger effect contributes to the trapping and recombination processes, in Ge and Si.

L. Bess[1] has already calculated about those processes, but we got different results although our method of calculation was almost same as his. We shall in the following show our results and the points that are essentially important.

Let us treat an n-type sample using the Born approximation method. The transition matrix element from $(\boldsymbol{K}_i,\ \boldsymbol{k})$-state to $(\boldsymbol{K}_f,\ t)$-state is got as

$$|\langle \boldsymbol{K}_f,\ t|H'|\boldsymbol{K}_i,\ \boldsymbol{k}\rangle| \propto [q^2+(\boldsymbol{K}_i-\boldsymbol{K}_f)^2]^{-1} \times [\alpha^2+(\boldsymbol{K}_i-\boldsymbol{K}_f+\boldsymbol{k})^2]^{-2}, \qquad (1)^{1)}$$

where

$$H'=(e^2/\kappa r_{12})\exp(-qr_{12}). \qquad (2)$$

In deriving (1), we used the hydrogen 1s type wave fn., $\sqrt{\alpha^3/\pi}\exp(-\alpha r)$, for trapped state, (t), and free wave approximation for conduction band electron, $(\boldsymbol{k})$, and another electron changed from its initial free state, $(\boldsymbol{K}_i)$, to final free state, $(\boldsymbol{K}_f)$.

Then we can know that

$$q^2,\ \boldsymbol{k}^2 \text{ and } \boldsymbol{K}_i^2 \ll \boldsymbol{K}_f^2,\ 2m_eE_t/\hbar^2, \qquad (3)$$

because of the equilibrium electron distribution (majority system); in (3), of course, is used the relation

$$|\boldsymbol{K}_f|^2 \sim 2m_eE_t/\hbar^2, \qquad (4)$$

which is resulted from the energy conservation law.

Now, we can easily show the capture cross section, after summing up all $\boldsymbol{K}_i$-states and averaging over all $\boldsymbol{k}$-states, as follows.

$$\sigma_e{}^0=\frac{8\pi^2e^4\beta^{1/2}\hbar^3}{\sqrt{3}\,\kappa^2m_e^{1/2}E_t^3}n_0, \qquad (5)$$

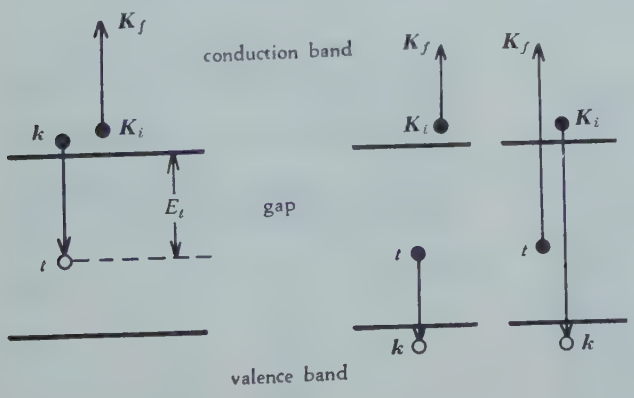

Fig. 1

(a) (b)

Fig. 2

where m_e is an effective mass of electron, $E_t \sim \hbar^2\alpha^2/2m_e$ and $\beta = 1/kT$.

Numerical result is got as

$$\sigma_e^0 \sim 2.17 \times 10^{-20}\ \mathrm{cm}^2, \text{ for Ge,} \quad (6)$$

where we used $n_0 = 10^{14}\ \mathrm{cm}^{-3}$, $m_e = 0.25\ m$, $\kappa = 16$, $E_t = 0.25$ eV and $T = 300°\mathrm{K}$.

In the case of the Auger effect, however, it should be noticed that we must take into account the exchange effect between those two electrons, but in this case the matrix element corresponding to (1) is got as in the same form as (1), on account of (2), and the result is not changed.

Next, we show that the exchange effects become essentially important in the case of hole capture (minority system), which is shown in Fig. 2. In this case, the trapped electron falls into $\boldsymbol{k}$-state hole in the valence band and make another electron accelerate towards the upper part in the conduction band, i. e., from $\boldsymbol{K}_i$ to $\boldsymbol{K}_f$.

When we neglect the exchange effect in this case (ref. Fig. 2a), matrix element becomes to be identical with (1), on account of (2). But when we do not neglect the exchange effects, the matrix element,

$$|\langle \boldsymbol{k},\ \boldsymbol{K}_f|H'|\boldsymbol{K}_i,\ t\rangle| \propto [q^2 + (\boldsymbol{k}-\boldsymbol{K}_i)^2]^{-1} \times [\alpha^2 + (\boldsymbol{k}-\boldsymbol{K}_i+\boldsymbol{K}_f)^2]^{-2}, \quad (7)$$

is got. Then we can find that (7) is much larger than (1), because the 1st factor of the right side of (7) is large. Nummerically σ_h becomes of the order of $10^{-16}\ \mathrm{cm}^2$ for Ge. This is the reason why Bess obtained the large capture probability.

Now, we must emphasize that $\boldsymbol{k}_0$ is not zero in the energy minimum points of conduction band but $\boldsymbol{k}_0$ is zero in the energy maximum points of full band, so we substitute $\boldsymbol{k}$, $\boldsymbol{k}_0+\boldsymbol{K}_i$, $\boldsymbol{k}_0+\boldsymbol{K}_f$ and $-i\boldsymbol{k}_0+\alpha$ in place of $\boldsymbol{k}$, $\boldsymbol{K}_i$, $\boldsymbol{K}_f$ and α in (7) respectively. Then we get

$$|\langle \boldsymbol{k},\ \boldsymbol{K}_f|H'|\boldsymbol{K}_i,\ t\rangle| \propto [q^2 + (\boldsymbol{k}-\boldsymbol{k}_0-\boldsymbol{K}_i)^2]^{-1} \times [\alpha^2 + (\boldsymbol{k}-\boldsymbol{k}_0-\boldsymbol{K}_i+\boldsymbol{K}_f)^2]^{-2}, \quad (8)$$

instead of (7), which is the essential point in our results. It is easily shown that (8) becomes much smaller than (7) because of $|\boldsymbol{k}_0| \gtrsim |\boldsymbol{K}_f|$ in such a case of Si and Ge.

Even when $\boldsymbol{k}_0 = 0$, σ_e^0 is very smaller than σ_h^0 as mentioned above in (5), and moreover σ_h^0 becomes large and σ_e^0 becomes small by the charge of trapping center. In such a case, therefore, recombination process cannot take place but the hole temporary trapping seems occur.[2]

We can conclude safely that the Auger effect can be neglected in the ordinary recombination phenomena compared with multiphonon process.[3),4)]

The auther is thankful to Asst. Prof. T. Ohta for valuable discussions.

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On the Auger Effect in the Deep Traps in Si

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December 23, 1957

In Si, the possibility of the Auger effect was suggested by the fact that the lifetime of the minority carrier trapped in the deep trap, which is resulted in the recombination with the majority carrier, is proportional to inverse square of the density of them.[1),2)]

It is the purpose of this note to show that the above is true, using the calculation method described in our previous note.[3)]

Let us consider about n-type sample, then it may be reasonable to regard that the deep trapping center of the hole, whose energy level lies near the bottom of the conduction band, has more negative charge than the ordinary acceptor. Therefore, before an electron is trapped in this center, there is still a potential barrier arround it. In such a case, it is difficult to calculate exactly the capture probability of an electron by the Auger effect. As an approximation, however, only the electron which has higher energy than a height of effective potential barrier, E^*, can be regarded as a plane wave state near this center and can approach there and can be captured. But another electron which is expected to be accelerated towards the upper free state by the Coulomb repulsion between this electron and the captured one need not jump over this barrier because of long range interaction between them (ref. Fig. 1).

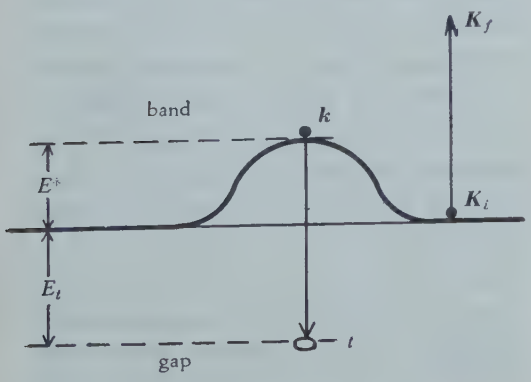

Fig. 1

It becomes somewhat complicated to calculate the electron capture cross section of this trap because we must use following conditions in such a case,

$$q^2,\ \boldsymbol{K}_i^2 \ll \boldsymbol{K}_f^2,\ \boldsymbol{k}^2 \text{ and } 2m_e E_t/\hbar^2, \quad (1)$$

$$\boldsymbol{K}_f^2 \sim 2m_e(E^*+E_t)/\hbar^2, \quad (2)$$

instead of (3) and (4), respectively, in our previous note. But, if $E^* \gg kT$, we get

$$\sigma_e = \frac{2}{\pi}\left(\frac{E^*}{kT}\right)^{1/2}\left[1+\frac{7}{3}\frac{E^*}{E_t} + \frac{4}{3}\left(\frac{E^*}{E_t}\right)^2\right]\left(1+\frac{E^*}{E_t}\right)^{-3} \times \sigma_e^{\,0}\exp(-\beta E^*) \quad (3)$$

as a good approximation. Where $\sigma_e^{\,0}$ is shown by (5) in the previous note.[3)] So, the probability that the trapped hole recombines with an electron (this electron is trapped into the center) can be expressed by $\bar{v}\sigma_e/V$, where $\bar{v}$ is the thermal velocity and V is the volume of the crystal. Multiplying this by the total number of electrons, the lifetime of trapped hole τ which is due to this process is got as

$$1/\tau = \bar{v}\sigma_e n_0 \quad (4)$$

where n_0 is the density of electrons.

For the deep trap in n-Si, it has been measured that $1/\tau$ has an activation energy of about 0.35 eV.[2] This value, however, contains not only E^* appeared in (3) but also the twice of the activation energy $\mathcal{E}$ of n_0, because σ_e has a factor of n_0 then $1/\tau$ is proportional also to n_0^2. We can estimate the value of $\mathcal{E}$ about 0.045 eV from the data of conductivity in such a sample. [Cf. the case (a) of Fig. 7 in ref. 2).] Then,

$$E^* = 0.35 - 2 \times 0.045 = 0.26 \ \text{(eV)}. \quad (5)$$

It is difficult to calculate the value of E^*, but the above value may be reasonable if the center has charge of $-2e$ or $-3e$.

In p-Si, there is no datum of E^*, but from the small value of $1/\tau$, E^* is regarded as larger than in n-Si.

For example, when $n_0 \sim 10^{14}\ \text{cm}^{-3}$, from (4) in the previous note,

$$\sigma_e^0 \sim 0.864 \times 10^{-20}\ \text{cm}^2 \quad (6)$$

is got, where we used following constants in Si, i.e., $m_e = 0.5\ m$, $v = 1.34 \times 10^7$ cm sec^{-1}, $\kappa = 12$, $E_t = 0.32$ eV[2] and $T = 300\,^\circ$K.

Then, inserting the above values into (4), we get

$$1/\tau = 0.00052\ \text{sec}^{-1}. \quad (7)$$

This value agrees with experimental one[2] in order, which is about 0.0002 sec^{-1} for the same electron density.

Nextly, let us consider about the multi-phonon capture in such a center. In this case, however, it is possible that there exists another activation energy of the phonon system, which is caused by the Franck-Condon principle,[4] in addition to the above E^*. Hence, the probability of this process will be much smaller than that by the Auger effect.

Therefore, it can be concluded that the Auger effect predominates such a phenomenon.

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Mass Levels and Relative Parities of Baryons*

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It seems tempting to speculate on the possibility whether one can explain the observed mass levels of baryons in terms of their self energies through the strong interactions with the π and K mesons, within the frame of the Nishijima and Gell-Mann scheme[1] of the strange particles.

In this note, we shall show the possibility that, unlike Schwinger's proposal,[2] the observed mass level ordering of Ξ, Σ, Λ and N comes chiefly from the charge independent baryon-K interaction, if a

* A brief account of this note has appeared in Memoirs of the Osaka Gakugei University, B. No. 6 (1957), 1.

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suitable (relative) parity for each baryon is assumed.*

Recent experiments on the K-N scattering and on the hyperfragments seem to give the coupling constant, $g_K^2/4\pi \sim 1$. We shall consider the baryon self energy due to the K- and π-interactions in the lowest order perturbation approximation, though this approximation is known to be unreliable for the $\pi-N$ interaction.

Assuming that all the baryons would have the equal rest mass in the hypothetical world where the K- and π-baryon interactions vanish, and using the charge independent interaction Hamiltonian given by d'Espagnat and Prentki,[3] we obtain the mass shifts of baryons,

$$\left.\begin{aligned}
\Delta M(N) &= g_K^2[3F(N\Sigma K) + F(N\Lambda K)] \\
&\quad + 3g_\pi^2 F(NN\pi) \\
\Delta M(\Lambda) &= 2g_K^2[F(\Lambda N\bar{K}) + F(\Lambda\Xi K)] \\
&\quad + 3g_\pi^2 F(\Lambda\Sigma\pi) \\
\Delta M(\Sigma) &= 2g_K^2[F(\Sigma N\bar{K}) + F(\Sigma\Xi K)] \\
&\quad + g_\pi^2[F(\Sigma\Lambda\pi) + 2F(\Sigma\Sigma\pi)] \\
\Delta M(\Xi) &= g_K^2[3F(\Xi\Sigma\bar{K}) + F(\Xi\Lambda\bar{K})] \\
&\quad + 3g_\pi^2 F(\Xi\Xi\pi)
\end{aligned}\right\} \qquad (1)$$

where $F(\cdots)$ stands for the contribution to each baryon self energy from the interaction $(\cdots)$, and all the K- and all the π-coupling constants are equally taken to be g_K and g_π respectively. If all the coupling types are assumed to be equally direct and/or derivative, a simple consideration from (1) shows that, *in order to get the observed $N-\Xi$ mass splitting, a relative parity between N and Ξ should be odd,** i. e., the N and Ξ form a parity doublet with the isospin $1/2$. Then we may examine only the following two cases.

Case (a). The relative parity of the N and Ξ is odd, but that of the Λ and Σ is even: eq. (1) gives the mass differences,

$$\left.\begin{aligned}
M(\Xi) - M(N) &= 2[M(\Lambda) - M(N)] \\
&= 4g_K^2(F_2^{(K)} - F_1^{(K)}), \\
M(\Sigma) - M(\Lambda) &= 0,
\end{aligned}\right\} \qquad (2a)$$

where $F_1^{(K)} \equiv F(N\Sigma K) = F(N\Lambda K)$ and $F_2^{(K)} \equiv F(\Sigma\Xi K) = F(\Lambda\Xi K)$.

For this case the π-interaction has no effect to the baryon mass difference in the lowest approximation. We have the interval ratio,

$$\frac{M(\Xi) - M(\Sigma, \Lambda)}{M(\Sigma, \Lambda) - M(N)} = 1, \qquad (3)$$

and the observed order of baryon mass levels under the condition $F_2^{(K)} - F_1^{(K)} > 0$, though the Σ and Λ are still degenerate (Fig. 1 (a)).

Case (b). The relative parity of the N and Ξ is odd, and that of the Λ and Σ is also odd: eq. (1) gives

* After this report was prepared, the new works by M. Gell-Mann (*Proc. of the 7th Rochester Conf.* (1957)) and by D. Kleitman (Phys. Rev. **107** (1957), 1453) came into our notice. Gell-Mann proposed the theory based on a global symmetry and obtained the relation, $1/2 \cdot [M(N) + M(\Xi)] = 1/4 \cdot [M(\Lambda) + 3M(\Sigma)]$. His theory, however, does not give the observed mass splitting in the lowest order approximation as was shown by Kleitman. On the other hand, in our model an appropriate parity assigned for each baryon gives the mass splitting in the lowest approximation.

* If the relative parity between N and Ξ is taken to be even, we get $F_1^{(K)} = F_2^{(K)}$ without respect to the parities of the other particles. This is the reason why the Gell-Mann theory of global symmetry does not give the level splittig in the lowest approximation.

$$M(\Xi)-M(N)=2g_K^2(F_2^{(K)}-F_1^{(K)})$$
$$M(\Lambda)-M(N)=g_K^2(F_2^{(K)}-F_1^{(K)})$$
$$-3g_\pi^2(F_1^{(\pi)}-F_2^{(\pi)})$$
$$M(\Sigma)-M(\Lambda)=g_\pi^2(F_1^{(\pi)}-F_2^{(\pi)}) \quad (2b)$$

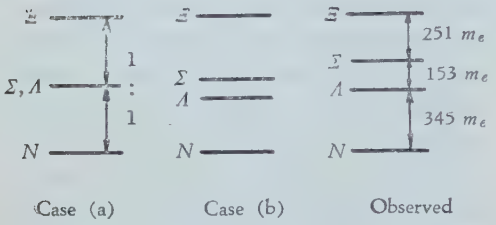

Fig. 1

where $F_1^{(K)}\equiv F(N\Sigma'K)=F(\Lambda\Xi K)$, $F_2^{(K)}\equiv F(N\Lambda K)=F(\Sigma'\Xi K)$, $F_1^{(\pi)}\equiv F(NN\pi)=F(\Sigma'\Sigma'\pi)=F(\Xi\Xi\pi)$ and $F_2^{(\pi)}\equiv F(\Lambda\Sigma'\pi)$. In the limit of vanishing π-interaction this case also gives the interval ratio (3). In the presence of π-interaction, however, we have the $\Sigma'-\Lambda$ level splitting, and the level ordering is in agreement with the observation under the condition $g_K^2(F_2^{(K)}-F_1^{(K)})>3g_\pi^2(F_1^{(\pi)}-F_2^{(\pi)})>0$ (Fig. 1 (b)).

The experimentally correct interval ratios might be obtained by a more reliable treatment of the strong interaction. The condition $F_2^{(K)}-F_1^{(K)}>0$, which has to be satisfied for the both cases, suggests that if the N and Σ' have the same (opposite) parities, the scalar K with direct (derivative) coupling and/or the pseudoscalar K with derivative (direct) coupling is preferable, as is inferred from the signs of the divergent integrals in the lowest order self energy expressions.[4] The future experiments would check the validity of the opposite parities of N and Ξ, and would determine the suitable parities of the other baryons and of K as well as the coupling types.

We wish to thank Professors H. M. Mahmoud and J. Joseph (I. S. C.), and Professors Y. Katayama and H. Enatsu for their critical discussions, and Professor G. W. Fox (I. S. C.) for his kind hospitality.

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Effect of Electron-Electron Scattering on Hall Mobility of Electrons in *n* Semiconductors

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The large ratio of Hall mobility to drift mobility ($\mu_I'/\mu_I=1.93$), predicted by the simple theory,[1] when lattice scattering is negligible, is in disagreement with experiments. The theory usually applied to transport phenomena in the impurity scattering region is oversimplified in other respects than in the neglect of electron-electron scattering and hence it is not to be anticipated that the inclusion of electron-electron scattering in the theory will give a complete explanation of all the deviations from the simple theory. Nevertheless since there is reason to believe that the electron-

electron scattering is by no means negligible it is desirable to include it in the theory.

The time of relaxation of an electron due to scattering by ionized impurity ions is given by

$$\tau_I = Bx^3 \tag{1}$$

where $x = (m/2kT)^{1/2} v$,

v is velocity of the electron,
m is the effective mass of electron,
k is Boltzmann constant,
T is the temperature of crystal,
and B is a constant at a given temperature, inversely proportional to impurity density.

Spitzer and Harm[2] have investigated the velocity distribution of electrons in the presence of an electric field in an ionized gas. Sodha and Eastman[3] have applied this analysis to semiconductors and shown that the time of relaxation of an electron due to both electron-ion and electron-electron scattering is given by

$$\tau = B\varphi(x)/x \tag{2}$$

where $\varphi(x) = ZD(x)/A$ is a function tabulated by Spitzer and Harm[2] for various values of mean ionic charge Z. Sodha and Eastman[3] have also pointed out that the case $Z=1$ applies to most of semiconductors of interest.

The velocity distribution of electrons in a nondegenerate semiconductor is given by

$$N(x)\,dx = Ax^2 \exp(-x^2)\,dx. \tag{3}$$

The drift and Hall mobilities are given by[1]

$$\mu = \frac{q}{m}\frac{\langle x^2\tau\rangle}{\langle x^2\rangle} \tag{4}$$

and

$$\mu' = \frac{q}{m}\frac{\langle x^2\tau^2\rangle}{\langle x^2\tau\rangle}. \tag{5}$$

From eqs. (1), (3), (4) and (5) we have[1]

$$\mu_I = \frac{8qB}{m\sqrt{A}} \tag{5A}$$

and

$$\mu_I'/\mu_I = 1.93. \tag{5B}$$

From eqs. (2), (3), (4) and (5) we have

$$\mu = \frac{qB}{m}\frac{\int_0^\infty x^3\varphi(x)\exp(-x^2)\,dx}{\int_0^\infty x^4\exp(-x^2)\,dx} = 0.582\,\mu_I \tag{6A}$$

and

$$\mu' = \frac{qB}{m}\frac{\int_0^\infty x^2\{\varphi(x)\}^2\exp(-x^2)\,dx}{\int_0^\infty x^3\varphi(x)\exp(-x^2)\,dx} = 0.685\,\mu_I. \tag{6B}$$

Hence

$$(\mu'/\mu) = 1.18. \tag{6C}$$

Both the integrals involving $\varphi(x)$ were numerically evaluated.

Thus we find that the inclusion of electron-electron scattering in the theory lowers the ratio of Hall to Drift mobility as anticipated. It is a strange coincidence that the ratio 1.18 is also obtained,[1] when scattering due to acoustical modes of lattice vibrations is taken as the sole mechanism of scattering.

The authors are grateful to Dr. G. M. Shrum for kind encouragement and to Professor R. E. Burgess for his kind interest in the investigation. Thanks are also due to National Research Council of Canada for award of a fellowship (M. S. S.) and scholarship (P. C. E.) to the authors.

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Density of the Normal Component for Solutions of He^3 and He^4

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Recently, Berezniak and Eselson[1] have reported measurements of the density of the normal component (fluid) for solutions of He^3 and He^4 for various concentrations of He^3. In the present note we have made use of a model recently proposed by us,[2)-8)] to explain the observed data reported by Berezniak and Eselson.

Assuming that He^3 atoms partake in the motion of the normal fluid, the density of the normal fluid, ρ_n, can be written as[5]

$$\rho_n = \rho[x(1-x_3) + x_3(\mu/m_3)], \qquad (1)$$

where ρ is the density of the solution, x the fraction of the total number of atoms of He^4 which constitute the normal fluid, while x_3 is the concentration (by mass) of He^3 in the solution. Here μ is the effective mass of the He^3 atoms in solution while m_3 is the actual mass.

We have calculated the ratio of the density of the normal fluid, ρ_n, to the density of the solution at its λ-temperature, $\rho_{\lambda X_L}$, as a function of absolute temperature, $T(^\circ K)$, on the basis of our model by making use of the linear and quadratic expressions[6] for G_4* and using $(\mu/m_3) = 2.17$, as required to explain the second sound velocity measurements in solutions.[3] The results thus obtained are normalized to unity at the λ-temperature of the solution and are compared in Fig. 1 with the experimental results.[1] We have ignored the volume change on mixing in order to calculate the density ρ of the solution and have made use of the experimental results of Keesom[9] and Kerr[10] for the densities of liquid He^4 and liquid He^3 respectively. As is evident from Fig. 1, the results calculated on the basis of the present model are in fairly good agreement with the experimental data.

Further, we have calculated the λ-temperature, $T_{\lambda X_L}$, of the solution of He^3 and He^4, as a function of the molar concentration of He^3, X_L, on the basis of the present model[6] using the linear and quadratic expressions of G_4. The calculated values are compared with the experimental results of Berezniak and Eselson[1] in Table I. As is evident from Table I, the results calculated on the basis of the present model are in fairly good agreement with the experimental results.

It should be mentioned that the linear as well as the quadratic expressions of G_4 lead to satisfactory results, in general agreement with the experimental data. This leads us to the conclusion that in all evaluations of the properties of the solution using classical thermodynamics, the predominating influence is that of the basic assumptions of the present model whereby

* G_4 is the Gibb's function for pure He^4.

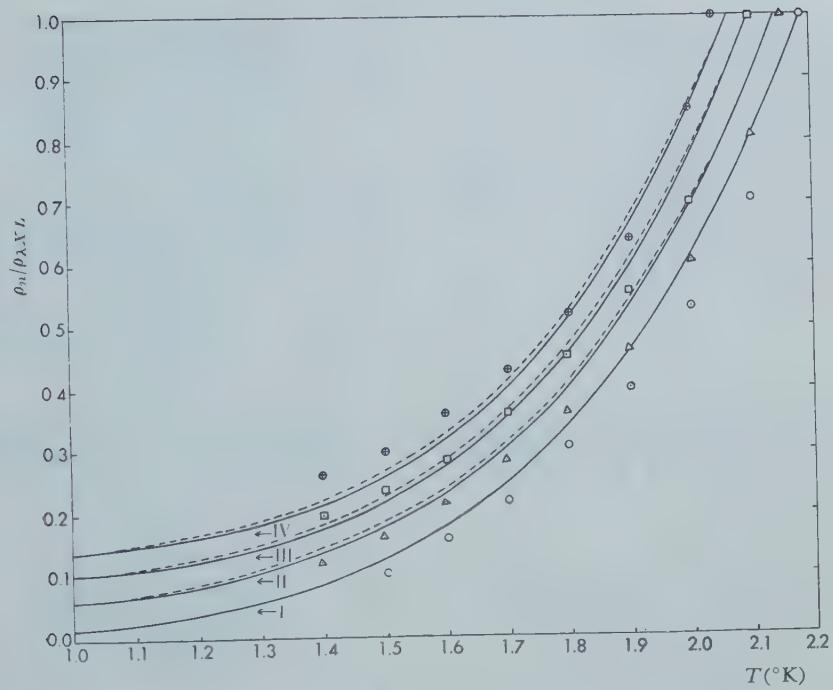

Fig. 1. The ratio of the density of the normal fluid, ρ_n, to the density of the solution at its λ-temperature, $\rho_{\lambda X_L}$, as a function of absolute temperature, T(°K).

---- : Present Model, Linear Expression for G_4.

——— : Present Model, Quadratic Expression for G_4.

Curve I: $X_L=0.00$; Curve II: $X_L=0.03$;

Curve III: $X_L=0.056$; Curve IV: $X_L=0.082$.

Smoothed experimental results of Berezniak and Eselson[1]:—

⊙: $X_L=0.00$; △: $X_L=0.03$; ⊡: $X_L=0.056$; ⊕: $X_L=0.082$.

Table I. λ-temperature of He^3 and He^4 solution, $T_{\lambda X_L}$, as a function of the molar concentration of He^3, X_L, in the solution.

X_L	$T_{\lambda X_L}$(°K)		
	Present Model using :—		Exptal. results
	Linear Expression of G_4	Quadratic Expression of G_4	
0.00	2.18	2.18	2.19
0.03	2.14	2.14	2.15
0.056	2.10	2.10	2.10
0.082	2.07	2.06	2.04

the quantum nature of the liquid He^4 is injected into the calculations, and whereby agreement with experiment is obtained irrespective of the assumed variations in the energy spectrum of pure liquid He^4.

My grateful thanks are due to Professor D. S. Kothari for his interest in this investigation.

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The Non-Indication of the "Anomalous" Scattering of Mu-Mesons

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January 9, 1958

The principal difficulties in the experimental study of the mu-meson scattering[1)~7)] to date are due to the fact that it is not certain whether particles other than mu-mesons are involved or not and to an uncertainty in the determination of the lowest momentum of mu-mesons detected by the apparatus. The present study of mu-meson scattering has been performed by means of the apparatus capable of selecting only mu-mesons, their momenta being determined with a thick absorber and a delayed coincidence method within a narrow range of (1 ± 0.15) GeV/C. The arrangement is illustrated in Fig. 1. The triggering method is AB (B or C or D)$_{\text{delay}}$ —D. The observed angular distribution of scattered mu-mesons has been found to be in good agreement with the Coulomb scattering theory for extended nuclei obtained by Cooper and Rainwater[8)] (see Fig. 2).

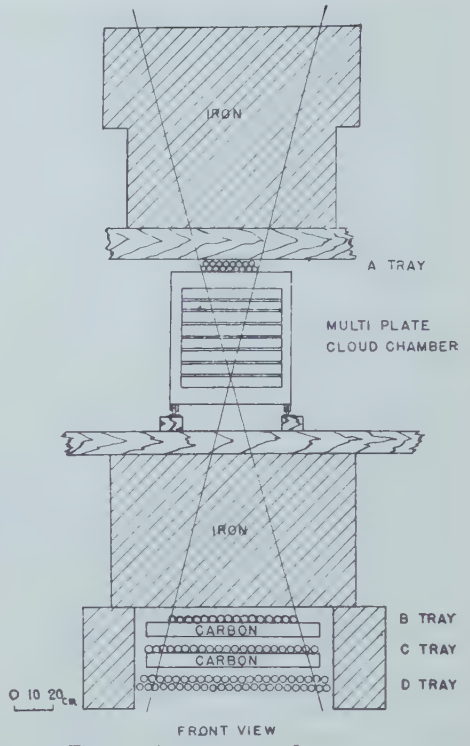

Fig. 1. Arrangement of apparatus.

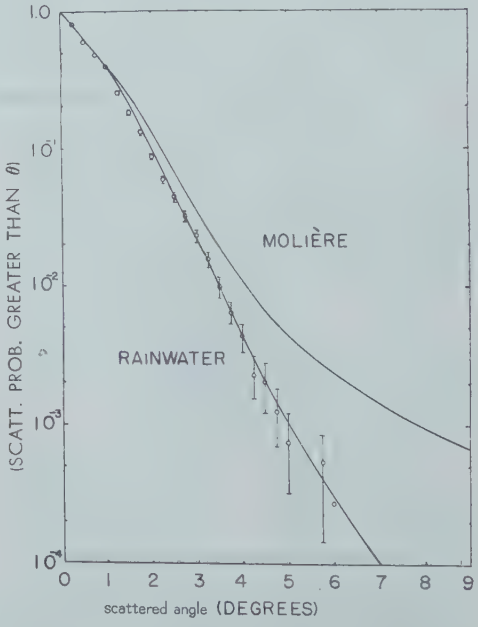

Fig. 2. Measured and calculated integral angular distribution of the scattered particles registered by the delayed coincidence.

It has thus been concluded by this observation that the possibility of the "anomalous" scattering of mu-mesons near 1 Ge V/C is excluded.

On the other hand, the angular distribution observed with the prompt coincidence (A. B. D. coincidence), which selects all the particles penetrating through the absorber, fits to Molière's theory[9] for point charge nuclei rather than Cooper and Rainwater's one as shown in Fig. 3. It seems that this fact can be explained mainly by an uncertainty in the determination of the lowest momentum of mu-mesons, because the contribution to large angle scatters comes chiefly from the mu-meson of the lower momenta. Indeed, if we assume the lowest momentum to be 0.8 Ge V/C, the observed angular distribution turns out to fit Cooper and Rainwater's theory quite well (see Fig. 4).

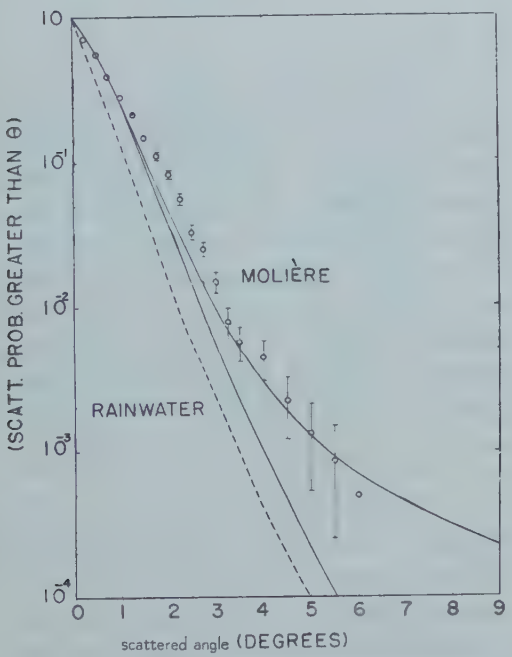

Fig. 3. Measured and calculated integral angular distribution of the scattered particles registered by the prompt coincidence. The dotted curve is the angular distribution uncorrected for the noiselevel scattering.

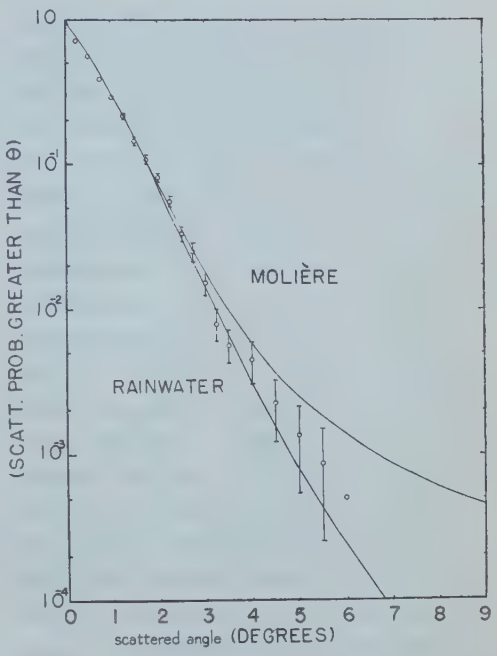

Fig. 4. Measured and calculated integral angular distributions of the scattered particles registered by the promt coincidence. The lowest momentum value of adopted particle's momentum spectrum is limited to 0.8 Be V/C

We shall further discuss the scattering of mu-mesons above 2 Ge V/C. In this case it can again be shown that the determination of momentum with a magnetic cloud chamber gives rise to the same kind of uncertainty. That is, a distortion which occured in the cloud chamber leads to a spurious curvature of track. In the case of no magnetic field tracks with a spurious curvature of about ten times the r. m. s. value are observed at a frequency of about 1/50.[10] The inaccuracy attributed to this distortion in determining momentum may give rise to the seeming existence of the "anomalous" scattering. Another cause,

to give rise to the "anomalous" scattering, may be the contamination of protons. Some "anomalous" scattering thus far observed can be attributed to the protons; our delayed coincidence method excludes this contamination. At high energies where the delayed coincidence method is not available, however, the proton contamination may play a significant role. It is interesting to note that the contamination is somehow related to high energy nucleons associated with extensive air showers. Since we have yet no definite experimental information on an elasticity in the nucleon-nucleus collision, we estimated it by referring to the nucleon cascade theory. A suitable choice of the elasticity predicts the proton contamination which can account for the "anomalous" scattering observed by Lloyd et al. It is thus hard to draw a very definite conclusion of existence of the "anomalous" scattering for mu-mesons with energies greater than 2 BeV from the experimental results to date.

We wish to express our sincere thanks to Prof. W. L. Kraushaar for his early suggestion on this approach and help in constructing the apparatus. We are also indebted to Prof. S. Hayakawa and the Underground Group in this laboratory for their stimulating discussions and criticisms.

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Ground State Energy of Bose System with Lennard-Jones Potential

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For the system of strongly interacting particles, e. g., the system with Lennard-Jones potential, the usual perturbation procedure cannot be applied, since the Fourier component of potential diverges. In this note, we shall propose one method which makes it possible to treat the singular potential by the perturbation method. Suppose that the Fourier component of such a potential can be expanded as

$$a\nu_1(\boldsymbol{k}) + a^2\nu_2(\boldsymbol{k}) + \cdots, \qquad (1)$$

where a is some expansion parameter peculiar to the potential, in particular the hard-sphere diameter for the hard-sphere potential. This power series should diverge when the sum is carried out to infinite orders. However, if one retains the finite terms, the usual perturbation method can be applied and an exact expression for the energy can

be obtained up to the specified orders. For example, if the first term is retained, the lowest term of order a can be obtained. Hence the problem is reduced to find an exact expression such as (1) for the Fourier component of potential.

By a simple consideration, we obtain an integral equation for t-operator which is equivalent to the potential as follows:

$$t(\alpha\beta, \lambda\gamma) = u(\alpha\beta, \lambda\gamma) - \frac{1}{2V}\sum_{\sigma\tau}\frac{u(\alpha\beta, \sigma\tau)\, t(\sigma\tau, \lambda\gamma)}{k^2 - k_{\sigma\tau}^2}, \qquad (2)$$

$$u(\alpha\beta, \lambda\gamma) = \delta(\alpha\beta, \lambda\gamma) \times \int e^{-i\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{x}}\, v(\boldsymbol{x})\, \psi_{\lambda\tau}(\boldsymbol{x})\, d\boldsymbol{x}, \qquad (3)$$

$$\left\{\nabla^2 + k^2 - \frac{v(\boldsymbol{x})}{2}\right\}\psi_{\lambda\tau}(\boldsymbol{x}) = (k^2 - k_{\lambda\tau}^2)\, e^{i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x}}, \qquad (4)$$

where $t(\alpha\beta, \lambda\gamma)$ is the Fourier component of t-operator, V volume, $k^2 = \mathcal{E} - (\boldsymbol{k}_\lambda + \boldsymbol{k}_\gamma)^2/4$, $(\mathcal{E} = 2mE/\hbar^2 N)$, $v(\boldsymbol{x}) = 2mV(\boldsymbol{x})/\hbar^2$, ($V(\boldsymbol{x})$: potential function), $\boldsymbol{k}_{\alpha\beta} = (\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta)/2$ and $\delta(\alpha\beta, \lambda\gamma)$ represents the momentum conservation. If $v(\boldsymbol{x})$ is regular, (4) can be solved by the perturbation method; then one may verify that the solution of (2) is just the Fourier component of $v(\boldsymbol{x})$. On the other hand, if $v(\boldsymbol{x})$ is singular, (2) is solved by iteration and the resulting equation becomes the power series like (1). It should be noted that $u(\alpha\beta, \lambda\gamma)$ does exist even if the $v(\boldsymbol{x})$ is singular.

It is convenient to use the method of partial waves in the practical calculations. If we confine ourselves to the lowest order term of the ground state energy for the Bose system, we have

$$\frac{E}{N} = \frac{\hbar^2 \rho \pi}{m}\int_0^\infty v(r)\, R(r)\, r^2\, dr \qquad (5)$$

where $R(r)$ is given by

$$\{(1/r^2)\, d/dr\,(r^2 d/dr) - v(r)/2\}\, R(r) = 0, \qquad (6)$$

$$R(r) \to 1, \quad (r \to \infty). \qquad (7)$$

One may easily verify that (5), (6), (7) lead to Lenz's formula[1] for the hard-sphere potential.

For the Lennard-Jones potential such as[2]

$$V(r) = V_0[(\sigma/r)^{10} - (\sigma/r)^6], \qquad (8)$$

we have

$$\frac{E}{N} = 4\pi\frac{2^{3/4}\Gamma(3/4)\,\Gamma(5/8 - \eta)}{\Gamma(1/4)\,\Gamma(3/8 - \eta)} \times m^{-7/8} V_0^{1/8} \sigma^{5/4} \hbar^{7/4} \rho, \qquad (9)$$

where $\eta = \sqrt{mV_0}\,\sigma/8\hbar$. This expression corresponds to Lenz's formula for the hard-sphere potential. The appearance of $V_0^{1/8}$ is interesting, since it denies clearly the use of conventional perturbation method in which E/N is expanded in powers of V_0. A full detail of this article will be published in the near future.

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π-N Interactions in Gev Region

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π-N interactions in an energy region near 1.0 Gev are studied by means of the simple ray optical model. The energy dependences of the optical parameters, the absorption coefficient K and the index of refraction k_1, are determined so as to reproduce the behaviours of the real and imaginary parts of the forward scattering amplitude, which are calculated from the observed total cross sections σ_T by making use of the dispersion relations. For convenience sake, the target radius of the nucleon is taken as 1.3, 1.5, or 1.6×10^{-13}cm and is assumed to have no energy dependence.

The energy dependence of K is similar to that of σ_T, but k_1 displays a very different dependence, which is characterized by the fact that k_1 takes a negative value in an energy region below the energy corresponding to the maximum σ_T. Using the obtained K and k_1, the elastic, inelastic, and exchange scattering cross sections for π-N collisions are calculated. According to our analysis, the choice for the nucleon radius as 1.6×10^{-13}cm seems to be consistent with experimental data. The bump which appears in the backward hemisphere of the angular distribution for the elastically scattered pions at 1.0 Gev, is explained by contributions from the spin flipped scattering in D wave.

§ 1. Introduction

For the interactions between pion and nucleon in Gev region, the recent experimental data indicate the following facts:

1) The total cross section for π^--p collisions has a maximum at an energy of about 1.0 Gev, and for π^+-p interactions such an energy is near 1.3 Gev.[1]

2) For π^--p interactions, the ratio of the elastic to inelastic cross sections $(\sigma_{el}/\sigma_{inel})$ is $\sim$0.7 and $\sim$0.35 at an incident energy 1.0[2] and 1.5 Gev[3] respectively, and for π^+-p interactions, estimating from π^--$(d-p)$ interactions, $\sigma_{el}/\sigma_{inel}\sim0.5$ at 1.37 Gev.[4]

3) In such a high energy region, the center of mass angular distribution of the elastically scattered π^--mesons rises up very strongly in the forward direction and the

existence of a small bump in the backward hemisphere seems to be plausible, especially at an energy 1.0 Gev.[2),3)]

Based on these pieces of experimental information, Takeda[5)] and Dyson[6)] independently suggested an idea of taking account of the π-π interactions to explain the maximum for $\sigma_T(\pi^-p)$. Ito and Minami[7)] analyzed the experiments under an assumption that in such a high energy region only the diffraction scattering is predominant in all elastic processes. This assumption seems to be correct at near or above 1.5 Gev, but in the energy region where σ_T is close to the maximum value, the refractive effects cannot be ignored as implied by the experiments.[2)] Thus it is necessary to reanalyze the experimental data without disregard of the refraction scattering. In the present note, we analyze the above mentioned three pieces of information in terms of the simple ray optical model.

§ 2. Determination of the optical parameters

There arises a question at first, whether the ray optical model, which was proposed by Fernbach et al.[8)] explain the scattering of the high energy neutrons by nucleus, is suitable to describing the π- nucleon collisions, since the boundary of a single nucleon is more diffuse than that of a nucleus. In spite of such a circumstance we tentatively adopt this model to understand the overall behaviours of the interactions.

According to the information given by Walker et al.[2)], the interaction radius of the nucleon is not so much dependent on the incident energies and always stays at equal to or greater than 1.1×10^{-13}cm. Thus assuming that the interaction radius of the nucleon, one of the optical parameters, is constant in the energy region concerned here, we determine the energy dependence of the other two parameters, the absorption coefficient K and the index of refraction k_1.

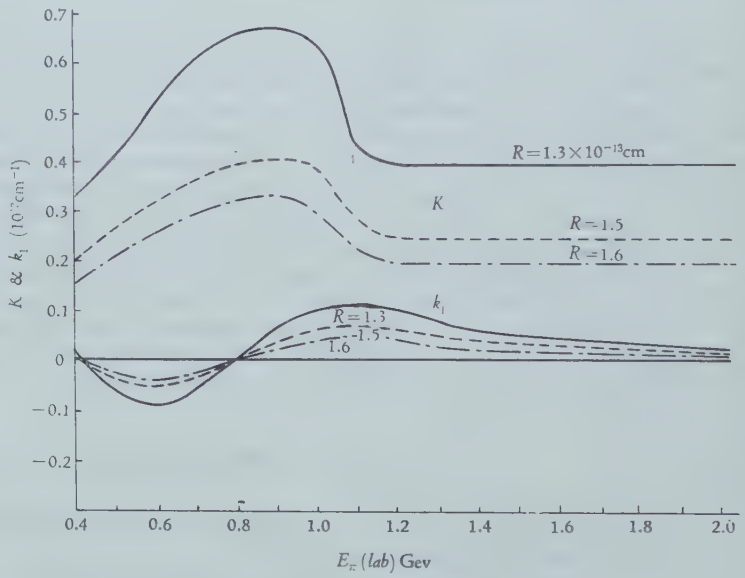

Fig. 1. The energy dependence of the calculated K and k_1 for $(\pi^- - p)$ interactions.

Usually, from the observed forward peak of the angular distribution for the elastic scattering and the measured value of σ_T or $\sigma_{el}/\sigma_{inel}$ the optical parameters are determined. Unfortunatly, in the present case, only the total cross section has been meassured in the whole energy region. Therefore, we determine the parameters so as to fit the curves of the real and imaginary parts of the forward scattering amplitudes which are calculated by

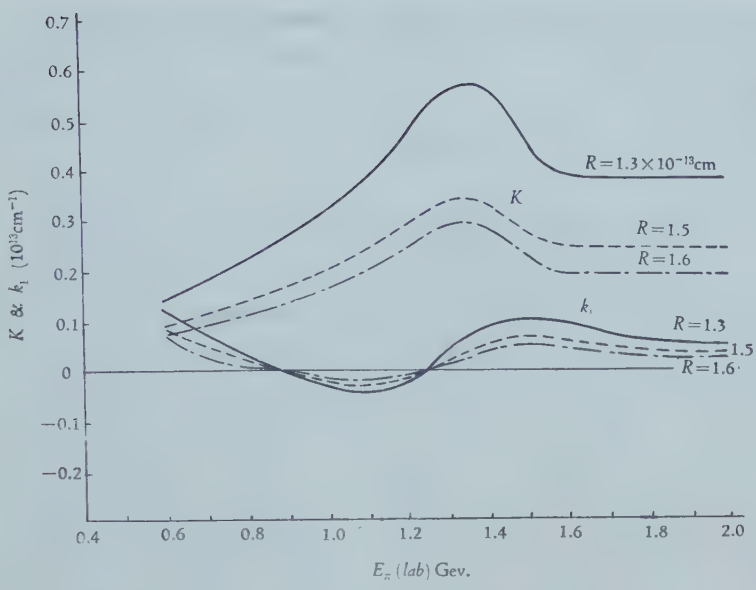

Fig. 2. The energy dependence of the calculated K and k_1 for (π^+-p) interactions.

Sternheimer[1] using the dispersion relations and the observed total cross section.

The energy dependences of K and k_1 obtained in this way are shown in Fig. 1 and Fig. 2 for π^--p interactions and π^+-p interactions respectively, in each case R being taken as 1.3, 1.5, or 1.6×10^{-13}cm following the suggestion by Walker et al. The fact that the values of K are greater than those of k_1 for all the cases indicates that the absorptive interactions play an important role in the energy region. Especially, the very small value of k_1 near and above 1.5 Gev confirms the validity of the assumption that was proposed by Ito and Minami to analyze the experimental data at 1.5 Gev.

As for the energy dependence of K and k_1, it is interesting to observe that, in contrast with the form of K being similar to that of σ_T both in π^--p and π^+-p collisions, the behavior of k_1 is considerably different from that of σ_T; that is, it becomes negative once in an energy region fairly below the energy corresponding to the maximum of σ_T, and reaches its maximum value at an energy slightly above it. These properties of K and k_1 are expected to be valuable tools for investigating pion-nucleon interactions.

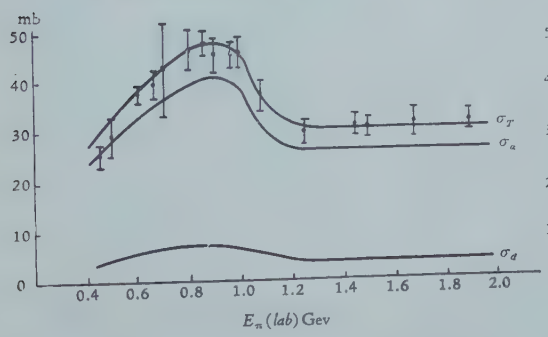

Fig. 3. The calculated (π^--p) total-, absorption- and diffraction cross section for the case of $R=1.6\times10^{-13}$cm.

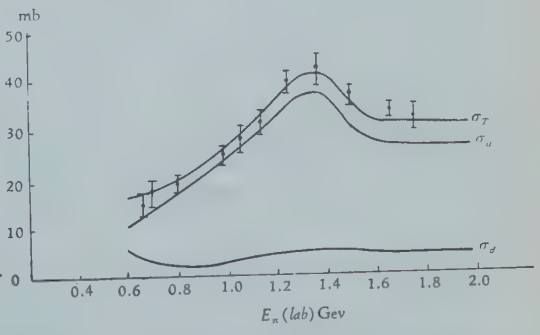

Fig. 4. The calculated (π^+-p) total-, absorption- and diffraction cross section for the case of $R=1.6\times10^{-13}$cm.

Next, we calculate the total cross sections in order to check to what extent σ_T can be reproduced by K and k_1 thus obtained. These results are shown in Fig. 3 and Fig. 4 with the experimental curves.

§ 3. The values of σ_{el} and σ_{inel}

Using the obtained values of K and k_1 and the equations obtained by Fernbach et al.[8], we can calculate the diffraction scattering cross section σ_d and the absorption cross section σ_a. The results are shown in Fig. 3 and Fig. 4. From these figures, we can see again that the absorption process is predominant in such high energy regions. The information from the experiments done in emulsion plates, however, are concerned with the elastic scattering cross section σ_{el} and the inelastic cross section σ_{inel} rather than σ_{el} and σ_a. Therefore, it is not reasonable to compare immediately the calculated σ_d and σ_a with the observed σ_{el} and σ_{inel}, respectively, since a part of the elastic process, the so-called compound elastic process, is included in the absorptive process. In the studies of the nuclear reactions, the separation of the compound elastic cross section σ_{ce} from σ_a is undertaken by the use of the principle of the detailed balance, though it is impossible in the present case. Thus we assume that for the decay of the compound system attained by the absorption of the incident pion beams, the Fermi's statistical theory may be applied. So we estimate σ_{ce} as if various decay processes are proportional to the

Table 1 Cross sections for $(\pi^- - p)$ interactions

	σ_d (mb)	σ_a (mb)	$\sigma_{c\cdot e}$ (mb)	σ_{el} (mb)	σ_{inel} (mb)	σ_{ex} (mb)	$\sigma_{el}/\sigma_{inel}$	*Experimental Values*			
								σ_{el}(mb)	σ_{inel}(mb)	σ_{ex}(mb)	$\sigma_{el}/\sigma_{inel}$
$E_\pi=0.8$ Gev											
$R=1.3\times10^{-13}$cm	9.9	34.1	13.9	23.8	20.2	11.1	1.2				
$R=1.5$	7.6	36.7	13.1	20.7	23.6	10.5	0.9	12 ± 5.4[9]	~30	~16.5	~0.33
$R=1.6$	7.2	38.9	12.9	20.1	26.0	10.3	0.8				
$E_\pi=1.0$ Gev											
$R=1.3\times10^{-13}$cm	11.2	34.5	12.0	23.2	22.5	9.6	1.0	20 ± 3[a]	26 ± 3	~5	0.77
$R=1.5$	8.0	36.7	10.6	18.6	26.1	8.5	0.7	18.6 ± 3[b]	27.4 ± 3	~7	0.68
$R=1.6$	6.4	37.9	10.0	16.3	28.0	8.0	0.6				
$E_\pi=1.2$ Gev											
$R=1.3\times10^{-13}$cm	4.4	24.9	7.4	11.8	17.5	5.9	0.7				
$R=1.5$	4.4	25.8	6.1	10.5	19.6	4.9	0.5				
$R=1.6$	2.9	26.3	5.6	8.5	20.7	4.5	0.4				
$E_\pi=1.5$ Gev											
$R=1.3\times10^{-13}$cm	4.8	25.2	6.0	10.7	19.2	4.8	0.6				
$R=1.5$	4.0	26.2	4.8	8.8	21.5	3.8	0.4	6.8	23.3	~6	0.35
$R=1.6$	3.4	26.8	4.3	7.6	22.6	3.4	0.3				

a) Diffusion chamber data
b) Emulsion data

magnitudes of the phase space volume of the final modes. The $\sigma_{el}(=\sigma_d+\sigma_{ce})$ and $\sigma_{inel}(=\sigma_a-\sigma_{ce})$ obtained by such manner are listed in Table I together with the experimental values. In the same manner, the cross sections for the charge exchange scattering $(\pi^-+p\to\pi^0+n)$, σ_{ex}, are also presented in this Table, in which the observed σ_{ex} are estimated from the data for the other processes using the assumption of charge independence.

From Table I, the target radius of the nucleon 1.6×10^{-13}cm seems to be compatible with the experimental data.

§ 4. Angular distribution of the elastically scattered pions

Adopting the value 1.6×10^{-13}cm for the interaction radius of the target nucleon which gives the ratio $\sigma_{el}/\sigma_{inel}$ consistent with the experimental data, we study the angular distribution of the elastically scattered pions. It is not reasonable to compare the curve $|f(\theta)|^2=d\sigma_d/d\Omega$, represented in ref. 8, with the observed one because σ_{el} is equal to the sum of σ_d and σ_{ce} as mentioned above. So $d\sigma_{ce}/d\Omega$ must be suitably added to $d\sigma_d/d\Omega$. Since the elastic processes through the compound states are incoherent, we may be allowed to superpose the both curves directly.

From the experimental results, the estimated value of the integral cross section for the backward scattering is about 7mb and its distribution curve seems to have a similar form to the Legendre function P_2^1. It suggests that the spin flipped reaction of the D-wave is predominant in this angular region.*) On

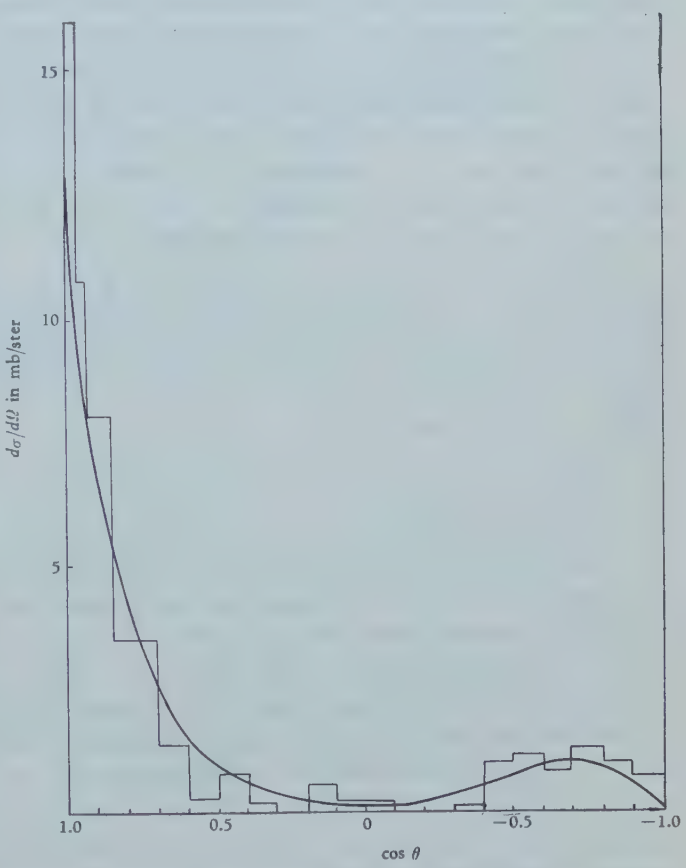

Fig. 5. Center of mass angular distribution of the elastically scattered π^- at 1.0 Gev. The curve represents the result of the calculation using our model, in which $R=1.6\times10^{-13}$cm is assumed.

* This is also suggested by Walker et al.[2]

the one hand, a half of the above calculated value of $\sigma_{c.e}$ is about 5mb for $R=1.6\times10^{-13}$cm. Then assuming the compound elastic process consists of only the D-wave spin flipped process and superposing the contribution from the diffractive parts, the angular distribution is estimated. The final result is shown in Fig. 5. The agreement with the experimental result is fairly good.

§ 5. Discussion

As is shown in previous sections, the several facts may be explained by the simple ray optical model. The main reason why the conventional results were obtained may be due to the fact that we determine the optical parameters by using the forward scattering amplitude, since the forward scattering is the characteristic feature in the optical model. However, in the present studies, there remain many problems which must be investigated more closely; for example, the method for the separation of σ_{ce} from σ_a, or the effects of the final state interactions of pions with nucleon, etc. In these problems the most serious one is the assumption that the absorption of D-wave is predominant in such processes. If we make the more detailed partial wave analysis for the diffraction scattering, one may hope that the "assumption of D-wave process being main" is examined. Though its effect was not considered in the present study, it may be necessary to investigate this effect upon the angular distribution.

Finally we conclude that in these high energy region the absorption is the main process and in which D-wave spin flipped scattering plays an important role, and that the value of the target radius of the nucleon is not so much dependent upon the incident energy and its consistent value is about 1.6×10^{-13}cm as far as we use the ray optical model.

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Parity-Nonconservation and the Spins of the Strange Particles

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Taking into account the possible parity-nonconservation in the weak interactions, the decay angular distributions of the strange particles produced by the pion-nucleon collisions are re-examined. It is shown that, when the parity-nonconservation effect is actually observed in the decay angular distribution of the hyperon, its spin value can be determined less ambiguously than in the case of the parity-conservation. Especially, in this case, it is possible in principle to decide completely whether the spin is 1/2 or not. The effect of the parity-nonconservation can appear also in the $K_{\mu 2}$- and $K_{\pi 3}$-modes of decay, unless the spin of K is zero. Taking this into account, also the possibility of determining the spin value of the K-meson is considered. In this case, however, the conclusion is not so decisive as the case of the hyperon.

§ 1. Introduction

Parity-nonconservation in the weak decay interaction seems to be an experimentally established fact for the decays accompanied by the neutrino.[1] However, this is not yet certain for other decay processes,[*] and accordingly the test on this point is a very important problem. Recently, Lee et al.[2] have shown a possible method of this test for the case of the decay of a hyperon into a nucleon and a pion, assuming that the spins of the hyperon and the K-meson are 1/2 and 0, respectively. Their arguments are essentially based on the fact that the produced particles in the pion-nucleon collisions are generally polarized along the direction perpendicular to their production plane. However, this fact holds independently of the spins of the produced particles, and thus the method of Lee et al. is applicable to any value of the hyperon spin. Moreover, it can be shown that the parity-test is possible (in principle) also for the $(\mu+\nu)$- and the 3π-decays of the K-meson, provided that the spin of K is not zero. What is needed for all of these conclusions is that the "parent" particle is sufficiently polarized.

Conversely, the existence of such effects of parity-nonconservation can be taken as a direct evidence for the sufficient polarization of the "parent" particle.

This is very useful for the practical determination of the spins of the strange particles. When the particle in question is sufficiently polarized, it would be possible to know its spin value from the angular correlation effects in its decay process. Along such a program, the method for the spin determination has already been considered by several authors[3),5)]. However, since the parity-conservation were regarded as an indisputable principle in those days, all of these authors have made use of this principle, and thus they have mentioned nothing of the method for judging whether the "parent" particle is polarized or not.

For example, when the parity conserves in the decay interaction, the spherically symmetric angular distribution of $Y \to N + \pi$ (in the rest system of Y) teaches us nothing on the spin value of Y: We cannot decide whether the spin of Y is $1/2$ or it is not $1/2$ but eventually unpolarized. However, we know nowadays that the parity-conservation law would possibly be violated in the weak decay interactions, especially in the one accompanied by the neutrino. It will be shown in the present paper that, making use of the angular correlation effects of parity-nonconservation, we have a possibility of distinguishing between the above-mentioned two cases, and thus nowadays we would be able to determine the spins of the strange particles more definitely than pointed out in the earlier papers.

Adair[4] has proposed a method for the spin determination, which is independent of the polarization of the "parent" particle and accordingly free from the above-mentioned defects. However, his method consists of examining the decay angular distribution of the hyperon such that it is produced by π-N collision into the forward or backward direction and the associated K-meson decays into two pions which have the relative momentum parallel to the direction of the incident pion beam. (If the spin of the K-meson be surely zero, then all the decay events of Y produced into the forward and backward directions are available.) Such a restriction of the events would decrease the abundance of the available data considerably. Thus, the method explained in the present paper would be worthy enough to be attempted. Moreover, we wish to emphasize here that our method is advantageous in that the determination of the spin and the test of the parity-nonconservation are simultaneously carried out.

Of the K-meson decays, we shall here treat only the modes of relatively abundant, namely $K_{\pi 2}$, $K_{\mu 2}$, and $K_{\pi 3}$. For this case, however, the conclusions are somewhat more ambiguous than in the case of the hyperon. This is mainly owing to the fact that the effects of the parity-nonconservation can occur only when the K-meson has a non-zero spin. Therefore, we cannot decide whether the K-spin is zero or not zero but eventually unpolarized.

Throughout the present paper, we do not take into account the possibility of the parity-doublet,[7] and postulate the following assumptions:

(1) The strict conservation of the total angular momentum.

(2) The conservation law of the parity holds for the production process (strong reaction), but may be violated in the decay process (weak reaction).

(3) The K-mesons are all the same particles except for their charge states.

The last assumption would be natural as the first step to the determination of the spin value of the K-meson, and seems to be plausible from our knowledge thus far obtained on this particle[6]. However, as occasion demands, it must be remembered that this assumption might be wrong.

§ 2. The hyperon decay, $Y \to N + \pi$

The decay of the hyperon

$$Y \to N + \pi \qquad (2 \cdot 1)$$

has already been investigated by many authors. Especially, in ref. 3) and 5), the explicit formulas of the decay angular distribution for several spin values are derived under the assumption of the parity-conservation. On the other hand, the angular distribution for the spin 1/2 hyperon is derived in ref. 2) for the case of possible violation of the parity law. For completeness, however, we shall here derive full expression for the decay angular distribution taking into account the possible parity-nonconservation.

As the preliminary step, we consider the process (2·1) in the rest system of the "parent" particle Y, and take an arbitrarily fixed direction as the z-axis in this frame. We assume here the conservation of the total angular momentum, but can no longer assume that of the parity. The derivation of the decay angular distribution is quite straightforward.

When the "parent" particle Y is in the spin state described by its spin function $\lambda(s_z)$, the wave function of the decay state has the form

$$g(s_z)=a\varphi_{s-1/2}(s,\ s_z)+b\varphi_{s+1/2}(s,\ s_z), \tag{2·2}$$

where s_z means the eigenvalue of the z-component of the hyperon spin in the state in question, and s is the magnitude of the spin of Y. $\varphi_l(s,\ s_z)$ denotes the simultaneous eigenfunction of the total angular momentum, its z-component, and the relative orbital angular momentum of the final $(N+\pi)$ system, belonging to the respective eigenvalues s, s_z and l. We have denoted the partial amplitudes of the orbital states $(s-1/2)$ and $(s+1/2)$ by a and b, respectively. Thus, either one of a and b is the parity-conserving amplitude, and the other the parity-reversing one. The eigenfunction φ can be represented as the linear combination of the products of the nucleon spin functions α("up") or β("down") and the eigenfunctions $Y(l,\ m)$ of the relative orbital angular momentum of the $(N+\pi)$ system:

$$\begin{aligned} g(s_z)=&\left[\{(s+s_z)/(2s)\}^{1/2}\cdot aY(s-1/2,\ s_z-1/2)\right.\\ &\left.-\{(s-s_z+1)/(2s+2)\}^{1/2}\cdot bY(s+1/2,\ s_z-1/2)\right]\alpha\\ &+\left[\{(s-s_z)/(2s)\}^{1/2}\cdot aY(s-1/2,\ s_z+1/2)\right.\\ &\left.+\{(s+s_z+1)/(2s+2)\}^{1/2}\cdot bY(s+1/2,\ s_z+1/2)\right]\beta. \end{aligned} \tag{2·3}$$

The angular distribution of the decay products is easily calculated by (2·3), and is given by, for $s_z>0$,

$$\begin{aligned} (1/2)\,d\sigma(s_z)/d(\cos\theta)=&\left[(s-s_z)!/(s+s_z)!\right]\\ &\times\left[|a|^2\{(s+s_z)^2[P(s-1/2,\ s_z-1/2)]^2+[P(s-1/2,\ s_z+1/2)]^2\}\right.\\ &+|b|^2\{(s-s_z+1)^2[P(s+1/2,\ s_z-1/2)]^2+[P(s+1/2,\ s_z+1/2)]^2\}\\ &+2\,\mathrm{Re}\,(a^*b)\{P(s-1/2,\ s_z+1/2)P(s+1/2,\ s_z+1/2)\\ &\left.-(s+s_z)(s-s_z+1)P(s-1/2,\ s_z-1/2)P(s+1/2,\ s_z-1/2)\}\right], \end{aligned} \tag{2·4}$$

where θ is the angle between the z-axis and the relative momentum of the decay products, and $P(l,\ m)$ means Legendre's associated polynomial of degree l and order m with respect

to the argument $\cos\theta$. Since both of s and s_z are half-integers, the coefficient of $2\mathrm{Re}(a^*b)$ in (2·4) is an odd function of $\cos\theta$ and those of $|a|^2$ and $|b|^2$ are the even functions of $\cos\theta$. Thus, as has been pointed out in ref. 2), the existence of the part of the angular distribution which is antisymmetric with respect to $\theta=90^\circ$ is taken as the evidence of the parity-nonconservation (precisely, parity-mixing) in this decay process. Moreover, it is obvious by (2·3) that, when the decay plane is taken to be x-z plane, the replacements $b\to -b$, $\alpha\to -\beta$, and $\beta\to\alpha$ in $g(s_z)$ yield the function $g(-s_z)$ except for the irrelevant sign factor. Thus, $d\sigma(-s_z)/d(\cos\theta)$ is obtained from $d\sigma(s_z)/d(\cos\theta)$ only by replacing b by $(-b)$. Therefore, the effect of the parity-mixing can be detected only when the expectation value of the spin of the "parent" Y has non-vanishing z-component.

We list the explicit forms of the angular distributions for several s- and s_z-values in Table I.

Table I. Angular distributions of $Y\to N+\pi$

s	s_z	$(1/2)\,d\sigma(s_z)/d(\cos\theta)$
1/2	$\pm 1/2$	$\lvert a\rvert^2+\lvert b\rvert^2\mp 2\mathrm{Re}(a^*b)\cos\theta$
3/2	$\pm 3/2$	$(3/2)\sin^2\theta\,[\lvert a\rvert^2+\lvert b\rvert^2\mp 2\mathrm{Re}(a^*b)\cos\theta]$
	$\pm 1/2$	$(1/2)\,[(\lvert a\rvert^2+\lvert b\rvert^2)(1+3\cos^2\theta)\pm 2\mathrm{Re}(a^*b)\cos\theta(5-9\cos^2\theta)]$
5/2	$\pm 5/2$	$(15/8)\sin^4\theta\,[\lvert a\rvert^2+\lvert b\rvert^2\mp 2\mathrm{Re}(a^*b)\cos\theta]$
	$\pm 3/2$	$(3/8)\sin^2\theta\,[(\lvert a\rvert^2+\lvert b\rvert^2)(1+15\cos^2\theta)\pm 2\mathrm{Re}(a^*b)\cos\theta(9-25\cos^2\theta)]$
	$\pm 1/2$	$(3/4)\,[(\lvert a\rvert^2+\lvert b\rvert^2)(1-2\cos^2\theta+5\cos^4\theta)\mp 2\mathrm{Re}(a^*b)\cos\theta(5-26\cos^2\theta+25\cos^4\theta)]$

In so far as it concerns the angular distribution defined above, the interference effects between the states with different s_z-values do not occur, so that, even if the state of the "parent" Y consisted of a certain superposition of the several spin states, resultant angular distribution could be obtained only by averaging the expressions corresponding to those s_z-values with the suitable weights (which are related to the degree of the polarization of the "parent" particle). It is obvious that, when the state of the "parent" Y is "unpolarized"*, the angular distribution of the decay products becomes necessarily spherically symmetric.

Now, we shall return to the original problem of the production and the subsequent decay of Y. It is wellknown that, in the center-of-mass frame of the production process, the expectation value of the spin of the particle (with non-zero spin) produced by the pion-nucleon collision can have non-vanishing component only along the direction perpendicular to the production plane. Even in the rest system of the produced particle, the situation is not much altered: The spin component along the direction perpendicular

* It should be noted that the vanishing of the expectation value of the spin does not necessarily mean that the particle is "unpolarized". However, when the particle is unpolarized, the expectation values of its spin components are always zero.

to the production plane can generally have a non-vanishing expectation value. In any case, this expectation value depends strongly on the production angle θ and the incident pion energy E in the production process. Therefore, it would be rather improbable that this expectation value vanishes identically over the considerably wide region of these parameters θ and E.

The foregoing analysis shows that the method of Lee et al.[2] for the parity-test can be applied independently of the spin value of the hyperon. Let us observe the chain of the processes

$$\pi+N\rightarrow Y+K, \quad Y\rightarrow N+\pi.$$

If the distribution of one of the final decay products is asymmetric with respect to the production plane of Y, then we can surely conclude that the parity is "mixed" in this decay process, and that the "parent" Y is certainly polarized.

Moreover, we may also obtain the information on the spin value of Y by expanding the observed decay angular distribution (in the rest system of Y) into the polynomial of $\cos\theta$ and comparing it with the expressions (2·4) or Table I. These show that there exists a certain correspondence between the above-mentioned polynomial and the spin value of Y. Alternatively, it is also good to examine the probability density $W(\theta, \tilde{\xi})$ defined in ref. 2). Following Lee et al., we denote the component, along the direction normal to the production plane of Y, of the momentum of one of the decay products (say, π) by R, and define the quantity $\tilde{\xi}$ by

$$\tilde{\xi}=R/(\text{maximum value of } R).$$

Then, $W(\theta, \tilde{\xi})\,d\tilde{\xi}\,d(\cos\theta)$ means the probability for finding the values of $\tilde{\xi}$ between $\tilde{\xi}$ and $\tilde{\xi}+d\tilde{\xi}$ in the decay of Y which is produced into the direction specified by the θ-values between θ and $\theta+d\theta$. In the following, we shall refer to $W(\theta, \tilde{\xi})$ simply as the "$\tilde{\xi}$-distribution". The correspondence between the form of $W(\theta, \tilde{\xi})$ and the spin value of Y is shown in Table II.

Table II. The form of $W(\theta, \xi)$ for the decay of Y

s	$W(\theta, \xi)$
1/2	$A_0(\theta)+A_1(\theta)\xi$
3/2	$B_0(\theta)+B_1(\theta)\xi+B_2(\theta)\xi^2+B_3(\theta)\xi^3$
5/2	$C_0(\theta)+C_1(\theta)\xi+\cdots+C_4(\theta)\xi^4+C_5(\theta)\xi^5$

The coefficients A's, B's and C's in Table II are determined by the degree of the polarization of the "parent" Y, and accordingly depend strongly on the production angle θ of Y and the incident pion energy E in the production process. There might eventually happen the case where the degree of the polarization of the produced Y makes unfortunately some of the coefficients (especially that of the highest even power term) vanishingly small. On such a reason, it has been believed that we could know only the lower limit of the spin value from the decay angular distribution.[3] This is true, when the parity conserves in the decay interactions. In this case, for example, $W(\theta, \tilde{\xi})$ is $\tilde{\xi}$-independent for both of two cases where the spin of Y is 1/2 and where the spin of Y is not 1/2 but it is unpolarized, and accordingly we can conclude nothing on the spin value of Y from the $\tilde{\xi}$-independent $W(\theta, \tilde{\xi})$. However, the situation

is very different for the case of the parity-nonconservation : $W(\theta, \hat{\xi})$ can be $\hat{\xi}$-dependent even if the spin of Y is equal to $1/2$.

Conversely, if $W(\theta, \hat{\xi})$ is $\hat{\xi}$-dependent, then it turns out that the "parent" Y is certainly polarized. In this case, we would be able to improve the information on the spin value of Y by repeating the measurements for various values of θ and E, since the degree of polarization of Y depends strongly on these two parameters.

Moreover, as can be seen from Table I, it is quite improbable in such a case that both of the coefficients of highest even power and highest odd power terms of $W(\theta, \hat{\xi})$ vanish simultaneously all over a certain finite region of the parameters θ and E. Particularly, for example, if the spin of Y were equal to $3/2$, then at least either one of the coefficients $B_2(\theta)$ and $B_3(\theta)$ in Table II would not vanish, in so far as the parity-mixing effect can be observed.

Thus, we can conclude as follows : When the effect of parity-mixing is actually observed, we shall be able to determine the spin value of Y, more definitely than pointed out in the earlier papers, by observing the form of $W(\theta, \hat{\xi})$. Particularly, in this case, we can surely judge whether the spin of Y is equal to $1/2$ or not.

Finally, we can relate the amplitudes a and b in Table I with the pion-nucleon scattering phase shifts and estimate the order of magnitude of the quantity $2\mathrm{Re}(a^*b)/(|a|^2+|b|^2)$ from the experimental data of the π-N phase shifts. Such considerations have already been done in ref. 2) and also by Minami and Yamaguchi[8], in connection with the problem of the charge-conjugation and the time-reversal invariances. However, since the procedure is quite same as those of the above-quoted papers, and the data for the phase shifts of the arbitrary (high) total angular momentum states are not yet known, we shall not touch on this problem here.

§ 3. The K-meson

The method in the preceding section can be used also for the determination of the spin value of the K-meson. For this case, however, the conclusions are not so decisive as the case of the hyperon. This is mainly owing to that we cannot in principle distinguish between the completely unpolarized boson and the one with 0-spin.

In this section, we shall first briefly summarize the distributions of the decay products in the relatively abundant modes of the K-meson decay, namely $K_{\pi 2}$, $K_{\mu 2}$, and $K_{\pi 3}$, and finally consider the possibility of the determination of the spin value of K.

(1) *Angular distribution of the $K_{\pi 2}$-decay*

The magnitude of the orbital angular momentum of the final 2π system (in the rest frame of the "parent" K) is uniquely determined by the spin value of K. Thus, the decay angular distribution does not contain the "parity-mixing" part and is quite same as the case of the parity-conservation. The angular distributions in the rest system of K for various spin states and the form of the $\hat{\xi}$-distribution of one of the final two pions in the chain of reactions $(\pi+N\to Y+K,\ K\to 2\pi)$ are given in Table III. Here,

t is the spin value of the K-meson and the meanings of the other notations are similar to those of § 2.

Table III. Angular and ξ-distributions for $K \to 2\pi$

t	t_z	$d\sigma(t_z)/d(\cos\Theta)$	$W(\theta, \xi)$
0	0	Θ-independent	ξ-independent
1	± 1	$(1/2)\sin^2\Theta$	$A_1'(\theta)+B_1'(\theta)\xi^2$
	0	$\cos^2\Theta$	
2	± 2	$(3/8)\sin^4\Theta$	$A_2'(\theta)+B_2'(\theta)\xi^2+C_2'(\theta)\xi^4$
	± 1	$(3/2)\sin^2\Theta\cos^2\Theta$	
	0	$(1/4)(1-6\cos^2\Theta+9\cos^4\Theta)$	

(2) *Angular distribution of the $K_{\mu 2}$-decay*

If we denote the eigenfunctions of the relative orbital angular momentum and the sum of the spin operators of the final $(\mu+\nu)$ system by $Y(l, l_z)$ and $X(j, j_z)$, respectively, then the wave function of the final state in the rest frame of K can be written as

$$f(t_z)=\sum a(j, l)\, C_{jl}(t, t_z; m, t_z-m)\, X(j, m)\, Y(l, t_z-m), \qquad (3\cdot 1)$$

where argument t_z of $f(t_z)$ denotes the spin state of the "parent" K, and C's are the Clebsch-Gordan coefficients.

(3·1) leads to the angular distribution

$$d\sigma(t_z)/d(\cos\Theta) \sim \sum' a^*(j, l)\, a(j, l')\, C_{jl}(t, t_z; m, t_z-m) \times C_{jl'}(t, t_z; m, t_z-m)\, \mathcal{Y}(l, t_z-m)\, \mathcal{Y}(l', t_z-m), \qquad (3\cdot 2)$$

where $\mathcal{Y}$ is the function of $\cos\Theta$ obtained by setting the azimuthal angle Φ as 0 in the function Y and $\sum'$ means the summation over all the possible sets of values (l, l', m) for $j=1$ and 0. (3·2) is derived basing on the ordinary 4-component theory of the neutrino. However, our essential conclusions hold also for the case of the 2-component neutrino.[9]

(3·2) is asymmetric with respect to $\Theta=90^\circ$ only when the contribution of the terms with $(l+l')=$odd does not vanish. The latter condition can be fulfilled only when $t_z \neq 0$ and the parity is mixed in this decay process. On the other hand, making use of the property of the Clebsch-Gordan coefficients

$$C_{jl}(J, -M; m, m') = (-)^{j+l+J+2M} C_{jl}(J, M; -m, -m'), \qquad (3\cdot 3)$$

it can be shown that $d\sigma(-t_z)/d(\cos\Theta)$ is obtained only by reversing the signs of the terms with $(l+l')=$odd in the expression (3·2) for $d\sigma(t_z)/d(\cos\Theta)$. Thus, when the angular distribution is asymmetric with respect to $\Theta=90^\circ$, it turns out that the parity is mixed in this decay process, and the spin of the "parent" K is certainly

non-zero and sufficiently polarized.

Explicit forms of the angular distribution for several t- and t_z-values and the form of the ξ-distribution of the final μ in the process $(\pi+N\to Y+K,\ K\to\mu+\nu)$ are given in Table IV. Here, the coefficients $A(t, t_z)$, $B(t, t_z)$, and so on, are the bilinear combinations of the partial amplitudes $a(j, l)$'s, whose explicit expressions will be given in Appendix.*

Table IV. Angular and ξ-distribution for $K_{\mu 2}$

t	t_z	$\sim d\sigma(t_z)/d(\cos\Theta)$	$W(\theta, \xi)$
0	0	$\lvert a(0,0)\rvert^2+\lvert a(1,1)\rvert^2$	ξ-independent
1	±1	$A(1,1)+B(1,1)\cos^2\Theta\mp A'(1,1)\cos\Theta$	$A_1(\theta)+B_1(\theta)\xi+C_1(\theta)\xi^2$
	0	$A(1,0)+B(1,0)\cos^2\Theta$	
2	±2	$A(2,2)+B(2,2)\cos^2\Theta+C(2,2)\cos^4\Theta\mp A'(2,2)\cos\Theta(1-\cos^2\Theta)$	$A_2(\theta)+B_2(\theta)\xi+C_2(\theta)\xi^2$ $+D_2(\theta)\xi^3+E_2(\theta)\xi^4$
	±1	$A(2,1)+B(2,1)\cos^2\Theta+C(2,1)\cos^4\Theta\pm A'(2,1)\cos\Theta(1-2\cos^2\Theta)$	
	0	$A(2,0)+B(2,0)\cos^2\Theta+C(2,0)\cos^4\Theta$	

(3) *Parity-nonconservation in the $K_{\pi 3}$-decay*

Parity-test in the $K_{\pi 3}$-decay is very interesting in connection with the so-called τ-θ dilemma. However, the parity-mixing part of the decay angular distribution can occur only in the case of the K-meson with non-zero spin, since, when the K-spin is zero, the parity of the final 3π system is necessarily odd.

Let us observe, for example, the distribution of one particularpion, which is distinguishable from the other two pions, in the 3π-decay of the K-meson produced by the π-N collision. When this distribution is asymmetric with respect to the production plane of K, it can be shown by the similar arguments as those used thus far that the parity does not conserve in this decay process and the spin of the "parent" K is certainly non-zero and sufficiently polarized.

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* Expressions given in Table IV and in Appendix are formally valid also for the case of the two-component neutrino. This is obvious, unless the conservation law of the lepton number is imposed. Moreover, within the frame of pure kinematics, this conservation law affects our formulas only through the following additional requirements:

$$a(1,1)\pm a(0,0)=0, \qquad \text{when } t=0,$$

$$\left.\begin{aligned} a(1,t+1)&=\pm[\sqrt{t}\,a(1,t)-\sqrt{t+1}\,a(0,t)]/\sqrt{2t+1}\\ a(1,t-1)&=\pm[\sqrt{t+1}\,a(1,t)+\sqrt{t}\,a(0,t)]/\sqrt{2t+1}\end{aligned}\right\}, \qquad \text{when } t\neq 0,$$

where the sign $(\pm)$ means $(+)$ for the neutrino process, and $(-)$ for the anti-neutrino process.

Thus, in any case, our conclusions are quite unaltered.

In addition to this, it should be noted that, in the case of the two-component neutrino, the problem of the parity, itself, becomes meaningless.

(4) *Determination of the spin value of the K-meson*

In so far as we restrict ourselves only to the $K_{\pi 2}$-decay, the parity-nonconservation does not cause anything new on the spin determination.

However, we want to notice here the following fact: If any ξ-dependence of $W(\theta, \xi)$ for $K_{\pi 2}$ or $K_{\mu 2}$, or the parity-mixing effect in $K_{\pi 3}$ are observed, we can conclude that the spin of K is not zero, and the experimental condition (the values of θ and E in the production process) of this case is just adequate for the production of sufficiently polarized K-mesons. Therefore, in such a case, there would be the possibility to improve the knowledge on the spin value by repeating the measurements of the ξ-distribution of $K_{\pi 2}$ for various values of θ and E near those of the above experiment.

On the other hand, when any effect of the parity-mixing is not found in $K_{\pi 3}$, and $W(\theta, \xi)$ are ξ-independent for both of $K_{\pi 2}$ and $K_{\mu 2}$, we cannot draw any definite conclusion on the spin value of K: We cannot decide whether the K-meson is a spinless particle or it has non-zero spin but completely unpolarized. Of course, the wider the region of θ and E where such a situation is observed, the more the possibility of K being spinless will become plausible. However, in such a case, it would be better to use the method of Adair[4].

Finally, we wish to add a remark on the $K_{\mu 2}$-decay. Since, in the rest system of K, the $K_{\mu 2}$-decay takes place on a sharply defined energy shell, each of the partial amplitudes $a(j, l)$'s has a certain fixed value. Thus, as will be seen from Appendix, the possibility may occur that these values of the amplitudes make some of the coefficients $A(t, t_z)$, $B(t, t_z)$, and so on, identically vanish. This can occur quite independently of the production process of K, and accordingly cannot be avoided by adjusting the experimental condition (θ- and E-values). In this respect, the situation is worse in $K_{\mu 2}$ than in $K_{\pi 2}$, although the former is the most abundant mode of K-decay. Thus, in the case where the parity-mixing effect cannot be observed, it is more convenient for the spin determination to use the $K_{\pi 2}$-mode. However, this is not necessarily the case, when the parity-mixing effect is actually observed in $K_{\mu 2}$. For example, let us consider the case where the spin of K is 2. The fact that the parity-mixing effect can be observed in $K_{\mu 2}$ means that $A'(2, 2) = A'(2, 1) \neq 0$ (see Table IV and Appendix), and thus, $W(\theta, \xi)$ would contain the term of ξ^3 for the suitable values of θ and E, which is quite in contrast with the case of the K-spin being 1. Therefore, in such a case, also the $K_{\mu 2}$-mode can be of service to the spin determination to some extent.

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Appendix. The expressions for the coefficients in Table IV

$$A(1, 1) = |a(1, 0)|^2 + (3/4)|a(1, 1)|^2 + (5/4)|a(1, 2)|^2 + (3/2)|a(0, 1)|^2 - (\sqrt{2}/2)\mathrm{Re}[a^*(1, 0)\, a(1, 2)],$$

$$B(1,1)=(3/4)\{|a(1,1)|^2-|a(1,2)|^2-2|a(0,1)|^2+2\sqrt{2}\,\mathrm{Re}[a^*(1,0)a(1,2)]\},$$

$$A'(1,1)=\mathrm{Re}[a^*(1,1)\{\sqrt{3}\,a(1,2)+\sqrt{2}\,a(1,0)\}],$$

$$B(1,0)+2B(1,1)=0,$$

$$A(1,0)+2A(1,1)=3\{|a(1,0)|^2+|a(1,1)|^2+|a(1,2)|^2+|a(0,1)|^2\};$$

$$A(2,2)=(3/2)|a(1,1)|^2+(5/4)|a(1,2)|^2+(13/8)|a(1,3)|^2$$
$$+(15/8)|a(0,2)|^2-(\sqrt{6}/4)\mathrm{Re}[a^*(1,1)a(1,3)],$$

$$B(2,2)=-(3/2)|a(1,1)|^2-(9/4)|a(1,3)|^2-(15/4)|a(0,2)|^2$$
$$+(3/2)\sqrt{6}\,\mathrm{Re}[a^*(1,1)a(1,3)],$$

$$C(2,2)=-(5/4)|a(1,2)|^2+(5/8)|a(1,3)|^2+(15/8)|a(0,2)|^2$$
$$-(5/4)\sqrt{6}\,\mathrm{Re}[a^*(1,1)a(1,3)],$$

$$A'(2,2)=\mathrm{Re}[a^*(1,2)\{\sqrt{15}a(1,1)+\sqrt{10}a(1,3)\}],$$

$$A(2,1)=(3/4)|a(1,1)|^2+(5/4)|a(1,2)|^2+(1/2)|a(1,3)|^2$$
$$+(1/2)\sqrt{6}\,\mathrm{Re}[a^*(1,1)a(1,3)],$$

$$B(2,1)=(3/4)|a(1,1)|^2-(15/4)|a(1,2)|^2+3|a(1,3)|^2$$
$$+(15/2)|a(0,2)|^2-(9/2)\sqrt{6}\,\mathrm{Re}[a^*(1,1)a(1,3)],$$

$$C(2,1)=-4C(2,2),\qquad A'(2,1)=A'(2,2),$$

$$2A(2,2)+2A(2,1)+A(2,0)=5\{|a(1,1)|^2+|a(1,2)|^2+|a(1,3)|^2+|a(0,2)|^2\},$$

$$2B(2,2)+2B(2,1)+B(2,0)=2C(2,2)+2C(2,1)+C(2,0)=0.$$

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Separable Dynamical Systems of Staeckel in Flat Space

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To help a study in orbits of a charged particle of high energy in electromagnetic fields, we determine coordinate systems and respective potential functions where the Hamilton-Jacobi equation of the particle is integrable by separation of variables.

§ 1. Canonical forms of h_i's

This paper is an appendix to our preceding paper[1] that has given a list of static electromagnetic fields where the Hamilton-Jacobi equation of an electron is integrable by separation of variables. In a Minkowski space the signature of ds^2 is $+++-$. But the signature has nothing to do with the separability of a dynamical system. So we shall determine separable dynamical systems in a euclidian space of 4 dimensions where the line element is given in a form $ds^2=\sum h_i dx_i^2$. Since the calculation goes quite similarly to that in Eisenhart's paper or in our preceding paper, the results only are listed.

There are eighteen canonical forms of the h_i's. The variables are not restricted to be real.

When the h_i's depend only on one variable

(A) $h_1=1,\ h_2=\sigma_1,\ h_3=\mu_1,\ h_4=\tau_1$

When the h_i's depend on two variables

(B) $h_1=1,\ h_2=\sigma_1,\ h_3=\sigma_1\sigma_2,\ h_4=\tau_1$

(C) $h_1=1,\ h_2=\sigma_1,\ h_3=\sigma_1\sigma_2,\ h_4=\sigma_1\tau_2$

(D) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=h_4=1$

(E) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=\sigma_1\sigma_2,\ h_4=1$

(F) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=\sigma_1\sigma_2,\ h_4=(\sigma_1+1)(1-\sigma_2)$

When the h_i's depend on three variables

(G) $h_1=1,\ h_2=\sigma_1,\ h_3=\sigma_1\sigma_2,\ h_4=\sigma_1\sigma_2\sigma_3$

(H) $h_1=1,\ h_2=h_3=\sigma_1(\sigma_2+\sigma_3),\ h_4=\tau_1$

(I) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=1,\ h_4=\sigma_3$

(J) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=\sigma_1\sigma_2,\ h_4=\sigma_1\sigma_2\sigma_3$

(K) $h_1=1,\ h_2=h_3=\sigma_1(\sigma_2+\sigma_3),\ h_4=\sigma_1\sigma_2\sigma_3$

(L) $h_i=\Pi(\sigma_i-\sigma_j),\ i=1,\ 2,\ 3,\ j\neq i,\ h_4=1$

(M) $h_i=\Pi(\sigma_i-\sigma_j),\ i=1,\ 2,\ 3,\ j\neq i,\ h_4=\sigma_1\sigma_2\sigma_3$

When the h_i's depend on four variables

(N) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=h_4=\sigma_3+\sigma_4$

(O) $h_1=1,\ h_2=\sigma_1,\ h_3=h_4=\sigma_1\sigma_2(\sigma_3+\sigma_4)$

(P) $h_1=h_2=\sigma_1+\sigma_2,\ h_3=h_4=\sigma_1\sigma_2(\sigma_3+\sigma_4)$

(Q) $h_i=\sigma_4\Pi(\sigma_i-\sigma_j),\ i=1,\ 2,\ 3,\ j\neq i,\ h_4=1$

(R) $h_i=\Pi(\sigma_i-\sigma_j),\ i=1,\ 2,\ 3,\ 4,\ j\neq i$

where $\sigma_1=\sigma_1(x_1),\ \sigma_2=\sigma_2(x_2)$, etc.

§ 2. Method of special values

The Hamiltonian of an electron is written as

$$H=\sum(p_i-b_i)^2/h_i,$$

p_i being the momentum conjugate to x_i. The condition of Levi-Civita gives a set of equations quadratic in p_i's, of which coefficients of the highest terms in p_i's are independent of b_i's, so that the h_i's can be determined independently of b_i's. The h_i's thus determined are listed in § 1. The b_i's may be determined from the vanishing of other terms, but as the procedure is very cumbersome, we take another approach.

When the Hamilton-Jacobi equation

$$\sum(\partial S/\partial x_i-b_i)^2/h_i=h=\text{const.}$$

admits a complete integral

$$S=S_1(x_1)+S_2(x_2)+S_3(x_3)+S_4(x_4),$$

we have an equation

$$\sum'(p_i-b_i)^2/h_i=h,\ p_i=dS_i/dx_i.$$

If we put $x_i=a_i$ (=const), we have a condition

$$\sum(\lambda_i-b_{io})^2/h_{io}=h$$

where $\lambda_i=p_{io}$, and the subscript o denotes the values at $x_i=a_i$. Next, we put $x_j=a_j$, $j\neq 1$, then we get

$$(p_1-\beta_1)^2=\psi_1(x_1,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4),$$

ψ_1 being quadratic in λ_i's and $\beta_1=b_1(x_1,\ a_2,\ a_3,\ a_4)$. Similarly we have

$$(p_j-\beta_j)^2=\psi_j(x_j,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4),$$

$$j=2,\ 3,\ 4.$$

and

$$p_j=\beta_j+\sqrt{\psi_j}.$$

Inserting these equations in the Hamilton-Jacobi equation, we have an equation

$$\sum'(b_i-\beta_i-\sqrt{\psi_i})^2/h_i=\sum(\lambda_i-b_{io})^2/h_{io}$$

which is to hold independently of λ_i's. This equation is rewritten as

$$\sum (b_i-\beta_i)\sqrt{\psi_i}/h_i=F,$$

F being a polynomial in λ_i's. In order that the left member which involves radicals $\sqrt{\psi_i}$ be a polynomial in λ_i's, each of $(b_i-\beta_i)\sqrt{\psi_i}$ must be a polynomial, so that either $b_i-\beta_i$ vanishes or ψ_i is a perfect square. Hence we have four cases.

I) All ψ_i's are perfect squares. In this case we have

$$\left.\begin{array}{l} h_i=\text{a function of } x_i \text{ only} \\ b_i= \quad ,, \\ \text{or} \quad =0 \end{array}\right\} \; i=1,\ 2,\ 3,\ 4$$

by virtue of a guage transformation.

II) ψ_2, ψ_3, ψ_4 are perfect squares, $b_1=\beta_1$

$$\left.\begin{array}{l} h_i=f_i(x_i)\cdot g_i(x_1) \\ b_i=b_i(x_1) \\ b_1=0 \end{array}\right\} \; i=2,\ 3,\ 4$$

f_i, g_i being any functions of their arguments.

III) ψ_3, ψ_4 are perfect squares, $b_1=\beta_1$, $b_2=\beta_2$

$$h_1=f_1(\sigma_1+\sigma_2),\quad h_2=f_2(\sigma_1+\sigma_2)$$

$$h_3=g_3^2(\sigma_1+\sigma_2)/(\rho_1+\rho_2),\quad h_4=g_4^2(\sigma_1+\sigma_2)/(\tau_1+\tau_2)$$

$$b_3=g_3(\lambda_1+\lambda_2)/(\rho_1+\rho_2),\quad b_4=g_4(\mu_1+\mu_2)/(\tau_1+\tau_2)$$

$$b_1=b_2=0.$$

$f_i=f_i(x_i)$, $\sigma_1=\sigma_1(x_1)$, etc.

IV) ψ_4 is a perfect square, $b_i=\beta_i$, $i=1,\ 2,\ 3$

$b_4=$function of $x_1,\ x_2,\ x_3$

$h_i=h_i(x_1,\ x_2,\ x_3)$, $i=1,\ 2,\ 3,\ 4.$

§ 3. Coordinate systems and potentials

The h_i's are determined by the condition $R_{ijkl}=0$, potentials b_i's by Maxwell equations in vacuo and the condition of separability.

(A) $ds^2=dx_1^2+dx_2^2+dx_3^2+dx_4^2$

$$H=p_1^2+(p_2-b_2)^2+(p_3-b_3)^2+(p_4-b_4)^2$$

$b_4,\ b_2,\ b_3$ are linear in x_1,

$X_i=x_i$, X_i being cartesian coordinates.

The field here is a constant electromagnetic field.

$$ds^2=dx_1^2+dx^2\cdot x_1^2+dx_3^2+dx_4^2 \tag{2}$$

$$H=p_1^2+(p_2-b_2)^2/x_1^2+(p_3-b_3)^2+(p_4-b_4)^2$$

$$b_2=\alpha x_1^2,\ b_3=\beta\log x_1,\ b_4=\gamma\log x_1$$

α, β, γ are arbitrary constants.

$$X_1=x_1\cos x_2,\ X_2=x_1\sin x_2,\ X_3=x_3,\ X_4=x_4.$$

(B) $$ds^2=dx_1^2+x_1^2dx_2^2+x_1^2\sin^2x_2dx_3^2+dx_4^2 \tag{3}$$

$$H=p_1^2+p_2^2/x_1^2+(p_3-b_3)^2/x_1^2\sin^2x_2+(p_4-b_4)^2$$

$$b_3=\beta\cos x_2+\beta_1\sin^2x_2/x_1\quad \alpha_1^2=\beta_1^2$$

$$b_4=\alpha/x_1+\alpha_1\cos x_2/x_1^2\qquad \alpha\alpha_1+\beta\beta_1=0$$

$$X_1=x_1\sin x_2\cos x_3,\ X_2=x_1\sin x_2\sin x_3,\ X_3=x_1\cos x_2,\ X_4=x_4$$

(C) $$ds^2=dx_1^2+dx_2^2+x_1^2dx_3^2+x_2^2dx_4^2 \tag{4}$$

$$H=p_1^2+p_2^2+(p_3-b_3)^2/x_1^2+(p_4-b_4)^2/x_2^2$$

$$b_3=x_1^2\{\alpha(x_1^2-2x_2^2)+\beta\}$$

$$b_4=\pm x_2^2\{\alpha(x_2^2-2x_1^2)-\beta\}$$

$$X_1=x_1\cos x_3,\ X_2=x_1\sin x_3,\ X_3=x_2\cos x_4,\ X_4=x_2\sin x_4$$

$$ds^2=dx_1^2+x_1^2dx_2^2+x_1^2\sin^2x_2dx_3^2+x_1^2\cos^2x_2dx_4^2 \tag{5}$$

$$X_1=x_1\sin x_2\cos x_3,\quad X_3=x_1\cos x_2\cos x_4$$

$$X_2=x_1\sin x_2\sin x_3,\quad X_4=x_1\cos x_2\sin x_4$$

$$H=p_1^2+p_2^2/x_1^2+(p_3-b_3)^2/x_1^2\sin^2x_2+(p_4-b_4)^2/x_1^2\cos^2x_2$$

$$\left.\begin{aligned} b_3&=(\alpha x_1^2+\beta/x_1^2)\sin^2x_2,\\ b_4&=\pm(\alpha x_1^2+\beta/x_1^2)\cos^2x_2 \end{aligned}\right\}$$

$$b_3=\alpha\log\cos x_2,\ b_4=\beta\log\sin x_2$$

(D) $$ds^2=(\sinh^2x_1+\sin^2x_2)(dx_1^2+dx_2^2)+dx_3^2+dx_4^2 \tag{6}$$

$$X_1=\cosh x_1\cos x_2,\quad X_3=x_3$$

$$X_2=\sinh x_1\sin x_2,\quad X_4=x_4$$

$$H=(p_1^2+p_2^2)/(\sinh^2x_1+\sin^2x_2)+(p_3-b_3)^2+(p_4-b_4)^2$$

$$\left.\begin{aligned} b_3&=\{(\beta_1+\beta x_1)\sinh 2x_1+(\beta_2+\beta x_2)\sin 2x_2\}/(\sinh^2x_1+\sin^2x_2)\\ b_4&=\pm b_3\sqrt{-1} \end{aligned}\right\}$$

$$\left.\begin{aligned} b_3&=(\beta_1\sinh 2x_1+\beta_2\sin 2x_2)/(\sinh^2x_1+\sin^2x_2)\\ b_4&=(\gamma_1\sinh 2x_1+\gamma_2\sin 2x_2)/(\sinh^2x_1+\sin^2x_2)\\ \beta_1^2+\gamma_1^2&=\beta_2^2+\gamma_2^2,\ \beta_1\beta_2+\gamma_1\gamma_2=0 \end{aligned}\right\}$$

$$ds^2=(x_1^2+x_2^2)\,(dx_1^2+dx_2^2)+dx_3^2+dx_4^2 \tag{7}$$

$$X_1=(x_1^2-x_2^2)/2,\quad X_2=x_1x_2,\quad X_3=x_3,\quad X_4=x_4$$

$$H=(p_1^2+p_2^2)/(x_1^2+x_2^2)+(p_3-b_3)^2+(p_4-b_4)^2$$

$$\left.\begin{aligned} b_3&=\{\beta(x_1^4-x_2^4)+\beta_1x_1+\beta_2x_2\}/(x_1^2+x_2^2)\\ b_4&=\pm b_3\sqrt{-1}\end{aligned}\right\}$$

$$\left.\begin{aligned} b_3&=(\beta_1x_1+\beta_2x_2)/(x_1^2+x_2^2)\\ b_4&=(\gamma_1x_1+\gamma_2x_2)/(x_1^2+x_2^2)\\ \beta_1^2+\gamma_1^2&=\beta_2^2+\gamma_2^2,\quad \beta_1\beta_2+\gamma_1\gamma_2=0\end{aligned}\right\}$$

(E) $$ds^2=(\sinh^2x_1+\sin^2x_2)\,(dx_1^2+dx_2^2)+\sinh^2x_1\sin^2x_2dx_3^2+dx_4^2 \tag{8}$$

$$X_1=\sinh x_1\sin x_2\cos x_3,\quad X_3=\cosh x_1\cos x_2$$

$$X_2=\sinh x_1\sin x_2\sin x_3,\quad X_4=x_4$$

$$\left.\begin{aligned} b_3&=(\alpha_1\cosh x_1\sin^2x_2+\alpha_2\cos x_2\sinh^2x_1)/(\sinh^2x_1+\sin^2x_2)\\ b_4&=(\beta_1\cosh x_1+\beta_2\cos x_2)/(\sinh^2x_1+\sin^2x_2)\\ \alpha_1^2+\alpha_2^2&=\beta_1^2+\beta_2^2,\quad \alpha_1\alpha_2+\beta_1\beta_2=0\end{aligned}\right\}$$

$$ds^2=(x_1^2+x_2^2)\,(dx_1^2+dx_2^2)+x_1^2x_2^2dx_3^2+dx_4^2 \tag{9}$$

$$X_1=x_1x_2\cos x_3,\quad X_3=(x_1^2-x_2^2)/2$$

$$X_2=x_1x_2\sin x_3,\quad X_4=x_4$$

$$\left.\begin{aligned} b_3&=\alpha x_1^2x_2^2+\alpha_1x_1^2/(x_1^2+x_2^2)\\ b_4&=\beta(x_1^2-x_2^2)+\beta_1/(x_1^2+x_2^2)\\ \alpha^2&=\beta^2,\quad \alpha_1^2=\beta_1^2\end{aligned}\right\}$$

(F) $$ds^2=(\sinh^2x_1+\sin^2x_2)\,(dx_1^2+dx_2^2)+\sinh^2x_1\sin^2x_2dx_3^2+\cosh^2x_1\cos^2x_2dx_4^2 \tag{10}$$

$$X_1=\sinh x_1\sin x_2\cos x_3,\quad X_3=\cosh x_1\cos x_2\cos x_4$$

$$X_2=\sinh x_1\sin x_2\sin x_3,\quad X_4=\cosh x_1\cos x_2\sin x_4$$

$$\left.\begin{aligned} b_3&=\alpha\sinh^2x_1\sin^2x_2\\ b_4&=\pm\alpha\cosh^2x_1\cos^2x_2\end{aligned}\right\}$$

(G) $$ds^2=dx_1^2+\sigma_1dx_2^2+\sigma_1\sigma_2dx_3^2+\sigma_1\sigma_2\sigma_3dx_4^2 \tag{11}$$

$$\sigma_1=x_1^2,\quad \sigma_2=\sin^2x_2,\quad \sigma_3=\sin^2x_3$$

$$X_1=x_1\sin x_2\sin x_3\cos x_4$$

$$X_2=x_1\sin x_2\sin x_3\sin x_4$$

$$X_3=x_1\sin x_2\cos x_3$$

$$X_4=x_1\cos x_2$$

$$b_4=\alpha\cos x_3$$

(H) $$ds^2=dx_1^2+\sigma_1(\sigma_2+\sigma_3)\,(dx_2^2+dx_3^2)+dx_4^2 \tag{12}$$

$$\sigma_1=x_1^2,\quad \sigma_2'^2+4\sigma_2^3+l\sigma_2^2+m\sigma_2+n=0$$

$$\sigma_3'^2+4\sigma_3^3-l\sigma_3^2+m\sigma_3-n=0$$

$$b_4=\alpha/x_1$$

(I) $$ds^2=(\sigma_1+\sigma_2)\,(dx_1^2+dx_2^2)+dx_3^2+\sigma_3\,dx_4^2$$

$$\sigma_1=\sinh^2 x_1,\ \sigma_2=\sin^2 x_2,\ \sigma_3=x_3^2 \tag{13}$$

$$X_1=\cosh x_1\cos x_2,\quad X_3=x_3\cos x_4$$

$$X_2=\sinh x_1\sin x_2,\quad X_4=x_3\sin x_4$$

$$b_4=\beta x_3^2$$

$$\sigma_1=x_1^2,\ \sigma_2=x_2^2,\ \sigma_3=x_3^2 \tag{14}$$

$$X_1=(x_1^2-x_2^2)/2,\quad X_3=x_3\cos x_4$$

$$X_2=x_1x_2,\quad X_4=x_3\sin x_4$$

$$b_4=\beta x_3^2$$

(J) $$ds^2=(\sigma_1+\sigma_2)\,(dx_1^2+dx_2^2)+\sigma_1\sigma_2 dx_3^2+\sigma_1\sigma_2\sigma_3\,dx_4^2$$

$$\sigma_1=\sinh^2 x_1,\ \sigma_2=\sin^2 x_2,\ \sigma_3=\sin^2 x_3, \tag{15}$$

$$X_1=\sinh x_1\sin x_2\sin x_3\cos x_4$$

$$X_2=\sinh x_1\sin x_2\sin x_3\sin x_4$$

$$X_3=\sinh x_1\sin x_2\cos x_3$$

$$X_4=\cosh x_1\cos x_2$$

$$b_4=\beta\cos x_3$$

$$\sigma_1=x_1^2,\ \sigma_2=x_2^2,\ \sigma_3=\sin^2 x_3, \tag{16}$$

$$X_1=x_1x_2\sin x_3\cos x_4,\quad X_3=x_1x_2\cos x_3$$

$$X_2=x_1x_2\sin x_3\sin x_4,\quad X_4=(x_1^2-x_2^2)/2.$$

$$b_4=\beta\cos x_3.$$

In other cases there are no fields possible.

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Perfect Imaging Dynamical Systems

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Perfect imaging dynamical systems are classified from the view-point of particle optics and their properties are studied. Examples realizing inversion, translation and rotation are given.

§ 1. Introduction

Perfect imaging systems in optics have recently attracted increasing attention and have been enriched with new examples.[1),2)] A dynamical system has one more degree of freedom than an optical system, that is, time and energy. Further the path of a charged particle in a magnetic field is not realizable in optical systems. Hence perfect imaging systems in particle optics display a variety of remarkable properties that have never been realized in optics. In optics the most interesting imaging is a conformal mapping realizable with Maxwell's fish eye for example. The translation and rotation are out of the scope of optical systems. In particle optics these two transformations are realizable by virtue of magnetic fields.

In particle optics it is possible to converge all particles from a source point to an image point irrespective of their initial velocity as well as to make an image of a source describe an energy spectrum.

In § 2 perfect imaging systems are classified according as how the position of an image and the time of travel vary with the particle energy. In § 3 a property of imaging when the Lagrangian involves the time is studied. A possible imaging is a similar mapping. In § 4 is shown that the advances of ignorable coordinates are independent of the position of source in perfect imaging. In § 5 is presented a fundamental formula in perfect imaging that the action over a whole cycle is linear in momenta conjugate to ignorable coordinates. In § 6 is given a general dynamical system that is axially symmetric and perfect imaging in plane. There are included Maxwell's fish eye,[3)] Stettler's generalized fish eyes,[2)] Luneberg lens, etc. A remark is added that the period of planetary motion depends solely on the energy of particle. In § 7 are given examples of systems performing translation and rotation. The two transformations are realizable in a plane with purely magnetic fields. In § 8 are added examples of periodic dynamical systems, among which are dynamical systems of constant period irrespective of their energy. A potential such as giving energy levels of hydrogen atom is also given in one dimension.

§ 2. Classification of perfect imaging systems

An imaging system is called perfect when all the rays or particles starting from a source A converge to an image A', the correspondence $A \to A'$ being one-to-one. The imaging thus achieved is called perfect.

Since the time and the energy of particle are indispensable factors in dynamical systems, perfect imaging systems may be classified as follows :

(1) a dynamical system which has the Lagrangian involving the time explicitly and make all particles starting from a point $A(x_1, x_2, x_3)$ at the time t converge to a point $A'(x_1', x_2', x_3')$ at the time t', the correspondence $(x_1, x_2, x_3, t) \to (x_1', x_2', x_3', t')$ being one-to-one. The imaging here is a similar mapping.

(2) a dynamical system which has the Lagrangian not involving the time explicitly and performs perfect imaging for particles of a definite energy.

(3) one which has the Lagrangian not involving the time explicitly and performs perfect imaging for particles of any energy (at least in a finite range of energy).

This type is subdivided into three types :

(α) a type in which not only the position of an image but the time of travel of a particle from a source to its image do not depend on the energy of the particle,

(β) a type in which not the position of the image but the time of travel depends on the energy,

(γ) a type in which the position of the image varies with the energy. If there exist the above dynamical systems, that of the type (α) will afford an effective apparatus for the determination of mass-to-charge ratio, that of the type (β) will be applied to the measurement of the energy of a particle by the time of travel, and that of the type (γ) will suggest the construction of an energy spectrograph.

§ 3. Properties of perfect imaging systems

We put the Lagrangian of a particle in the euclidian space of 3 dimensions as

$$L = 1/2 \cdot \dot{x}_\nu \dot{x}_\nu + B_\nu(x)\dot{x}_\nu - B(x)$$

where (B_1, B_2, B_3) may be interpreted as vector potentials, B as the scalar potential of an electromagnetic field, the mass-to-charge ratio of the particle being taken for unity. Referring to §§ 69, 70 of Caratheodory's Geomestrische Optik,[1] especially to the equations (69.5), (69.9) we have, for a perfect imaging system of the type (1), a relation

$$L(t', x', dx'/dt')\,dt' = L(t, x, dx/dt)\,dt + dS, \qquad (3\cdot1)$$

dS being a perfect differential. So we have a relation

$$\{\dot{x}_\nu' \dot{x}_\nu' + 2B_\nu(x')\dot{x}_\nu' - 2B(x')\}\,dt' = \{\dot{x}_\nu \dot{x}_\nu + 2B_\nu(x)\dot{x}_\nu - 2B(x)\}\,dt + 2dS. \qquad (3\cdot2)$$

Since the imaging is perfect, the correspondence $(x, t) \to (x', t')$, or

$$x_\nu' = x_\nu'(x, t), \quad t' = t'(x, t) \qquad (3\cdot3)$$

is to be consistent with $(3\cdot2)$. If we regard dx_ν, dt as independent variables and write

them as $\tilde{\xi}_r$, $\tilde{\xi}_0$, we get from (3·3) a linear transformation

$$\tilde{\xi}_r' = a_{rs}\tilde{\xi}_s + a_{r0}\tilde{\xi}_0$$
$$\tilde{\xi}_0' = a_{0s}\tilde{\xi}_s + a_{00}\tilde{\xi}_0 \qquad (3\cdot4)$$

where $a_{rs} = \partial x_r'/\partial x_s$, etc. and from (3·2)

$$\{\tilde{\xi}_r'\tilde{\xi}_r' + 2B_r(x')\tilde{\xi}_r'\tilde{\xi}_0' - 2B(x')\tilde{\xi}_0'^2\}\,\tilde{\xi}_0' = \{\tilde{\xi}_r\tilde{\xi}_r + 2B_r(x)\tilde{\xi}_r\tilde{\xi}_0 - 2B(x)\tilde{\xi}_0^2\}/\tilde{\xi}_0 + 2s_r\tilde{\xi}_r + 2s_0\tilde{\xi}_0$$

where $s_r = \partial S/\partial x_r, \quad s_0 = \partial S/\partial t.$

This relation will be

$$\{\tilde{\xi}_r'\tilde{\xi}_r' + 2B_r(x')\tilde{\xi}_r'\tilde{\xi}_0' - 2B(x')\tilde{\xi}_0'^2\}\,\tilde{\xi}_0 = \{\tilde{\xi}_r\tilde{\xi}_r + 2(B_r(x) + s_r)\tilde{\xi}_r\tilde{\xi}_0 - 2(B(x) - s_0)\tilde{\xi}_0^2\}\,\tilde{\xi}_0'. \qquad (3\cdot5)$$

The equations (3·4), (3·5) are to be consistent. Both members of the eq. (3·5) are homogeneous expressions of 3 degrees. Therefore, if the eq. (3·5) is valid for a finite range of $\tilde{\xi}$, it is valid identically in $\tilde{\xi}$. The left member is divisible by $\tilde{\xi}_0$, so the right member must be divisible by $\tilde{\xi}_0$. Hence we get

$$\tilde{\xi}_0' = a_{00}\tilde{\xi}_0, \quad a_{0s} = \partial x_0'/\partial x_s = \partial t'/\partial x_s = 0.$$

In other words, t' depends not on x_r but t, that is, $t' = t'(t)$.

Comparing both members of (3·5) after dividing by $\tilde{\xi}_0$, we get

$$\tilde{\xi}_r'\tilde{\xi}_r' = a_{00}\tilde{\xi}_r\tilde{\xi}_r \qquad (3\cdot6)$$

$$\left.\begin{aligned} B_r(x')\tilde{\xi}_r' &= (B_r(x) + s_r)\tilde{\xi}_r \\ a_{00}B(x') &= B(x) - s_0 \end{aligned}\right\}. \qquad (3\cdot7)$$

The relation (3·6) implies

$$dx_r'\,dx_r' = a_{00}\,dx_r\,dx_r.$$

Because of the relation above obtained $a_{00} = dt'/dt$, this will be

$$dx_r'\,dx_r'/dt' = dx_r\,dx_r/dt \qquad (3\cdot8)$$

that is, the magnification ratio is a function of t only. The eq. (3·7) represent a coordinate transformation combined with a gauge transformation of vector and scalar potentials. The above reasoning reveals a possibility of pure magnification.

The properties of perfect imaging systems of which the Lagrangian does not involve the time explicitly were studied hitherto.[5],[6] The relations obtained there are

$$(h - B(x'))\,dx_r'\,dx_r' = (h - B(x))\,dx_r\,dx_r \qquad (3\cdot9)$$

$$B_r(x')\,dx_r' = B_r(x)\,dx_r + d\psi. \qquad (3\cdot10)$$

The relation (3·9) admits only conformal transformations. For perfect imaging systems of the types (α) and (β), the position of the image point does not depend on the energy of particles. So the relation (3·9) must be valid for any value of h. Therefore we have

$$dx_r' dx_r' = dx_r dx_r \tag{3·11}$$

$$B(x') = B(x) \tag{3·12}$$

The relation (3·11) admits only congruent transformations, that is, rotations and translations.

§ 4. Ignorable coordinates

When the Hamiltonian of a dynamical system referred to an orthogonal curvilinear coordinate system $(q_1, q_2, \cdots, q_n)$

$$H = 1/2 \cdot \sum (p_r - B_r(q))^2 / h_r + B(q) \tag{4·1}$$

does not contain k coordinates $q_1, \cdots, q_k$ explicitly, there exist k integrals $p_i = \text{const}$, $i = 1, \cdots, k$, p_i being the momentum conjugate to an ignorable coordinate q_i. If the imaging by the dynamical system is perfect for particles of energy h, there is a property that the respective advances $Q_1, \cdots, Q_k$ of the coordinates $q_1, \cdots, q_k$ are independent of the position of source. We take (q_r, p_r) for canonical variables of a source, (q_r', p_r') for ones of the corresponding image. Since the imaging is perfect, there are the relations

$$q_r' = q_r'(q_1, \cdots, q_n), \quad r = 1, 2, \cdots, n \tag{4·2}$$

and, from the constancy of momenta conjugate to ignorable coordinates,

$$p_1' = p_1, \cdots, p_k' = p_k. \tag{4·3}$$

If we represent the canonical transformation $(q_r, p_r) \to (q_r', p_r')$ with a characteristic function $U(q, p')$ taking q_r, p_r' for independent variables, as

$$p_r = \partial U / \partial q_r, \quad q_r' = \partial U / \partial p_r' \tag{4·4}$$

the above relations (4·2) permit us to put

$$U = p_r' q_r'(q) + f(q). \tag{4·5}$$

To ensure the relations (4·3), we have to put

$$\left.\begin{aligned} q_i' &= q_i + Q_i(q_{k+1}, \cdots, q_n), \quad i = 1, \cdots, k \\ q_s' &= q_s'(q_{k+1}, \cdots, q_n) \qquad s = k+1, \cdots, n \\ f(q) &= f(q_{k+1}, \cdots, q_n) \end{aligned}\right\}. \tag{4·6}$$

Hence we have

$$U = p_1'(q_1 + Q_1) + \cdots + p_k'(q_k + Q_k) + p_{k+1}' q_{k+1}' + \cdots + p_n' q_n' + f. \tag{4·7}$$

Accordingly we have

$$p_{k+1} = \partial U / \partial q_{k+1} = \sum_{i=1}^{k} p_i' \partial Q_i / \partial q_{k+1} + \sum_{s=k+1}^{n} p_s' \partial q_s' / \partial q_{k+1} + \partial f / \partial q_{k+1}, \text{ etc.} \tag{4·8}$$

Since the characteristic function must satisfy two equations,

$$H(q_r,\ p_r)-h=0,\quad H(q_r',\ p_r')-h=0, \tag{4·9}$$

where p_r, q_r' are to be given by (4·4) and (4·7), the relations (4·4), (4·5), (4·7), (4·9) must be compatible.

Therefore if we eliminate p_r from the first equation of (4·9), the two equations must be proportional to each other, for they are both of the second degree in p_r'. Since $H(q',\ p')$ lacks such terms $p_1'p_{k+1}'$, $\cdots$, $p_1'p_n'$, $\cdots$, $p_k'p_n'$, that is, cross terms of p_1', $\cdots$, p'_k with p'_{k+1}, p_n', $H(q,\ p)$ also, p_r being expressed by p_r', must lack such terms. So the coefficients of $p_1'p_{k+1}'$, $\cdots$, $p_1'p_n'$ in $H(q,\ p)$ are to vanish,

$$\left.\begin{aligned}&\sum_{s=k+1}^{n}\frac{1}{h_s(q)}\frac{\partial Q_1}{\partial q_s}\frac{\partial q'_{k+1}}{\partial q_s}=0\\ &\qquad\cdots\cdots\cdots\cdots\cdots\cdots\\ &\sum_{s=k+1}^{n}\frac{1}{h_s(q)}\frac{\partial Q_1}{\partial q_s}\frac{\partial q_n'}{\partial q_s}=0\end{aligned}\right\}. \tag{4·10}$$

This is $n-k$ linear equations in $\partial Q_1/\partial q_{k+1}$, $\cdots$, $\partial Q_1/\partial q_n$. Since the imaging is perfect, $\partial(q_1',\ \cdots,\ q_n')/\partial(q_1,\ \cdots,\ q_n)=\partial(q'_{k+1},\ \cdots,\ q_n')/\partial(q_{k+1},\ \cdots,\ q_n)\neq 0$.

So the determinant formed with the coefficients of the equations (4·10) does not vanish. Therefore we have

$$\partial Q_1/\partial q_{k+1}=\cdots=\partial Q_1/\partial q_n=0,$$

that is, Q_1 is independent of q_{k+1}, $\cdots$, q_n. The same conclusion prevails also with other Q_i's. The Q_i's are functions of h at most.

§ 5. Separable dynamical systems

When the Hamilton-Jacobi equation $\partial W/\partial t+H(q,\ \partial W/\partial q)=0$ of a dynamical system is integrable by separation of variables, the integration method of Jacobi gives a path by

$$\partial W/\partial\alpha_r=-\beta_r,\quad r=2,\ 3,\ \cdots,\ n \tag{5·1}$$

and the position of particle along the path by

$$\partial W/\partial\alpha_1=-\beta_1 \tag{5·2}$$

where W is a complete integral involving n arbitrary constants $\alpha_1=h$, α_2, $\cdots$, α_n and is written as

$$W=-\alpha_1 t+\sum_r\int p_r\,dq_r \tag{5·3}$$

$p_r=$function of q_r alone, $\quad r=1,\ 2,\ \cdots,\ n$.

We consider the case where each q_r of variables is contained in an interval and oscillates cyclically. We define the action J over a whole cycle by

$$J=\sum_r\oint p_r\,dq_r, \tag{5·4}$$

$\oint$ denoting the integration over a whole cycle. The integration constants α_r, β_r introduced above serve to determining the initial values q_r^0, p_r^0 so that we can consider the constants β_r's to be functions of the constants α_r's and initial values q_r^0's. If the β_r's assume the same values as before after each of all variables q_r completes its respective cycle, the motion of the particle is periodic since the dynamical system has the same α_r, β_r after a cycle.

The condition for the above periodic motion is

$$\partial J/\partial \alpha_r = 0, \quad r = 2, 3, \cdots, n \tag{5·5}$$

while the period T is given as

$$T = \partial J/\partial \alpha_1 = \partial J/\partial h \tag{5·6}$$

by the condition that β_1 assumes the same value after a cycle.

When the coordinates $q_1, \cdots, q_k$ are ignorable, the momenta $p_1, \cdots, p_k$ are constants

$$p_i = \alpha_{i+1}, \quad i = 1, 2, \cdots, k$$

and a complete integral is written

$$W = -\alpha_1 t + \alpha_2 q_1 + \cdots + \alpha_{k+1} q_k + \sum_s \int p_s dq_s .$$

When the motion is cyclic with respect to a set of other variable $q_{k+1}, \cdots, q_n$, the action defined as before

$$J = \sum_s \oint p_s dq_s \tag{5·7}$$

gives the relations

$$\left.\begin{aligned} &\partial J/\partial \alpha_1 = T, \quad \partial J/\partial \alpha_{i+1} = \partial J/\partial p_i = -Q_i, \quad i = 1, \cdots, k \\ &\partial J/\partial \alpha_{s+1} = 0, \qquad\qquad\qquad s = k+1, \cdots, n-1 \end{aligned}\right\} \tag{5·8}$$

where Q_i is the advance of the coordinate q_i over a whole cycle. The relation takes a form

$$dJ = T dh - Q_1 dp_1 - \cdots - Q_k dp_k \tag{5·9}$$

in its differential form.

If the motion is cyclic with respect to the set of $q_{k+1}, \cdots, q_n$ after the variable q_s completes ν_s cycles, $s = k+1, \cdots, n$, the action is to be taken as

$$J = \sum_s \nu_s \oint p_s dq_s .$$

We combine the result obtained here with that of § 4. If the perfect imaging is such as

$$\begin{aligned} &q_i' = q_i + Q_i, \quad i = 1, 2, \cdots, k \\ &q_s' = q_s, \qquad\quad s = k+1, \cdots, n \end{aligned}$$

where the advance Q_i of the variable q_i is a function of h alone, we have a fundamental formula

$$J=f(h)-Q_1(h)p_1-\cdots-Q_k(h)p_k. \tag{5·10}$$

In short, for a perfect imaging and separable dynamical system, the action over a whole cycle is a linear function of the momenta conjugate to ignorable coordinates, its coefficients depending on the energy alone. The classification of § 2 may be written in terms of J as

(2) $J=$a linear function of $p_1, \cdots, p_k$ for a certain energy h.

(3) (α) $J=bh-a_1p_1-\cdots-a_kp_k$, $b, a_1, \cdots, a_k$ being any constants

(β) $J=f(h)-a_1p_1-\cdots-a_kp_k$, $f'(h)\neq 0$

(γ) $J=f(h)-Q_1(h)p_1-\cdots-Q_k(h)p_k$

one of $Q_i'(h)$ at least not vanishing,

a prime denoting the differentiation with respect to h.

§ 6. Imaging by the reversible system

When the Lagrangian L lacks the term linear in $\dot{q}_r$, the L is invariant under a transformation $t\rightarrow -t$, $\dot{q}_r\rightarrow -\dot{q}_r$. So the equation of motion is also invariant under the transformation. Hence a path reversed in its direction is also another path. Therefore, if A' is an image of A, A is an image of A'. So the imaging repeated twice is an identical mapping. Accordingly the advance repeated twice $2Q_r$ of the coordinate q_r must be congruent to 0. Hence Q_r must be 0 unless the coordinate q_r is a many-valued function of the position of a source. Rotations by an angle other than $k\pi$ and translations are not realizable by a reversible system.

When the coordinate θ is ignorable with respect to the polar coordinate in plane, the advance Θ of θ must satisfy the condition.

$$2\Theta=2k\pi, \quad k=1, 2, \cdots$$

or $\Theta=k\pi$.

For the simplest Hamiltonian H

$$H=1/2\cdot(p_r^2+p_\theta^2/r^2)+B(r)=h \tag{6·1}$$

we have an integral $p_\theta=p$ and the action J

$$J=\oint p_r dr=\oint\sqrt{2h-2B(r)-p^2/r^2}\,dr. \tag{6·2}$$

The imaging is realizable when the action J is a linear function in p as was seen above.

Putting $r=a\exp x$, $2(h-B(r))r^2=f(x)$, we transform J into

$$J=\oint\sqrt{f(x)-p^2}\,dx.$$

We choose the constant a to be the value of r where $f(x)$ is maximum at $x=0$, assuming that $f(x)$ has a maximum. We put then

$$f(x)=l^2-\tau^2$$

$$l^2=f(0), \quad l^2-p^2=\mathcal{E}^2$$

and

$$x=\varphi(\tau)=\sum_{n=1}^{\infty}c_n\tau^n,$$

and get

$$J=\oint\sqrt{\mathcal{E}^2-\tau^2}\,\varphi'(\tau)\,d\tau=\oint\frac{\varphi(\tau)\,\tau}{\sqrt{\mathcal{E}^2-\tau^2}}d\tau$$

$$=2\pi\sum_{n=1}^{\infty}c_{2n-1}(-1)^n\binom{-1/2}{n}\mathcal{E}^{2n}$$

where the bracket expression is the coefficient of x^n in the expansion of $(1+x)^{-1/2}$.

On the other hand, J must be linear in p and satisfy the relation $\partial J/\partial p=-\theta=-k\pi$, $(k=1, 2, \cdots)$, so that J may be put

$$J=b-\pi kp=b-\pi k\sqrt{l^2-\mathcal{E}^2}$$

$$=b-\pi kl+\pi kl\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\binom{-1/2}{n}\frac{\mathcal{E}^{2n}}{l^{2n}}.$$

Comparing the two expressions of J with each other, we get

$$b-\pi kl=0$$

$$c_{2n-1}=\frac{k}{2(2n-1)}\cdot\frac{1}{l^{2n-1}}, \quad n=1, 2, 3, \cdots$$

c_{2n} : arbitrary.

Therefore we get

$$\left.\begin{aligned} &x=k/2\cdot\operatorname{arctanh}(\tau/l)+c_2\tau^2+c_4\tau^4+\cdots\\ &J=\pi k(l-p), \quad \theta=k\pi. \end{aligned}\right\} \qquad (6\cdot3)$$

So there appear numerous examples of perfect imaging systems among which are included various known systems, such as Maxwell's fish eye, Luneberg lens, generalized fish eye obtained by R. Stetller and so forth.

The inversions $r'=a^2/r$, $\theta'=\theta+k\pi$, $(k=1, 2, \cdots)$ are realizable only when $c_2=c_4=\cdots=0$. They are, however, perfect only for a certain energy.

There are only two cases where the perfect imaging is possible for any energy h. They are that of $B(r)=\omega^2r^2/2$ and that of $B(r)=-c/r$.

For $B(r)=\omega^2r^2/2$ we have

$$J=\oint\sqrt{2(h-\omega^2r^2/2)-p^2/r^2}\,dr=\pi(h/\omega-p), \quad p>0$$

$$\theta = -\partial J/\partial p = \pi, \quad T = \pi/\omega.$$

For $B(r) = -c/r$ we have

$$J = \oint \sqrt{2(h + c/r) - p^2/r^2}\, dr = 2\pi (c/\sqrt{(-2h)} - p)$$

$$p > 0, \quad h < 0,$$

$$\theta = 2\pi, \quad T = 2\pi c (-h)^{-3/2}.$$

The imaging is an identical mapping. The time of travel depends, however, on the energy. This is the only case of type (β) ever known. In space the potential $B(r) = -c/r$ is realizable electrically.

§ 7. Imaging by irreversible systems

1) Translations. When the coordinate y is ignorable with respect to a cartesian coordinate in a plane, the Lagrangian and Hamiltonian may be put as

$$\left.\begin{aligned} L &= 1/2 \cdot (\dot{x}^2 + \dot{y}^2) + b(x)\dot{y} - B(x) \\ H &= 1/2 \cdot p_x^2 + 1/2 \cdot (p_y - b(x))^2 + B(x) \end{aligned}\right\}. \tag{7·1}$$

Since $\dot{y} = p_y = p$ is an integral, the action is

$$J = \oint \sqrt{2h - 2B(x) - (b(x) - p)^2}\, dx. \tag{7·2}$$

If we put $B(x) = -\alpha x$, $b(x) = \beta x$, we have

$$J = 2\pi (\alpha p/\beta^2 + \alpha^2/2\beta^3 + h/\beta)$$

so that the translation is realizable with the result

$$Y = -\partial J/\partial p = -2\pi\alpha/\beta^2$$

$$T = \partial J/\partial h = 2\pi/\beta.$$

Y is the advance of the coordinate y. The advance as well as the time do not depend on the energy. This dynamical system is of the type (α), and gives nothing but the field invented by Bleakney—Hipple[7] for a mass-spectrometer.

The translation is realizable with $b(x)$ only. Dropping off $B(x)$, we have

$$J = \oint \sqrt{2h - (b(x) - p)^2}\, dx. \tag{7·3}$$

Changing the variable from x to z by $b(x) = z$, $x = \varphi(z)$, further from z to u by $z - p = u$, we have

$$J = \oint \sqrt{2h - (z-p)^2}\, \varphi'(z)\, dz$$

$$= \oint \sqrt{2h - u^2}\, \varphi'(p+u)\, du$$

$$= 2\pi \{h\varphi'(p) + h^2 \varphi'''(p)/4 + \cdots\}.$$

The prime here denotes differentiation. To ensure that J be a linear function in p, $\varphi(p)$ must be quadratic in p. So we can put

$$x=\varphi(z)=\alpha(z+\beta)^2/2+\gamma.$$

Since an additive constant to $b(x)$ is indifferent to the field, the constant β may be dropped. The constant γ also may be dropped by the change of the coordinate x. Hence we get

$$b(x)=\sqrt{(2x/\alpha)} \tag{7·4}$$

and

$$J=2\pi\alpha hp$$

$$Y=-2\pi\alpha h, \quad T=2\pi\alpha p.$$

The advance Y is proportional to the energy. Therefore an energy-spectrum of a charged particle in this magnetic field is a straight line parallel to the y axis. This is an example of the type (γ) in § 2.

An example of translation in space is given by a dynamical system with its Hamiltonian

$$2H=p_x^2+p_y^2+(p_z-\lambda x)^2+\lambda^2y^2-2\alpha x.$$

There are two integrals:

$$p_z=p$$

$$p_y^2+\lambda^2y^2=q.$$

J is calculated to be

$$J=\oint\sqrt{\{2h-p^2-q+2(\lambda p+\alpha)x-\lambda^2x^2\}}\,dx+\oint\sqrt{(q-\lambda^2y^2)}\,dy$$

$$=\pi(2\alpha p+2\lambda h)/\lambda^2+\pi\alpha^2/\lambda^3.$$

Hence $\partial J/\partial q=0$, and J is linear in p.

$$Z=-\partial J/\partial p=-2\pi\alpha/\lambda^2, \quad T=2\pi/\lambda.$$

This dynamical system is however not realizable with the electromagnetic field in vacuo.

2) Rotations. A general Lagrangian not involving θ with respect to the polar coordinate in a plane may be put

$$L=(\dot r^2+r^2\dot\theta^2)/2+b(r)\dot\theta-B(r). \tag{7·5}$$

Hence the Hamiltonian and the action become

$$H=p_r^2/2+(p_\theta-b(r))^2/2r^2+B(r)$$

$$J=\oint\sqrt{\{2h-2B(r)-(b(r)-p)^2/r^2\}}\,dr.$$

For the field $B(r)=\alpha r^2$, $b(r)=\beta r^2$, we have

$$J=\pi[(\beta p+h)/(2\alpha+\beta^2)^{1/2}-p],\quad p>0$$
$$\theta=-\partial J/\partial p=\pi(1-\beta(2\alpha+\beta^2)^{-1/2})$$
$$T=\partial J/\partial h=\pi(2\alpha+\beta^2)^{-1/2}.$$

θ as well as T do not depend on the energy.

Next we determine $b(r)$ dropping off $B(r)$. To expand J in powers of h, we introduce new variables z, ζ defined by

$$b(r)=z,\quad r=\varphi(z)$$

and

$$b(r)-p=r\zeta$$

or

$$z=p+\zeta\varphi(z).$$

From the Lagrange's theorem[8] concerning the expansion of inverse function we have

$$r=\varphi(z)=\varphi(p)+(\varphi^2)'\zeta/2!+(\varphi^3)''\zeta^2/3!+(\varphi^4)'''\zeta^3/4!+\cdots$$

so that we get

$$J=\oint\sqrt{(2h-(b(r)-p)^2/r^2)}\,dr$$
$$=\oint\sqrt{2h-\zeta^2}\,dr$$
$$=2\pi\sum_{n=1}^{\infty}\frac{(\varphi^{2n})^{(2n-1)}}{n!\,n!}\left(\frac{h}{2}\right)^n=2\pi\{(\varphi^2)'h/2+(\varphi^4)'''h^2/16+\cdots\}.$$

In order to ensure that J be linear in p for any energy, φ^2 must be quadratic in p. So we put

$$\varphi^2=\alpha(p+\beta)^2/2+\gamma.$$

As in (7·4) we may put $\beta=0$ and get

$$\left.\begin{aligned}J&=2\pi(1/\sqrt{1-\alpha h}-1)p\\ \theta&=2\pi(1-1/\sqrt{1-\alpha h})\\ T&=2\pi\alpha p(1-\alpha h)^{-3/2}\\ b(r)&=\sqrt{\{2(r^2-\gamma)/\alpha\}}\end{aligned}\right\}.\qquad(7\cdot6)$$

When $\gamma=0$, the field $B(r)=-c/r$ may be added without spoiling the perfect imaging.

Two dynamical systems $B(x)=0$, $b(x)=\sqrt{(2x/\alpha)}$ in translations and $B(r)=-c/r$, $b(r)=r\sqrt{(2/\alpha)}$ in rotations seem to be the first examples of perfect imaging energy spectrograph.

Examples in a foregoing paper[5] perform the rotation in space.

A dynamical system defined by

$$L=(\dot r^2+r^2\dot\theta^2+\dot z^2)/2+\beta r^2\dot\theta-\beta^2(2z^2-r^2)/3$$

also performs the rotation around the z axis, whereby $J=\pi((\sqrt{3}-1)p+h\sqrt{3})$, $\theta=\pi(1-\sqrt{3})$, $T=\pi\sqrt{3/\beta}$.

A dynamical system

$$2H=p_r^2+p_0^2/r^2+(p_z-\beta r\sin^2\theta)^2/r^2\sin^2\theta-\beta^2\sin^2\theta$$

with respect to the polar coordinate in space gives an example where the advance of φ varies with the energy, though it is not realizable with electromagnetic field in vacuo.

§ 8. Periodic dynamical systems

A dynamical system may be called periodic if all paths of a particle of a certain energy are periodic. So periodic dynamical systems also perform perfect imaging though they are of rather less interest in view of particle optics, since the imaging is an identical mapping.

We show here, however, few examples of periodic dynamical systems as an application of the fundamental formula (5·5).

A dynamical system of 1 degree of freedom $H=1/2\cdot p_x^2+f(x)$ has an energy integral $p_x^2+2f(x)=2h$ and the action

$$J=\oint\sqrt{2h-2f(x)}\,dx.$$

The motion is periodic within a valley of potential. Assuming $f(x)$ to be minimum at $x=0$ and $f(0)=0$, we change the variable from x to τ by $2f(x)=\tau^2$ or

$$x=\varphi(\tau)=\beta\tau+\beta_1\tau^2+\beta_2\tau^3+\cdots$$

and get

$$\begin{aligned}J&=\oint\sqrt{2h-\tau^2}\,\varphi'(\tau)\,d\tau\\&=2\pi\left\{\frac{1}{2}\beta\cdot 2h+\frac{1\cdot 3}{2\cdot 4}\beta_2(2h)^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\beta_4(2h)^3+\cdots\right\}.\end{aligned}\tag{8·1}$$

$$T=\partial J/\partial h=2\pi\{\beta+1\cdot 3\beta_2 h+1\cdot 3\cdot 5/(1\cdot 2)\cdot\beta_4 h^2+\cdots\}$$

If the period T or the action J is given as a function of h, a potential is determined except for the constants β_1, β_3, $\cdots$. For example, a potential for which the period does not depend on the energy is given when

$$x=\varphi(\tau)=\beta\tau+\beta_1\tau^2+\beta_3\tau^4+\cdots\tag{8·2}$$

If we put $x=\beta\tau+\beta_1\tau^2$, we have $f(x)=x/2\beta_1-\beta(\sqrt{\beta^2+4\beta_1 x}-\beta)/4\beta_1^2$. This is not symmetric with respect to x, though the period $2\pi\beta$ is constant. Next we determine a potential such that

$$J=\oint\sqrt{2h-2f(x)}\,dx=2\pi(-a+b/\sqrt{(-2h)}).$$

This is suggested by a classical relation between the energy and the action in hydrogen

atom. Assuming $h<0$, f to be minimum at $x=0$, and $2f(0)=-l<0$, we put $2f(x)=-l+\tau^2$, $x=\varphi(\tau)$, $l+2h=\varepsilon$ and we get an equation in ε

$$J=\oint\sqrt{\varepsilon-\tau^2}\,\varphi'(\tau)\,d\tau=2\pi\,(-a+b/\sqrt{(l-\varepsilon)}).$$

Expanding both members in ε and equating coefficients of ε^n, we have

$$x=\varphi(\tau)=a\tau/(l-\tau^2)+\beta_1\tau^2+\beta_3\tau^4+\cdots, \tag{8·3}$$

β_1, β_3, $\cdots$ being arbitrary. This arbitrariness suggests us the difficulty in determining a potential from its energy levels in quantum mechanics.

A separable dynamical system referred to the cartesian coordinate in a plane is written

$$2H=p_x^2+p_y^2+2f(x)+2g(y),$$

whereby $p_y^2+2g(y)=p$ is an integral and the action becomes

$$J=\oint\sqrt{\{2h-p-2f(x)\}}\,dx+\oint\sqrt{\{p-2g(y)\}}\,dy.$$

The condition of periodicity is, from (5·5), $\partial J/\partial p=0$. Hence we must determine $f(x)$, $g(y)$ to satisfy the condition.

Assuming $f(x)$, $g(y)$ to be minimum at $x=0$, $y=0$, and $f(0)=g(0)=0$, we change variables from x, y to σ, τ by

$$2f(x)=\sigma^2,\quad 2g(y)=\tau^2$$

or

$$x=\varphi(\sigma),\quad y=\psi(\tau).$$

Then we have

$$J=\oint\sqrt{2h-p-\sigma^2}\,\varphi'(\sigma)\,d\sigma+\oint\sqrt{p-\tau^2}\,\psi'(\tau)\,d\tau$$
$$=2\pi\,\Phi(2h-p)+2\pi\,\Psi(p),$$

Φ, Ψ being determined as in (8·1). The condition $\partial J/\partial p=0$ gives a relation between Φ and Ψ. If we require the dynamical system to be periodic for any energy, there must be an identity

$$\Phi'(2h-p)=\Psi'(p)$$

for any p and for any h, consequently, both Φ' and Ψ' must be constant. Therefore $f(x)$ must be one of the potentials obtained above (8·2), $g(y)$ the same or another.

The above process may be applied to other coordinate systems with increasing difficulties.

A thorough list of periodic systems is far from completion.

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Elastic Scattering of Deuterons by Heavy Nuclei*

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In the present paper the elastic scattering of deuterons by heavy nuclei is investigated through the analysis of the available data at 11 MeV and 15.2 MeV. Because $Ze^2/\hbar v \gg 1$ and the reduced wave-length of the deuteron is much shorter than the nuclear dimensions, the center-of-mass motion of the deuteron can be described by the classical approach. The wave mechanical description is applied only to the internal motion of the deuteron. It is shown that for the deuteron scattering at 11 MeV the departure from the Rutherford scattering can be completely attributed to the electric breakup of the deuteron. The result of the analysis of the deuteron scattering at 15.2 MeV shows that although at larger scattering angles the effect of the nuclear absorption due to the diffractional scattering of deuterons cannot be neglected, the main contribution to the departure from the Rutherford scattering comes from the effect of the electric breakup of the deuteron. The calculation of the cross section of the electric breakup by the semi-classical method of the virtual quanta modified for the non-relativistic case is performed to compare with the result obtained from the analysis. The limitation of this method is also discussed.

In conclusion, it is emphasized that the mechanism of the elastic scattering of deuterons by heavy nuclei is quite different from those of the elastic scattering of alpha-particles by nuclei and of deuterons by light nuclei.

§ 1. Introduction

Within the past few years various experimental investigations were performed concerning the elastic scattering of alpha-particles[1] and deuterons[2)3)4)] by nuclei. One of the most striking features of these data is the fact that the scattered intensity shows the departure from the Rutherford value beyond a certain angle at a fixed energy. Such a character in the elastic scattering of alpha-particles could be successfully explained for overall nuclei as a result of their interactions with the nuclear optical potential.[5)6)] However, the satisfactory interpretation of the deuteron scattering is so far unknown. It may be supposed that the cause of the departure in the deuteron scattering will not necessarily be similar to that of alpha-particles because of the marked difference of structures between them. However, some evidences[2)] of the elastic scattering of deuterons by light nuclei suggest that the departure from the Rutherford scattering is due to their nuclear interactions (e.g. the existence of the diffraction pattern in the angular distribution). Then a question arises about whether the departure in the deuteron scattering may be explained for overall nuclei by only nuclear interactions of deuterons. In order

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to answer this question it is particularly necessary to study the elastic scattering of deuterons by "heavy nuclei" as a limiting case.

The main purpose of the present paper is to investigate in detail the characters of the elastic scattering of deuterons by heavy nuclei through the analysis of the available data,[3)4)] and especially to clarify whether there exists any essential difference between the mechanisms of the anomalous Coulomb scattering of deuterons and alpha-particles.

Some years ago few theoretical approaches were attempted concerning the elastic scattering of deuterons by heavy nuclei. French and Goldberger[7)] calculated the magnitude of the departure from the Rutherford scattering expected because of the deuteron being a structure with non-coincident centers of mass and charge. They found that this effect was negligible for $Ze^2/\hbar v \gg 1$.

Malenka, Kruse and Ramsey[8)] calculated the effect of the electric polarizability of the deuteron upon its elastic scattering for $Ze^2/\hbar v \gg 1$, and showed that the magnitude of the departure was negligible even at the backward scattering.

Therefore these effects will be useless to account for the considerable departure shown in the available data in which $Ze^2/\hbar v \gg 1$ is satisfied.

With a classical procedure Porter[9)] attempted to explain the deuteron data[3)] at 15.2 MeV by assuming that the departure from the Rutherford scattering is mainly due to the absorption of deuterons by nuclei. However, the best fit to the above data led to an unsatisfactory result that the mean free path of the deuteron near the center of the nucleus is much longer than that of the alpha-particle obtained in the similar manner. The situation will become worse if the effect of the electric breakup of the deuteron is considered. Now, the reason of his failure, I believe, comes from the fact that he treated the nuclear absorption of deuterons and alpha-particles with the similar method. It is especially important to discuss the nuclear absorption of deuterons taking account of their loosely bound characters. In addition, the effect of the electric breakup of the deuteron which was not so far considered should be discussed in detail. Therefore the considerations of these problems constitute the essential part of the present paper.

In section 2 it is discussed which role the diffractional scattering[10)~13)] of deuterons by nuclei plays in the present analysis, and the difference from the scattering of alpha-particles is emphasized. It is also shown that the effect of the nuclear absorption through the diffractional scattering is too small to explain the available data. In section 3 the formal description of the deuteron scattering with the electric breakup is presented. It is shown that such a scattering is equivalent to the scattering by the Coulomb potential Ze^2/r plus the *electric* optical potential (function of r only), where r is the distance between centers of deuteron and nucleus. In section 4 a simple classical model for the deuteron scattering with the electric breakup is proposed, and the best fit to the experimental data is attempted in such a way that the effect of the electric breakup covers the part which could not be explained by the diffractional scattering. The result obtained in such a way shows that this absorptive field is weak enough to be called soft medium for the incident deuteron. In section 5 the differential cross section of the electric breakup along the Coulomb orbit is estimated only for the case of the dipole transition

by the semi-classical method of the virtual quanta modified for the application to the energy region concerned. It is shown that the effect of the higher multipole transitions and the effect of the electric breakup of higher order will be necessary to obtain the cross section of the electric breakup obtained in section 4. Finally, in section 6 the conclusions obtained in the present work are summarized.

§ 2. Diffractional scattering of deuterons by nuclei

In this section we discuss the nuclear interaction of the deuteron with the target nucleus. The available data are concerned with the elastic scattering of 15.2 MeV deuteron[3] by Pb^{208}_{82} and Bi^{209}_{83}, and of 11 MeV deuteron[4] by Ta^{181}_{73}, Au^{197}_{79}, Bi^{209}_{83}, and U^{238}_{92}. Therefore $Ze^2/\hbar v \gg 1$, and the reduced wave-length of the deuteron is much shorter than the nuclear dimensions. Then the center-of-mass motion of the deuteron may be described approximately by the classical Coulomb orbit. In all nuclei under investigation, the apsidal distance corresponding to the largest angle of the observation exceeds the nuclear radius $R=r_0 \cdot A^{1/3}$ (where r_0 is assumed to be 1.33×10^{-13}cm). So we may think that the nuclear interaction of the deuteron takes place when it grazes the nucleus. Now, in order to see what takes place in this encounter we discuss both experiments individually.

(*a*) *Deuteron scattering at* 15.2 MeV

Here the incident deuteron energy is slightly over the barrier height Ze^2/R. In this case the period of the internal motion of the deuteron is fairly longer than the time which the deuteron requires to graze the nucleus. So it has an approximate meaning to speak of the instantaneous positions of the nucleons in the deuteron during the collision with the nucleus. Now we assume the target nucleus as a black sphere for the incident nucleon. Then the decrease of the deuteron intensity by the nuclear absorption might be understood as a result of either neutron or proton in the deuteron being absorbed when the deuteron grazes the nucleus. Here we notice that under such a situation the incident deuteron suffers the diffractional scattering[10)~13)] by the nucleus.

Glauber's procedure[10] for the diffractional scattering can be straightforwardly applied to obtain the probability with which the deuteron survives after it has grazed the nucleus. Assumiug the zero-range force between the neutron and the proton in the deuteron, the probability in question $T_1(\theta)$ for the scattering with the apsidal distance $q(\theta)$ is given by the following expression :

$$T_1(\theta)=1-exp\left[-4\alpha(q-R)\right]-4\alpha(q-R)Ei\left[-4\alpha(q-R)\right] \quad (q>R) \tag{1}$$

where

$$q(\theta)=\frac{Ze^2}{2Mv^2}\left(1+\mathrm{cosec}\,\frac{\theta}{2}\right) \qquad (\theta: \text{scattering angle})$$

$$\alpha=(MB_d/\hbar^2)^{1/2} \qquad (B_d: \text{binding energy of the deuteron})$$

$$Ei(-x)=-\int_x^\infty \frac{\exp(-t)}{t}\,dt \qquad (x>0)$$

The effect of non-zero range of the force between the neutron and the proton in the

deuteron upon the diffractional scattering is negligible.[13] The relation between $T_1(\theta)$ and the scattering angle θ is schematically shown in Fig. 1. Clearly, the effect of the diffractional scattering is too small especially at smaller angles to explain the magnitude of the departure from the Rutherford scattering shown in the experiment (see Fig.4).* Moreover, it will be evident that at smaller angles T_1 is insensitive to the magnitude of the nuclear radius R.

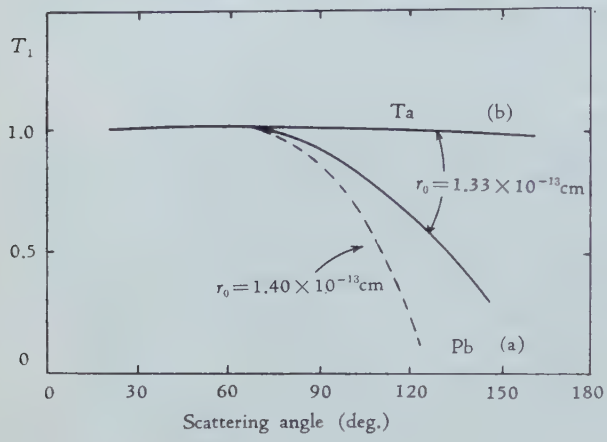

Fig. 1. The schematic representations of $T_1(\theta)$ for the diffractional scattering of (a) 15.2 MeV deuteron by Pb, and of (b) 11 MeV deuteron by Ta.

Finally, it might be pointed out that the behavior of the nuclear interaction of the alpha-particle will be different from that of the deuteron at the intermediate energies. The period of the internal motion of the alpha-particle is fairly short compared with the time which the alpha-particle requires to pass through the distance equal to the nuclear diameter. So there is a good reason to discuss the nuclear absorption of the alpha-particle, as the Porter theory does, using the average quantity like the radius or the interaction radius. Because of the remarkable contrast of the situation, it will be clear in the nuclear interaction of the deuteron that the introduction of the average quantity like the radius or the interaction radius is meaningless.

(*b*) *Deuteron scattering at* 11 MeV

Here the incident deuteron energy is below the barrier height Ze^2/R. So we may think that in this case the nuclear absorption of the deuteron takes place as a result of the only neutron in the deuteron being absorbed by the nucleus when the deuteron grazes the nucleus. Then the probability with which the deuteron survives after it has grazed the nucleus will be instead of (1) given by

$$T_1(\theta)=1-\tfrac{1}{2}\exp[-4\alpha(q-R)]-2\alpha(q-R)Ei[-4\alpha(q-R)] \qquad (2)$$

The magnitude of $T_1(\theta)$ given by (2) is also schematically shown in Fig. 1. Clearly, the result tells us that the effect of the nuclear interaction of the deuteron is quite negligible in the elastic scattering of deuteron at 11 MeV.

As a result of the individual discussion, it is concluded that the departure from the Rutherford scattering shown in both experiments can never be explained by the

* If the effect of the nuclear transparency were considered, the magnitude of the departure by the diffractional scattering would become smaller.

nuclear interaction of the deuteron only. Under such a situation, it will become essential to discuss the effect of the electric breakup of the deuteron which has not so far been considered.

§ 3. Formal description of deuteron scattering with electric breakup

In this section the effect of the electric breakup[14,15] of the deuteron upon its elastic scattering is formally discussed. For the present, the nuclear interaction of the deuteron will be put aside. Such an approach will be justified for 11 MeV deuteron, but it will mean for 15.2 MeV deuteron that the interference between the effects of the electric breakup and the nuclear interaction is neglected.

The Schrödinger equation for the system is

$$\left[-\frac{\hbar^2}{2M}(\Delta_p+\Delta_n)+V+\frac{Ze^2}{r_p}-E_d+B_d\right]\psi_d=0 \tag{3}$$

Here V is the n-p potential, E_d the incident deuteron energy, and B_d is the binding energy of the deuteron.

If the coordinates $\boldsymbol{r}$ and ρ, where $\boldsymbol{r}=\frac{1}{2}(\boldsymbol{r}_n+\boldsymbol{r}_p)$ and $\rho=\boldsymbol{r}_n-\boldsymbol{r}_p$, are introduced, the equation for ψ_d is rewritten as

$$\left[-\frac{\hbar^2}{4M}\Delta_r+\frac{Ze^2}{|\boldsymbol{r}-\frac{1}{2}\rho|}-\frac{\hbar^2}{M}\Delta_\rho+V(\rho)-E_d+B_d\right]\psi_d(\boldsymbol{r},\rho)=0 \tag{4}$$

Now, $\psi_d(\boldsymbol{r},\rho)$ can be expanded in terms of the orthogonal two-particle eigenfunction $\chi_n(\rho)$.

$$\psi_d(\boldsymbol{r},\rho)=\sum_n \chi_n(\rho)\,\Omega_n(\boldsymbol{r}). \tag{5}$$

where

$$\left[-\frac{\hbar^2}{M}\Delta_\rho+V(\rho)-W_n\right]\chi_n(\rho)=0 \quad (W_0=-B_d) \tag{6}$$

From (4) (5) (6), the equations for $\Omega_m(\boldsymbol{r})$ are written as

$$(H-E_d)\,\Omega_0(\boldsymbol{r})=\sum_n\langle 0|U|n\rangle\Omega_n(\boldsymbol{r}) \tag{7}$$

$$(H-E_m)\,\Omega_m(\boldsymbol{r})=\sum_n\langle m|U|n\rangle\Omega_n(\boldsymbol{r}) \quad (m\neq 0) \tag{8}$$

In the above equations the following abbreviations were adopted.

$$H\equiv-\frac{\hbar^2}{4M}\Delta_r+\frac{Ze^2}{r} \tag{9}$$

$$E_m\equiv E_d-W_m+W_0\,. \tag{10}$$

$$\langle m|U|n\rangle\equiv Ze^2\int\chi_m^*(\rho)\left(\frac{1}{r}-\frac{1}{|\boldsymbol{r}-\frac{1}{2}\rho|}\right)\chi_n(\rho)\,d\rho \tag{11}$$

Then, $\Omega_m(\boldsymbol{r})$ satisfying the boundary condition is from (8) written as

$$\Omega_m(\boldsymbol{r}) = -\frac{1}{E_m+i\varepsilon-H}\left[\langle m|U|0\rangle - \sum_{l\neq 0}{}'\langle m|U|l\rangle\frac{1}{E_l+i\varepsilon-H}\langle l|U|0\rangle + \cdots\right]\Omega_0(\boldsymbol{r}) \tag{12}$$

Introducing (12) into (7), the equation for $\Omega_0(\boldsymbol{r})$ is

$$(H-E_d)\,\Omega_0(\boldsymbol{r}) = \left[\langle 0|U|0\rangle - \sum_{m\neq 0}{}'\langle 0|U|m\rangle\frac{1}{E_m+i\varepsilon-H}\langle m|U|0\rangle\right.$$

$$\left.+\sum_{m\neq 0}{}'\sum_{l\neq 0}{}'\langle 0|U|m\rangle\frac{1}{E_m+i\varepsilon-H}\langle m|U|l\rangle\frac{1}{E_l+i\varepsilon-H}\langle l|U|0\rangle - \cdots\right]\Omega_0(\boldsymbol{r}) \tag{13}$$

As mentioned previously, the effect of only the term $\langle 0|U|0\rangle\Omega_0(\boldsymbol{r})$ on the right side of (13) upon the elastic scattering of the deuteron was estimated by French and Goldberger. It turns out to be negligible for the present analysis and it may be dropped out in the following discussion. The other terms on the right side of (13) manifest the effect of the electric breakup of the deuteron owing to the existence of $\langle m|U|0\rangle$ $(m\neq 0)$.

(13) is rewritten as follows:

$$(T_d+V_{eff}-E_d)\,\Omega_0(\boldsymbol{r}) = 0 \tag{14}$$

Here

$$V_{eff} = \frac{Ze^2}{r} + \sum_{m\neq 0}{}'\langle 0|U|m\rangle\frac{1}{E_m+i\varepsilon-H}\langle m|U|0\rangle$$

$$-\sum_{m\neq 0}{}'\sum_{l\neq 0}{}'\langle 0|U|m\rangle\frac{1}{E_m+i\varepsilon-H}\langle m|U|l\rangle\frac{1}{E_l+i\varepsilon-H}\langle l|U|0\rangle + \cdots \tag{15}$$

Note that V_{eff} is not Hermitian as would be expected. Therefore one may be led to the result that the deuteron scattering with the electric breakup is equivalent to the scettering by the Coulomb potential plus the *electric* optical potential. Clearly, this optical potential depends on the incident deuteron energy as well as the charge of the target nucleus. In addition, it will be a decreasing function of r.

However, it is particularly important to mention about the physical significance of this potential. To be sure, it might be that part of the potential which the deuteron feels in the complex processes connected with its electric breakup because of the existence of the term $\langle m|U|0\rangle$ $(m\neq 0)$. The incident deuteron will suffer a virtual electric breakup and then will be recombined to go away as the outgoing deuteron. This corresponds to the process which includes an intermediate state in which the deuteron is virtually broken up. Moreover, there will be higher order processes which include two or more intermediate states in which the deuteron is virtually broken up. The imaginary part of the electric optical potential will show the order of magnitude of the deuteron falling off on the way of these processes. First of all, it will include the effect of the incident deuteron suffering a real electric breakup before being recombined. Moreover,

it will also include the effect of the incident deuteron suffering a real electric breakup through one virtual electric breakup or more. So the real part of this optical potential may be considered as derived from the higher order effect than the imaginary one. Therefore, it will be reasonable, in contrast to the *nuclear* optical potential of Feshbach, Porter and Weisskopf,[16] to think that the magnitude of the real part of this *electric* optical potential will be at most of a comparable order with that of its imaginary part. Moreover, the effect of the real part of the electric optical potential will become fairly obscure in V_{eff} because of the existence of the strong Coulomb potential.*

§ 4. Classical model for deuteron scattering with electric breakup

It was already shown that the deuteron scattering with the electric breakup might be treated in principle by solving the equation (14). In a more approximate treatment of the scattering problem, however, the difficulty of solving (14) can be avoided by noting that $Ze^2/\hbar v \gg 1$ is satisfied and so the classical approach is possible. Moreover, referring to the discussion in the last paragraph of section 3 we can assume that the departure of the deuteron scattering from the Rutherford formula by the electric breakup is interpretable as a result of the absorption of the deuteron by the electric medium along the Coulomb orbit. The distortion of the Coulomb orbit by the electric medium should be considered, but might be neglected if this medium were soft. Fortunately, this neglect will prove to be valid later.

At first, we shall begin from the determination of the extension of the electric medium using the classical approach. The determination of the minimum radius a of this region is straightforward. It is given by the radius corresponding to the classical turning point for 11 MeV deuterons, and for 15.2 MeV deuterons it may be a nuclear radius. For the latter case, the effect of the electric optical potential within the nucleus will become obscure owing to the strong nuclear potential. The determination of the maximum radius b can be done as follows. When the deuteron (mass $2M$) passes by the nucleus of charge Ze with the apsidal distance q and the velocity v, it gains the energy of Z^2e^4/Mv^2q^2 by the Coulomb field of the nucleus.[17] This energy will be transfered to the relative motion of the neutron and the proton in the deuteron. Therefore it is necessary for this energy to exceed the binding energy of the deuteron in order that its electric breakup may be possible. This gives the maximum radius $b=Ze^2/M^{1/2}vB_d^{1/2}$. Then the critical angles at which the departure from the Rutherford scattering begins, *independently of the nuclear charge*, will be 27.4° for 15.2 MeV deuterons and 33.8° for 11 MeV deuterons.** These are consistent with the experimental data. The values of

* As is well known, the meaning of an optical potential will be obscure when the magnitude of its imaginary part is fairly larger than that of its real one. In the present case, however, the separation of the Coulomb potential and the remaining terms in V_{eff} was made only for a convenience of explanation. Exactly speaking, V_{eff} itself is the optical potential and most of its real part will be the Coulomb potential.

** The evaluation of the critical angle was performed under the assumption of the straight line path. However, it will be a good approximation because the critical angle thus obtained is fairly small.

a and b for the nuclei concerned are shown in Table I. For comparison the values of the observed critical apsidal distance for elastic scattering of alpha-particles are also shown in Table I.

Table I. The calculated values of a and b for the deuteron scattering at 11 MeV and 15.2 MeV and the corresponding values of the observed critical apsidal distance b for the alpha-particle scattering at 22 MeV.

target nuclei	calculated values for deuteron scattering		observed values for α-particle scattering
	a	b	b †)
Ta_{73}^{181}	9.56	21.22	12.2
Au_{79}^{197}	10.35	22.97	~13
Pb_{82}^{208}	7.88*	20.31*	
Bi_{83}^{209}	10.87	24.13	13.1
U_{92}^{238}	12.05	26.75	14.5

Each value is written with a unit of 10^{-13}cm. The letters with the asterisk represent the values for 15.2 MeV deuterons.
†) The data for the alpha-particle scattering were taken from the experiment by Rees and Sampson.

Clearly, each value of the critical apsidal distance for the deuteron scattering is fairly larger than the corresponding one for the scattering of alpha-particles. Moreover, it is fairly larger than the magnitude of the interaction radius of the deuteron. These facts will establish the difference between the mechanisms of the anomalous Coulomb scattering of the deuteron and the alpha-particle.

Next, we shall proceed to obtain the "absorption coefficient" for the deuteron within the electric medium in question. Introducing the "mean free path" $l(x)$ along the Coulomb orbit (where x is a path coordinate), the magnitude of the departure from the Rutherford scattering by the electric breakup is written as

$$T_2(\theta)=\exp\left[-\int\frac{dx}{l(x)}\right]. \tag{16}$$

To calculate $T_2(\theta)$, we use the oversimplified model in which the Coulomb orbit is replaced by the straight line path with the same apsidal distance. Without this approximation, the subsequent calculations would be obliged to be performed by the numerical integrations.

Then, in the similar way to Porter's procedure[9] applied to the nuclear absorption of deuterons, we obtain the following expression:

$$\int\frac{dx}{l(x)}=2\int_q^b\frac{dx}{dr}\frac{1}{l(r)}dr=2\int_q^b\frac{r}{\sqrt{r^2-q^2}}\frac{1}{l(r)}dr \tag{17}$$

Here $q(\theta)$ is the apsidal distance corresponding to the scattering angle θ. In order to find the detailed form of $l(r)$, we should compare the total magnitude $T(\theta)$ of the departure from the Rutherford scattering by the nuclear absorption and the electric breakup, i.e.

$$T(\theta)=T_1(\theta)\cdot T_2(\theta) \tag{18}$$

(See (1) (2) (16)) with the experimental magnitude.

The fits to the experimental curves were performed for the several probable forms of $l(r)$.

As the result, when the absorption coefficient was chosen as

$$l(r)^{-1}=c(b-r)^3 \qquad a\leq r\leq b \tag{19}$$

(where c is an adjustable parameter), the fairly good fits could be obtained. The details are shown in Figs. 2—6.* Moreover, the relations between the absorption coefficients and the distance from the nucleus were shown in Fig. 7 for the nuclei concerned. Fig. 7 shows surely that in the region far from the nucleus the absorption coefficients

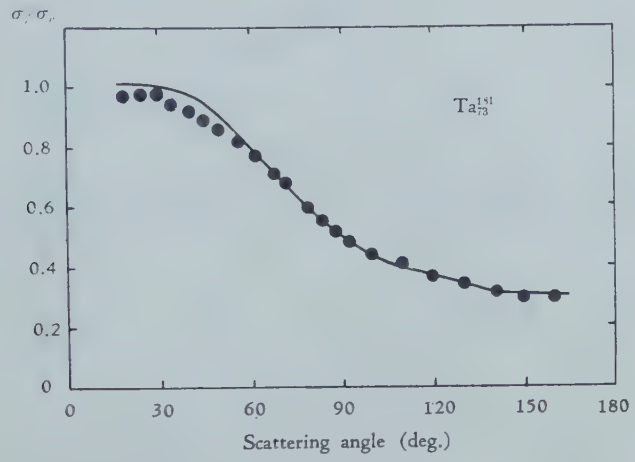

Fig. 2. Angular distribution of the elastic scattering of 11 MeV deuteron by Ta. •••••• experimental points; ——— theoretical curve. $c=0.51\times10^{48}\mathrm{cm}^{-4}$.

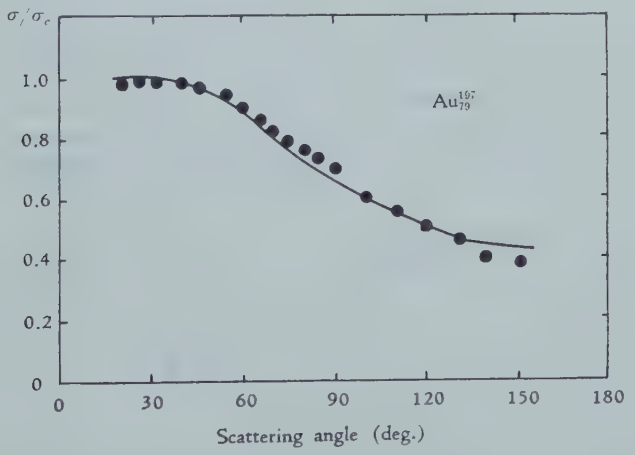

Fig. 3. Angular distribution of the elastic scattering of 11 MeV deuteron by Au. •••••• experimental points; ——— theoretical curve. $c=0.25\times10^{48}\mathrm{cm}^{-4}$.

* It will be difficult to determine to what extent the effect of the nuclear transparency is necessary for the diffractional scattering of 15.2 MeV deuterons by Pb, because the present approximation will be less accurate at larger angles.

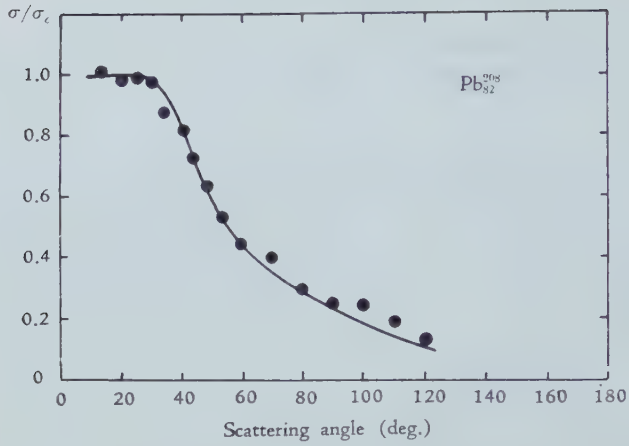

Fig. 4. Angular distribution of the elastic scattering of 15.2 MeV deuteron by Pb. •••••• experimental points; ——— theoretical curve. $c=0.77\times10^{48}\mathrm{cm}^{-3}$.

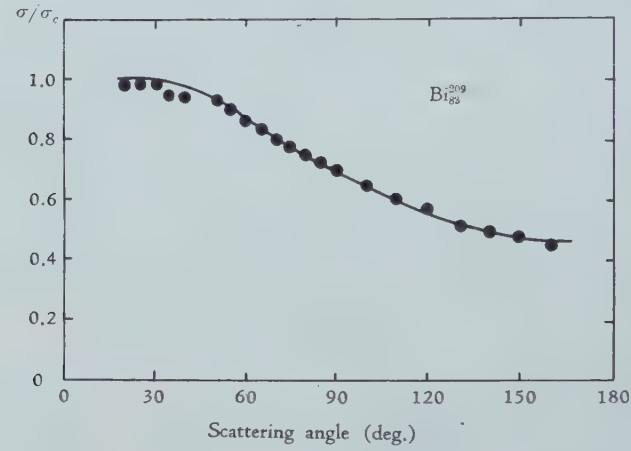

Fig. 5. Angular distribution of the elastic scattering of 11 MeV deuteron by Bi. •••••• experimental points; ——— theoretical curve. $c=0.18\times10^{48}\mathrm{cm}^{-3}$.

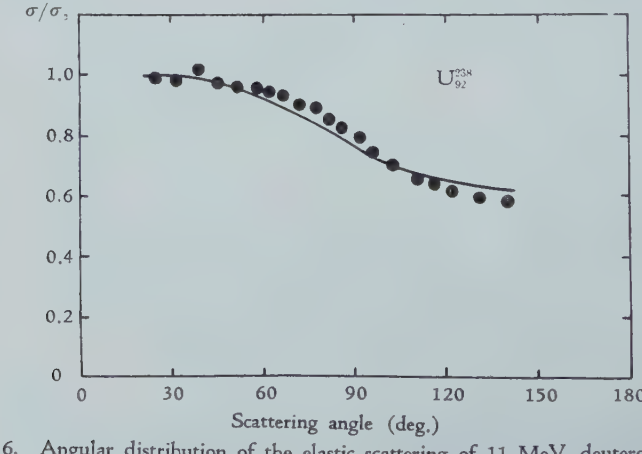

Fig. 6. Angular distribution of the elastic scattering of 11 MeV deuteron by U. •••••• experimental points; ——— theoretical curve. $c=0.08\times10^{48}\mathrm{cm}^{-3}$.

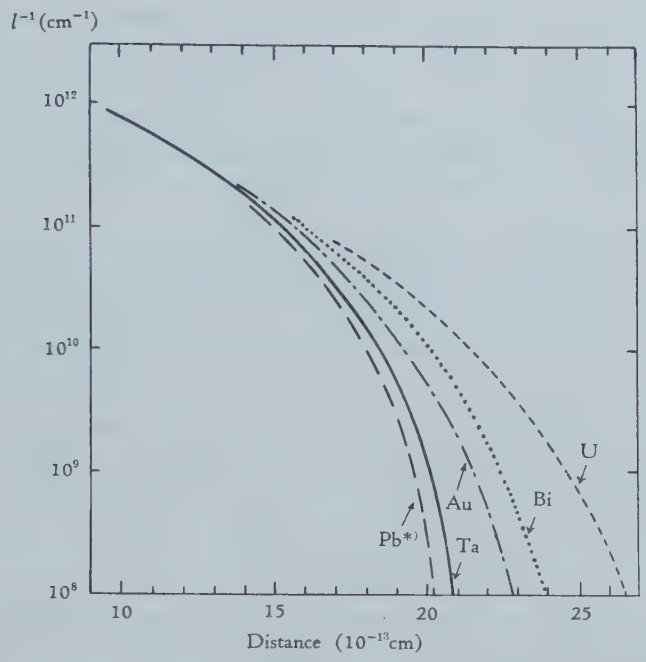

Fig. 7. The schematic representations of the absorption coefficients as functions of r for the deuteron scattering at 11 MeV and 15.2 MeV.
*) represents the curve at 15.2 MeV.

increase with the increase of the nuclear charge and the decrease of the incident deuteron energy. In the region near the nucleus, however, the present calculation is not so accurate to find such a definite conclusion as in the region far from the nucleus, because the approximation of the straight line path will be less accurate there. Therefore only a representative curve was shown in that region in Fig. 7. However, it will be clear that the mean free path is much longer than the nuclear dimensions even near the nucleus and the electric medium may be quite enough to be called soft.

Here it is very interesting to compare the magnitudes of both effects which bring the departure from the Rutherford scattering. Since the departure at smaller scattering angles proved to be mainly due to the electric breakup of the deuteron, it will be clear that for the energy region concerned the cross section of the electric breakup is much larger than that of the nuclear absorption. This fact would confirm the prediction by Mullin and Guth[15] that in the collisions of 15 MeV deuterons with "heavy" nuclei the cross section of the electric breakup may be larger than that of the stripping reaction.

Finally, the validity of the approximation of the straight line path must be discussed. To be sure, this approximation can be criticized at larger scattering angles. However, we may realize that the absorption coefficient (19) obtained with this approximation is a rapidly decreasing function of r, and also $\lim_{r\to\eta} dx/dr=\infty$ is satisfied for the straight line path as well as for the Coulomb orbit. So, the main contribution to the integral (17) may be considered as coming from the neighborhood of the apsidal distance.

The approximation of the straight line path will not, therefore, be so rough even at larger scattering angles.

§ 5. Calculation of the electric breakup cross sections by the semi-classical method of the virtual quanta and its limitation

In section 4 we regarded the majority of the departure from the Rutherford scattering as due to the electric breakup of the deuteron. In this section we shall proceed to study how far the theoretical approach can explain the above result for the electric breakup. For a while, therefore, we shall put aside the effect of the diffractional scattering.

Some of the deuterons which pass by the nucleus along the classical Coulomb orbit will break up due to the Coulomb force of the nucleus. Consequently, the cross section of the elastic scattering will be smaller by the cross section of the electric breakup than that of the Rutherford scattering. With $q(\theta)$, the departure from the Rutherford scattering due to the electric breakup will be "classically" given by

$$T_2(q) = 1 - \sigma_b(q) / \sigma_c(q) \tag{20}$$

Here $\sigma_c(q)\,dq$ is the differential cross section within $(q, q+dq)$ of the Rutherford scattering and, as is well known, is given by

$$\sigma_c(q)\,dq = 2\pi\,(q - Ze^2/2Mv^2)\,dq. \tag{21}$$

Therefore it is necessary to calculate the differential cross section $\sigma_b(q)\,dq$ of the electric breakup.

As in section 4 we use the oversimplified model* in which the Coulomb orbit is replaced by the straight line path with the same apsidal distance. When the deuteron passes by the nucleus with the impact parameter q, the emission probability $p(\omega)\,d\omega$ of the Brehmsstrahlung with the frequency $(\omega, \omega+d\omega)$ is given by

$$p(\omega)\,d\omega = \frac{\pi Z^2 e^6}{12\hbar c^3 M^2 q^2 v^2} F(\omega q/v)\,d\omega/\omega \tag{22}$$

Here

$$F(x) = 2[f(x)^2 + g(x)^2], \quad \int_0^\infty F(x)dx = 1. \tag{23}$$

$$f(x) = -xH_1(ix), \quad g(x) = ixH_0(ix). \tag{24}$$

These functions are schematically shown in Fig. 8.

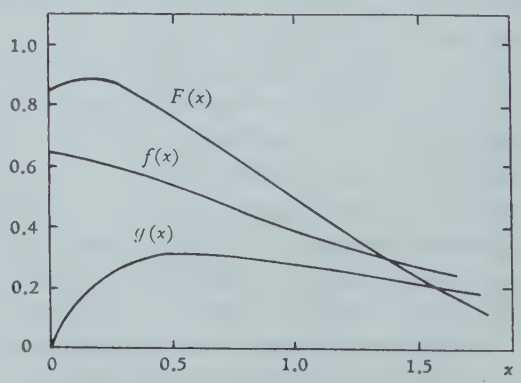

Fig. 8. The schematic representations of $F(x)$, $f(x)$, $g(x)$ as functions of x.

* In practice, this approximation does not seem to be so rough even at larger scattering angles, because the main contribution to the following calculation comes from the neighborhood of the apsidal distance.

The frequency distribution of the Brehmsstrahlung $p(\omega)d\omega$ can also be derived with the concept of the virtual quanta.[15)17)] Since the velocity of the deuteron is fairly small compared with the light velocity, the change of the frequency due to the Doppler effect may be neglected at the transformation from the system referred to the nucleus to the system referred to the deuteron. Moving nucleus acts for the deuteron as the electromagnetic pulse. The virtual photons correspond to this pulse. The production of the Brehmsstrahlung can be understood as a result of the virtual photons having changed to the real photons due to the Thomson scattering by the deuteron. With $N(\omega\,;\,q)d\omega$, the number of virtual photons per unit area with the frequency $(\omega,\ \omega+d\omega)$ at the impact parameter q, the probability of the virtual photons suffering the Thomson scattering by the deuteron is given by

$$\sigma_T \cdot N(\omega\,;\,q)\,d\omega \tag{25}$$

Here σ_T is the total cross section of the Thomson scattering for the deuteron and is given by

$$\sigma_T = 8\pi/3 \cdot (e^2/2Mc^2)^2. \tag{26}$$

Therefore we obtain the following relation :

$$8\pi/3 \cdot (e^2/2Mc^2)^2 \cdot N(\omega\,;\,q)\,d\omega = \frac{\pi Z^2 e^6}{12\hbar c^3 M^2 q^2 v^2} F(\omega q/v)\,d\omega/\omega. \tag{27}$$

With $\sigma(\omega)$, the cross section of the photodisintegration of the deuteron by the photons of energy $\hbar\omega$, the cross section of the electric breakup within $(q,\ q+dq)$ is given by

$$\sigma_b(q)\,dq = 2\pi q dq \int_{B_d/\hbar}^{\infty} \sigma(\omega)\,N(\omega\,;\,q)\,d\omega \tag{28}$$

With correspondence to the derivation of (22), we consider only the dipole transition for the photodisintegration of the deuteron. With zero range approximation, the cross sections for the electric and the magnetic dipole transitions are respectively given by[18)]

$$\sigma^{ed} = \frac{8\pi}{3} \cdot \frac{e^2}{\hbar c} \gamma^{-2} \left(\frac{k\gamma}{k^2+\gamma^2} \right)^3 \tag{29}$$

$$\sigma^{md} = \frac{2\pi}{3} \cdot \frac{e^2}{\hbar c} \left(\frac{\hbar}{Mc} \right)^2 (\mu_n - \mu_p)^2 \frac{k\gamma (1-\gamma a_s)^2}{(k^2+\gamma^2)(1+k^2 a_s^2)}. \tag{30}$$

By the numerical integration (28) was calculated for the scattering of 15.2 MeV deuterons by Pb. Main contribution to the integral comes from the part of ω near the threshold of the photodisintegration. The numerical values used in the calculation were taken from the text of Blatt and Weisskopf.[18)] The magnitude $T_2(\theta)$ thus obtained is compared with the corresponding one obtained in section 4 in Fig. 9. It will be observed that there is a considerable difference between them. The consideration of the effective range in (29) (30) leads only to the result that $T_2(\theta)$ is lowered by several percent than zero range value. It will be improbable that most part of this difference will come

from the approximation of the straight line path, because this approximation is used in both calculations of section 4 and section 5, and in addition may be valid at smaller scattering angles.

Most part of this difference, we suppose, may be due to the effect of the higher multipole transitions and the effect of the electric breakup of higher order. In order to investigate the effect of the higher multipole transitions, it is necessary to calculate the corresponding number of the virtual quanta. However, it seems to be difficult to calculate it using the semi-classical method similar to the case of the dipole transition.

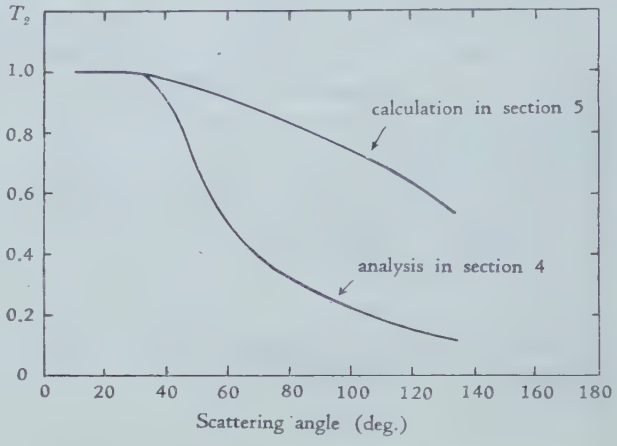

Fig. 9. Comparison of $T_2(\theta)$ between the analysis in section 4 and the calculation in section 5 for the scatttering of 15.2 MeV deuteron by Pb.

The latter effect must be considerd in connection with the discussion of section 3. To be sure, the present calculation has been performed only for the direct electric breakup. The effect of the process which includes one virtual electric breakup or more has not been considered. With respect to this point, therefore, the present calculation will only give the first approximation to the effect of the electric breakup.

§ 6. Concluding remarks

We have pursued the mechanism of the elastic scattering of deuterons by heavy nuclei using the available data at 11 MeV and 15.2 MeV. It was shown that for the deuteron scattering at 11 MeV the departure from the Rutherford scattering can be completely attributed to the electric breakup of the deuteron. Moreover, it was shown that for the deuteron scattering at 15.2 MeV the main contribution to the departure from the Rutherford scattering comes from the effect of the electric breakup of the deuteron, although at larger scattering angles the effect of the nuclear absorption in the diffractional scattering of the deuteron must be considered.

Especially, it might be noticed that the critical angle at which the departure from the Rutherford scattering begins, can be straightforwardly derived from a simple classical consideration associated with the electric breakup. This critical angle depends only on the incident deuteron energy, and does not on the nuclear charge.

New electric medium which causes the electric breakup of the deuteron was introduced and the "mean free path" of the deuteron in the region was determined. We were led to the definite conclusion that in the region far from the nucleus, the "absorption coefficient" increases with the increase of the nuclear charge and the decrease of the

incident deuteron energy.

Calculation of the magnitude of the electric breakup by the semi-classical method of the virtual quanta modified for the application to the present case was performed to compare with the result obtained from the analysis. The result shows that the effects of the higher multipole transitions and the electric breakup of higher order should be considered.

In conclusion, it was found that the mechanism of the elastic scattering of deuterons by heavy nuclei is quite different from those of the elastic scattering of alpha-particles by nuclei and of deuterons by light nuclei.

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Low Levels of Some Nuclei in the $d_{3/2}$—$f_{7/2}$ Shell

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Excitation energies of some low states of the nuclei Cl^{35}, A^{37}, Cl^{39} and K^{41} are calculated in the *jj* coupling scheme, independently of the details of the two-body nuclear interaction. Some pertinent experiments are suggested to check the calculations.

§ 1. Introduction

The properties of the low lying states of some nuclei in the region A∼40 have recently been analysed in terms of the *jj* coupling shell model.[1),2)] These investigations, based only on the assumption of the angular momentum coupling scheme and free of the uncertainties inherent in the knowledge of the two-body nuclear forces,* show that the low states of these nuclei have fairly pure *jj* coupling wave functions. In view of these results, it is desirable to analyse the low levels of other nuclei in this region also by the use of similar techniques.

It has been shown[3)] that with suitable assumptions* the excitation energies of the levels of a configuration $(\pi j_1)^{n_1}\ (\nu j_2)^{n_2}$, where π and ν denote the proton and the neutron configurations respectively, can be expressed in terms of the energies of the levels of the simple two-particle configurations $(j_1)^2$, $(j_2)^2$, and $(\pi j_1)\ (\nu j_2)$. Where the latter level energies are experimentally known, we may use them directly to evaluate the level scheme of $(\pi j_1)^{n_1}\ (\nu j_2)^{n_2}$ instead of using the unknown two body nuclear interaction. Such a calculation then depends only on the assumption of the *jj* coupling scheme. This approach has been used to evaluate the binding energies of various nuclei[4)] and also to study the level scheme of $(f_{7/2})^{\pm 2}$, $(f_{7/2})^{\pm 3}$ configurations.[5)] We present here further results for some nuclei in the $d_{3/2}$—$f_{7/2}$ shells in the hope that they may stimulate some experiments to study these level schemes.

§ 2. Results

In the framework of the simple *jj* coupling scheme the low levels of the nuclei Cl^{35}, A^{37}, Cl^{39}, and K^{41} may be represented in terms of the configurations $(d_{3/2})\ (d_{3/2})^2$,

* Such analysis, however, does make the reasonable assumption, as we shall also do in this paper, that the parameters of the single particle potential well, as well as those of the two body nuclear interaction, are the same for all nuclei in this region. Also, many-body forces are not considered.

$(d_{3/2})^2$ $(d_{3/2})^{-1}$, $(d_{3/2})$ $(f_{7/2})^2$, and $(d_{3/2})^{-1}$ $(f_{7/2})^2$, where the proton and the neutron configurations are indicated in separate brackets. It also follows that such levels of Cl^{35} and A^{37} should have identical excitation energies. In our approach[3] (where neutrons and protons are considered separately, i. e. isotopic spin formalism is not used), the energies of the levels of spin J of the configurations $(j_1)^{\pm 1}(j_2)^2$ may be obtained as eigenvalues of the matrix

$$E[(j_1)^{\pm 1}(j_2)^2 : J]_{xx'} \text{ given by,}$$

$$E[(j_1)^{\pm 1}(j_2)^2 : J]_{xx'} = E[(j_2)^2 : x]\delta_{xx'}$$

$$+2[(2x+1)(2x'+1)]^{1/2}\sum_y (2y+1) W(j_2 j_2 J j_1 : xy) W(j_2 j_2 J j_1 : x'y) E[(j_1)^{\pm 1}(j_2) : y] \tag{1}$$

where $W(abcd : ef)$ denotes a Racah coefficient, and $E[x]$, $E[y]$ denote the excitation energies of the levels of spin x and y respectively, of the indicated configurations.

For the purpose of our calculations, we take the energies $E[(d_{3/2})\ (d_{3/2}) : y]$ and $E[(d_{3/2})^{\pm 1}\ (f_{7/2}) : y]$ to be given by the lowest four levels of Cl^{34}, Cl^{38} and K^{40} respectively.* For $E[(f_{7/2})^2 : x]$ we consider only $x=0,2$ and take these energies as 0 and 1.5 Mev from the known levels of Ca^{42}. The effect of neglecting $x=4$, 6 is discussed in the next section.

The results thus obtained for Cl^{35}, A^{37}, Cl^{39} and K^{41} are summarised in Table I.

Table I. Levels of the configurations $(d_{3/2})$ $(d_{3/2})^2$, $(d_{3/2})^{\pm 1}$ $(f_{7/2})^2$ as obtained from equation (1)

J	Cl^{35}, A^{37} $E[(d_{3/2})\ (d_{3/2})^2]$ Mev.	Cl^{39} $E[(d_{3/2})\ (f_{7/2})^2]$ Mev.	K^{41} $E[(d_{3/2})^{-1}\ (f_{7/2})^2]$ Mev.
3/2	0	0	0
1/2	3.4	2.3	1.0
3/2	3.6	2.1	2.1
5/2	2.6	1.4	2.2
7/2	2.0	2.2	1.8

§ 3. Discussion

Cl^{35}, A^{37}: Very little is known about the spins and parities of the excited states of these nuclei. In Cl^{35} the first excited state occurs at 1.2 Mev., and about eight more states are reported between 1.75 and 4.2 Mev.[6] Below excitation energy of ~ 4 Mev, one may expect in Cl^{35} besides the states of $(d_{3/2})$ $(d_{3/2})^2$ configuration, additional states due to excitation of a nucleon from the $s_{1/2}$ shell to the $d_{3/2}$ shell, as well as odd parity

* For details see S. P. Pardya; reference 2). Though the spin assignments $J=1$ in Cl^{34}, $J=0$, 1, 3 in Cl^{36} and $J=3$, 4 in Cl^{38} have not been experimentally confirmed, it appears safe to assume them as correct. In this connection, it would be very valuable to obtain the spins and energies of the lowest four levels of K^{38}, which on the basis of jj coupling should be identical with those of Cl^{34}.

states due to excitation of a $d_{3/2}$ nucleon into the $f_{7/2}$ or the $p_{3/2}$ shell. For excitation energy $\gtrsim 2$ Mev, configuration interaction would be expected to cause some displacements of the levels from their calculated positions. Further experiments Cl^{35} $(pp'\gamma)$ or $S^{34}(p\gamma)$ and determination of spins and parities of these levels would help very much in deciding the degree of validity of jj coupling here.

The energy levels of A^{37} are also as yet not precisely known.[7] The present evidence appears insufficient to make a suitable comparison between the levels of A^{37} and Cl^{35}. Experiments such as $A^{36}(dp)$ or $A^{36}(n\gamma)$ would be very desirable.

Cl^{39}: No information is available regarding the level scheme of this nucleus.

K^{41}: In calculating the low levels of Cl^{39} and K^{41} we neglected the $J=4,6$ states of $(f_{7/2})^2$ configuration. Since these states lie at energies $\gtrsim 2.5$ Mev, they would not affect very much the calculated energies; in any case, their effect would be only to depress slightly the levels considered here.

We should also note that since the energy levels of the configurations $(f_{7/2})^2$ and $(f_{7/2})^4$ should be identical in jj coupling,* the low-lying energy levels of K^{43} should be identical to those of K^{41}. While levels in K^{43} have been observed at 0.65 and 1.20 Mev,[7),8)] levels in K^{41} at 0.6 and 1.4 Mev have also been reported. The results of the $K^{40}(n\gamma)$ experiment indicate γ rays from the capture state to the 0.6 and 1.6 Mev states. Our calculation would then suggest the spins of these states to be $7/2^+$ and $5/2^+$ respectively. Further careful experiments $A^{40}(dn)$ and $K^{40}(n\gamma)$ would be very useful.

Recently the collective model has been applied successfully to the nuclei $A=25$, 28, 29 and collective effects have been suggested to occur in the nuclei $A=31,33$.[9] In view of this it would be desirable to check carefully the validity of jj coupling for nuclei $A=34-40$. An experimental check of the results of this paper may provide one way to obtain some information on this problem.

Part of this work was done during the author's stay at the University of Rochester. I am indebted to Dr. French for several helpful discussions.

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* $J=2$ state in Ca^{44} is at 1.15 Mev, compared to 1.5 Mev for $J=2$ state in Ca^{42}. We also note that in Ca^{42} and Ca^{44} a level at 1.84 Mev is observed which clearly does not belong to the $(f_{7/2})^2$ configuration. We do not know yet the implication of this level on the low levels of K^{41}, K^{43}.

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On the Form Factor of Liquid He^4 at Absolute Zero II,

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A refinement of the approximations made in I is carried out and it is shown that the contradictions concerning the behaviours of the form factor $S(k)$ near at $k=0$ involved in the previous treatment can be removed. An integral equation for $S(k)$ is solved numerically for the hard-sphere potential and for the modified Lennard-Jones potential. These solutions are compared with the experimental data. Under our approximations, the cohesive energy is shown to be expressed as a function of the sound velocity and $S(k)$, and is calculated using the theoretical and the experimental value of $S(k)$, respectively. The method is applied to a dilute hard-sphere system and the results are compared with those corresponding to exact one.

§ 1. Introduction

In the previous paper, I,[1] a theory of the form factor of liquid He^4 at absolute zero was developed. There we have encountered with the difficulties associated with the normalization for the radial distribution function or with the behaviour of the form factor near at the origin, on account of the inappropriateness of the approximations employed. In this paper we shall undertake a refinement of the method in order to remove the contradictions mentioned above.

In § 2 we introduce the chain approximation,[2] familiar in the theoretical treatment of the classical liquids, in connecting the wave function with the radial distribution function (this approximation together with the pair and superposition approximation will be denoted briefly as PSCA). Under PSCA an integro-differential equation for the radial distribution function which generalizes the one given in I is obtained. It is shown that the form factor $S(k)$ should be a linear function of k near at $k=0$, corresponding to the phonon excitation in liquid He^4. In § 3 the integral equation for $S(k)$ is obtained and in § 4 it is solved numerically for the hard-sphere potential. In § 5 the method is generalized to the more realistic potential which includes the attractive forces besides the hard-core repulsion and $S(k)$ is calculated in the same manner as in § 4. It can be shown that the inclusion of the attractive forces favours to reduce the discrepancy between the theoretical value of $S(k)$ and the experimental one. In § 6 an expression for the ground state energy is given under PSCA, which provides a simple functional relation of the energy with the sound velocity and $S(k)$. We calculate this for the calculated and experimental $S(k)$, respectively. In § 7 PSCA is applied to a dilute hard-sphere system and results are compared with those corresponding to exact one.

Throughout this paper we shall measure the distance in units of a (a: some standard

measure of the distance, in particular the hard-sphere diameter for the hard-sphere potential) and the wave number in $1/a$, unless the particular notice is given. Notations are the same as were used in I.

§ 2. Euler's equation in PSCA

In I we have derived Euler's equation for the radial distribution function under the following three assumptions: (1) pair approximation, (2) superposition approximation, (3) a simple functional relation connecting $g(x)$ with $u(x)$. There, however, we have encountered with some difficulties on account of the assumption (3). In this section we wish to refine this assumption by using the chain approximation[2] and show that the difficulties involved in I can be removed.

As we have remarked in I, the expectation value of the kinetic energy can be written in a form of the density expansion under the assumptions (1) and (2) mentioned above. From eq. (2·3b) of I, the coefficient of ρ^n is proportional to

$$\int \{\Gamma_1 g(1,2)\cdot\Gamma_1 g(1,n+2)\}\, g(n+1,n+2) \prod_{i=3}^{n+1} \{g(1,i)\, g(i-1,i)\}\, dr_3 dr_4 \cdots dr_{n+2}.$$

This integral is a kind of the cluster integrals (see Fig. 1). Since it is expected that g approaches 1 when two particles are far apart from each other say by several times a, and that there may be few chances that all particles come nearer to the particle 1 at once, g connecting 1 with the other particles may be approximated by 1 without serious errors.

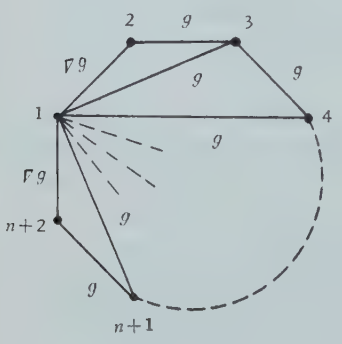

Fig. 1

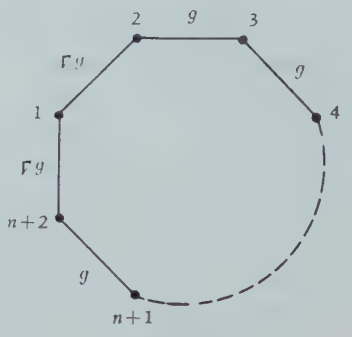

Fig. 2

In other words, the cluster shown in Fig. 1 is replaced by a chain cluster shown in Fig. 2. This approximation is the chain approximation which we shall employ in the following. Then the power series in terms of the density, eq. (2·3b) of I, can be summed up in a closed form and the expectation value of the Hamiltonian becomes under PSCA

$$\langle H'\rangle = \int_0^\infty g'^2/g\cdot x^2\,dx + (4ma^2/\hbar^2)\int_0^\infty V(x)\, g(x)\, x^2\,dx - 8\pi^3\rho a^3\int_0^\infty k f^3(k)/S(k)\cdot dk. \qquad (2\cdot 1)$$

The derivation of this equation is given in the appendix.

If we take the variation of $g(x)$, we have (see appendix)

$$\frac{2}{\sqrt{g}}\frac{d^2}{dx^2}(x\sqrt{g})-\frac{2ma^2}{\hbar^2}xV(x)+4\pi^2\rho a^3\int_0^\infty\frac{kf^2(k)}{S^2(k)}\{1+2S(k)\}\sin kx\,dk=0, \quad (2\cdot2)$$

which is Euler's equation for $g(x)$ in PSCA.

Now we consider the hard-sphere system and split $\varphi(x)$ into two parts as we have done in eq. (3·5) of I. Further we approximate for $|x|>1$

$$\frac{2}{\sqrt{g(x)}}\frac{d^2}{dx^2}(x\sqrt{g(x)})\doteqdot\varphi_1''(x). \quad (2\cdot3)$$

This approximation together with PSCA will be denoted as PSCA′.

In PSCA′ we have from eq. (2·2)

$$\varphi_1(x)=0, \quad |x|<1 \quad (2\cdot4a)$$

$$-\varphi_1(x)+2\pi^2\rho a^3\int_{-\infty}^{\infty}\frac{f^2(k)}{kS^2(k)}\{1+2S(k)\}\sin kx\,dk=A\pi+Bx, \quad |x|>1 \quad (2\cdot4b)$$

where the integral over k is extended to the negative side [$f(-k)=-f(k)$, $S(-k)=S(k)$] and A, B are the integral constants. It can be seen easily that $B=0$ if one considers the limit $x\to\infty$. However A is not always zero.

In order to clarify the physical meaning of A, we consider the limiting case $x\to\infty$. By using the equation

$$\lim_{x\to\infty}\sin kx/k=\pi\delta(k),$$

we find

$$A=(1/8\pi^2\rho a^3)\{k^2[1-S(k)]^2[1+2S(k)]/S^2(k)\}_{k=0}.$$

From this equation it follows that $A=0$ if $S(0)=$const., $A=$finite if $S(k)\sim k$, $A=\infty$ if $S(k)\sim k^2$. On the other hand, according to Feynman,[3] $S(k)$ is given by

$$S(k)\simeq\hbar k/2mca \quad (2\cdot5)$$

at small k, where c is the sound velocity. From eq. (2·5) it follows

$$A=(3/2\lambda\pi)(2mca/\hbar)^2 \quad (2\cdot6)$$

where λ was defined by eq. (3·14) of I.

From the above considerations it can be seen that $S(0)=0$ and $S(k)\sim k$ at small k in PSCA, which removes the contradiction we have encountered in I. The constant A or the sound velocity is determined by the condition that $g(1)=0$. If the $g(x)$ in the denominator of the first term in eq. (2·2) is approximated by 1, we have after the integration of eq. (2·2) by x twice,

$$c^2=(4\hbar^2\pi\rho a/m^2)+(\hbar^2/2\pi m^2 a^2)\int_0^\infty k\{S(k)-1\}^2\{1+2S(k)\}/S^2(k)\cdot\sin k\,dk \quad (2\cdot7)$$

for the hard-sphere potential.

§ 3. Integral equation for $S(k)$

Eqs. (2·4a) and (2·4b) can be united into a single integral equation, as we have done in I. By using

$$\int_{-\infty}^{\infty}\frac{\cos k\sin kx}{k}dk=\pm\pi,\quad |x|>1\quad (\pm \text{ is according to } x\gtrless 0) \qquad (3\cdot1a)$$

$$=0,\quad |x|<1$$

$$\int_{-\infty}^{\infty}\frac{\sin k\cos kx}{k}dk=\pi,\quad |x|<1 \qquad (3\cdot1b)$$

$$=0,\quad |x|>1$$

we have an equation for $f_1(k)$:

$$-f_1(k)+2\pi^2\rho a^3\frac{f^2(k)}{kS^2(k)}\{1+2S(k)\}=\frac{2\pi^2\rho a^3}{\pi}\int_{-\infty}^{\infty}\frac{f^2(k')\{1+2S(k')\}}{k'S^2(k')}$$

$$\times\frac{\sin(k-k')}{k-k'}dk'+A\frac{\cos k}{k}. \qquad (3\cdot2)$$

If we set the $S(k)$ in the denominators to be 1, and assume $1+2S(k)=3$, $A=0$, eq. (3·2) is reduced to eq. (3·12) of I. The essential difference between them lies in the fact that there appears $S(k)$ in the denominators in eq. (3·2), which makes the analytic treatment of the equation difficult.

Substituting eqs. (3·8), (3·9) and (3·10) of I in eq. (3·2) we have

$$\frac{1}{S^2(k)}=\frac{1}{\pi k}\int_{-\infty}^{\infty}\frac{k'\{1-S(k')\}^2\{1+2S(k')\}}{S^2(k')}\frac{\sin(k-k')}{k-k'}dk'+1-\frac{2\lambda}{3}G(k)$$

$$+(2mca/\hbar)^2\cos k/k^2, \qquad (3\cdot3)$$

which is the integral equation for $S(k)$ in PSCA′. Here λ and $G(k)$ are given by

$$\lambda=12\rho\pi a^3$$

$$G(k)=(k\cos k-\sin k)/k^3.$$

§ 4. Numerical solution of the integral equation

In § 3 we have derived an equation for $S(k)$. In this section the numerical solution of an equation will be discussed.

If we transform the integration in eq. (3·3) from 0 to ∞ and define $K(k, k')$ and $\gamma(k')$ in the following manner,

$$K(k, k') = \sin(k-k')/(k-k') - \sin(k+k')/(k+k'), \qquad (4\cdot1)$$

$$\gamma(k') = k'\{1-S(k')\}^2\{1+2S(k')\}/S^2(k'), \qquad (4\cdot2)$$

eq. (3·3) is reduced to

$$1/S^2(k) = (1/\pi k)\int_0^\infty K(k, k')\gamma(k')\,dk' + 1 - (2\lambda/3)G(k) + (2mca/\hbar)^2\cos k/k^2. \qquad (4\cdot3)$$

In solving this equation we assume that $\gamma(k')$ for $k' > 6A^{-1}$ is zero. The reason for this assumption lies first in the fact that the experimental form factor is not observed for $k' > 6A^{-1}$ which denies the comparison with the calculated one, and secondly in the fact that the preliminary calculations of $\gamma(k)$ by the use of $S(k)$ obtained in I show that $\gamma(k)$ for $k > 6A^{-1}$ is negligibly small and hardly gives contributions to the integral in eq. (4·3). The oscillating behaviour of the kernel $K(k, k')$ is favourable for justifying this neglect.

As we have mentioned in § 2, the sound velocity should be determined by the requirement that the $\eta(x)$ is zero at $x=1$. The calculations taking account of this requirement are extremely tedious, hence we shall carry out the computation substituting the observed value for the sound velocity in eq. (4·3).

With these remarks in mind, we express the integral in eq. (4·3) by means of Simpson's formula deviding the region $0 \leq k \leq 6A^{-1}$ into 24 parts at the interval $0.25\,A^{-1}$. Then eq. (4·3) is reduced to

$$1/S_i^2 = (1/\pi k_i)\sum_{j=0}^{24} K_{ij}\gamma_j + f_i, \quad (i=1, \cdots, 24), \qquad (4\cdot4)$$

which is a set of 24-dimensional non-linear simultaneous equations. Here K_{ij} and f_i are the known numbers. Further the integrand in eq. (4·3) corresponding to $j=0$ is the known number, as can be seen from the following equation

$$[K(k, k')\gamma(k')]_{k'=0} = 2(2mca/\hbar)^2(\sin k - k\cos k)/k^2.$$

Instead of solving the integral equation (4·3), we solve eq. (4·4) by an iteration procedure. In order to make the iteration procedure easier, we solve first the subset of eq. (4·4) corresponding to the region $0.25\,A^{-1} \leq k \leq 1.50\,A^{-1}$, assuming γ_j's outside this region to be zero. Secondly the values of $\gamma(k)$ for $0.25\,A^{-1} \leq k \leq 1.50\,A^{-1}$ are substituted in the equations for $1.75\,A^{-1} \leq k \leq 2.75\,A^{-1}$ and the 5-dimensional simultaneous equations are solved. Finally the values of $\gamma(k)$ obtained for $0.25\,A^{-1} \leq k \leq 2.75\,A^{-1}$ are substituted in the equations for $3.00\,A^{-1} \leq k \leq 6.00\,A^{-1}$ and they are solved. In general, the values of $S(k')$ $(k' < k)$ have serious effects on the determination of $S(k)$, on the other hand those of $S(k')$ $(k' > k)$ hardly affect the value of $S(k)$. It may be recognized easily that the manner of calculations we have performed is based on the above general properties of the integral equation.

The form factor thus calculated is presented in Fig. 3, together with the experimental data obtained at $T=2.06°\mathrm{K}$.[4] The essential difference between the $S(k)$ obtained here and the one given in I lies in the linear dependence of $S(k)$ on k near at $k=0$. However, as in I, the theoretical curve deviates towards the rightward direction and the maximum height of the theoretical one is smaller than that of the experimental data.

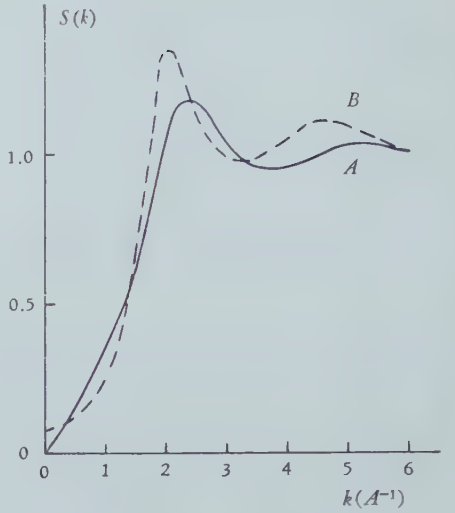

Fig. 3. Curve A: Calculated form factor for the hard-sphere potential. Curve B: From Goldstein and Reekie[4] at $T=2.06°\mathrm{K}$.

§ 5. Equations for modified Lennard-Jones potential

In § 4 we have considered the hard-sphere system. In this section the method will be generalized to the more realistic potential which includes the attractive forces besides the hard-core repulsions. If one wishes to calculate $S(k)$ for the Lennard-Jones potential in PSCA, one should solve the integro-differential equation (2·2), which seems to be difficult to carry out. Hence we modify the Lennard-Jones potential in the following manner: the potential is a sum of the hard-sphere one and $V(r)$ which is defined by

$$V(r)=\varepsilon[(r_0/r)^{12}-2(r_0/r)^6], \quad r>a \tag{5·1}$$

$$=0, \qquad r<a.$$

Expressing $V(r)$ in the reduced form, we write it $\varepsilon\gamma(x)$, i.e.,

$$V(x)=\varepsilon\gamma(x). \tag{5·2}$$

Repeating the procedure leading us to eq. (3·3), we obtain the equation

$$1/S^2(k)=(1/\pi k)\int_0^\infty K(k,k')\gamma(k')\,dk'+1-(2\lambda/3)G(k)+(2mca/\hbar)^2\cos k/k^2$$

$$-(4\lambda/3\Lambda^2)J(k), \tag{5·3}$$

where

$$\Lambda=\hbar/a\sqrt{m\varepsilon}, \tag{5·4}$$

which corresponds to de Boer's quantum mechanical parameter.[5]

Further $J(k)$ is given by

$$J(k)=\int_0^\infty U(x)\,(\sin kx/k)\,dx \tag{5·5}$$

where

$$U''(x) = x\gamma''(x), \quad |x| > 1 \tag{5·6a}$$

$$U(x) = 0, \quad |x| < 1, \qquad U(\infty) = 0. \tag{5·6b}$$

From eq. (5·1) it follows

$$U(x) = (\alpha/6)\,(\alpha/15x^9 - 1/x^3), \quad |x| > 1 \tag{5·7a}$$

$$U(x) = 0, \qquad |x| < 1, \tag{5·7b}$$

where

$$\alpha = (r_0/a)^6. \tag{5·8}$$

From eqs. (5·5), (5·7a) and (5·7b) we have

$$J(k) = (\alpha/6)\,\{(\alpha/15)\,A(k) - (1 + \alpha k^6/302400)\,B(k)\}, \tag{5·9}$$

where

$$A(k) = \sin k/8k + \cos k/56 - k\sin k/336 - k^2\cos k/1680 + k^3\sin k/6720 + k^4\cos k/20160,$$

$$B(k) = \sin k/2k + \cos k/2 - (k/2)\int_k^\infty \sin t/t \cdot dt.$$

The graphical representation of the function $(-4\lambda/3\Lambda^2)J(k)$ is given in Fig. 4.

Now the calculation of the $S(k)$ for the modified Lennard-Jones potential can be carried out in the same manner as we have done in § 4. The form factor thus obtained is presented in Fig. 5. From this figure we see that the positions at which the $S(k)$ becomes maximum coincide fairly with the experimental ones, though the maximum values are smaller than the observed ones as in § 4. Therefore it follows that the deviation of $S(k)$ towards the rightward direction for the hard-sphere potential is due to the neglect of attractive forces. The leftward deviation of $S(k)$ associated with the inclusion of attractive forces can be seen qualitatively in the following way: from Fig. 4 we see that $(-4\lambda/3\Lambda^2)J(k)$ is negative for $1.0\,\Lambda^{-1} \lesssim k \lesssim 2.0\,\Lambda^{-1}$ and hence the right-hand side of eq. (5·3) is smaller than that for the simple hard-sphere potential, provided that the change of the first term on the right can be neglected; this yields the increase of $S(k)$ for $1.0\,\Lambda^{-1} \lesssim k \lesssim 2.0\,\Lambda^{-1}$ which leads to the leftward deviation of $S(k)$. At the same time, the maximum value of $S(k)$ becomes larger than that for the simple hard-sphere potential.

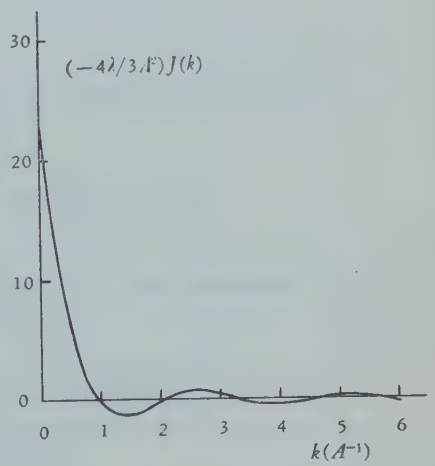

Fig. 4. Graphical representation of $(-4\lambda/3\Lambda^2)J(k)$.

We conclude this section by tabulating the numerical values we have used (in c. g. s. units) :

$m=6.64\times10^{-24}$, $c=2.37\times10^{4}$,

$\hbar=1.05\times10^{-27}$, $a=2.44\times10^{-8}$,

$\lambda=12.0$, $r_0=2.95\times10^{-8}$,

$\varepsilon=8.27\times10^{-16}$, $\alpha=3.12$,

$\Lambda=0.583$, $(2mca/\hbar)^2=53.0$,

$(4\lambda/3\Lambda^2)=47.1$.

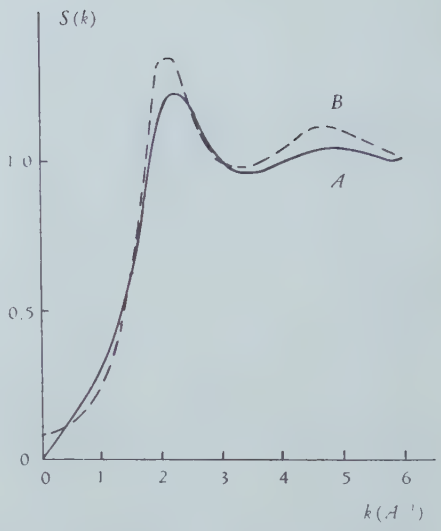

Fig. 5. Curve A: Calculated form factor for the modified Lennard-Jones potential. Curve B: Experimental.

§ 6. The cohesive energy

In the previous sections, we have dealt with the form factor function. In this section we wish to consider the cohesive energy, i. e., the ground state energy of liquid He^4.

If one solves the integro-differential equation (2·2) and substitutes the solution in eq. (2·1), one can calculate the ground state energy E_0 in PSCA. It is shown that such a procedure yields a simple functional relation of E_0 with the form factor and the sound velocity.

By using the equation

$$\int_0^\infty \frac{g'^2}{g}x^2\,dx=-4\int_0^\infty x\sqrt{g}\,\frac{d^2}{dx^2}(x\sqrt{g})\,dx$$

and eliminating $(d^2/dx^2)(x\sqrt{g})$ with the aid of eq. (2·2), we get

$$E_0\Big/\Big(\frac{\pi\hbar^2V_0^2a}{2m}\Big)=8\pi^2\rho a^3\int_0^\infty\int_0^\infty xg(x)\,\frac{kf^2(k)}{S^2(k)}\{1+2S(k)\}\sin kx\,dk\,dx$$

$$-8\pi^3\rho a^3\int_0^\infty\frac{kf^3(k)}{S(k)}\,dk. \qquad (6\cdot1)$$

If we substitute $g(x)=1+\varphi(x)/x$ in the first term on the right and use the equations

$$\int_0^\infty\varphi(x)\sin kx\,dx=\pi f(k),\quad \int_0^\infty x\sin kx\,dx=-\pi\delta'(k),$$

$$f(k)=(k/4\pi^2\rho a^3)\{S(k)-1\},$$

and eq. (2·5), we have

$$\frac{E_0}{N}=\frac{mc^2}{2}+\frac{3\hbar^2}{4m\pi a^2\lambda}\int_0^\infty \frac{k^4\{1+S(k)\}\{S(k)-1\}^3}{S^2(k)}dk. \qquad (6\cdot 2)$$

The first term on the right-hand side in the above expression has a simple physical meaning; it is the kinetic energy of the particle moving with the sound velocity and is, of course, positive. On the other hand, the second term can be negative for an appropriate potential function, on account of the negative values of integrand in the region $S(k)<1$. It should be noted that eq. (6·2) is valid for an arbitrary potential function in PSCA. It depends, however, on the potential function indirectly through eq. (2·2) whether E_0 is negative or not.

Substituting the experimental value of c and the calculated value of $S(k)$ obtained in § 5, we have

$$E_0/N=-1.9\times 10^{-16} \text{ erg/atom.}$$

On the other hand, the corresponding experimental value is -9.7×10^{-16} erg/atom. Hence the calculated value is about 1/5 of the observed one. One of the origin of this rather large discrepancy seems to lie in the approximation we have employed in eq. (2·3); clearly this approximation is bad near the point $x=1$, where $\varphi_1(x)$ differs from 0 considerably. Another origin seems to be the rather arbitrary determination of the sound velocity; the values of $S(k)$ which give the most contributions to the second term on the right-hand side in eq. (6·2) are those corresponding to $k\simeq 1\,A^{-1}$, which are considerably sensitive to the value of sound velocity.

However, eq. (6·2) itself, which represents the functional relation of E_0, c and $S(k)$, which are all the experimentally observable quantities, can fairly account for the observed value of cohesive energy. Substituting the experimental value of $S(k)$ and using the theoretical formula (2·5) near at $k=0$, we have

$$E_0/N=-8.0\times 10^{-16} \text{ erg/atom,}$$

which is about 80% of the experimental one. A fairly good coincidence of E_0 seems to indicate the validity of PSCA in a certain sense, since eq. (6·2) is the necessary consequence of our approximations.

§ 7. Application to the dilute hard-sphere system

Returning now to the hard-sphere system, in this section we wish to apply PSCA to the dilute hard-sphere system and compare the results with those corresponding to exact one. Such a comparison will serve to clarify the meaning of our approximations.

First of all, we wish to tabulate the formulas which are exact up to the specified orders :[6),7),8)]

$$S(k)=k/\sqrt{k^2+2k_0^2}, \quad k_0^2=8\pi a\rho, \qquad (7\cdot 1)$$

$$\frac{E_0}{N}=\frac{2\hbar^2\pi\rho a}{m}\left\{1+\frac{128\,(\rho a^3)^{1/2}}{15\sqrt{\pi}}\right\}, \qquad (7\cdot 2)$$

$$c^2=\frac{4\hbar^2\pi\rho a}{m^2}\left\{1+\frac{16(\rho a^3)^{1/2}}{\sqrt{\pi}}\right\}. \tag{7·3}$$

In eq. (7·1), k is measured in the ordinary units and not in $1/a$. In the following calculations, it is convenient to measure k in the ordinary units. This means simply to replace k in the previous sections by ka. After this replacement, we expand the terms including ka in powers of a and calculate E_0 and c, retaining the lowest and the next order terms.

From eq. (2·7), the lowest order term of c becomes

$$c^2=4\hbar^2\pi\rho a/m^2 \tag{7·4}$$

which is identical with the corresponding term of the exact one. Substituting this in eq. (3·3), we have

$$S(k)=k/\sqrt{k^2+2k_0^2} \tag{7·5}$$

as the lowest order term. This is identical with eq. (7·1).

Next, substituting eq. (7·5) in eq. (2·7), we have

$$\begin{aligned}c^2&=\frac{4\hbar^2\pi\rho a}{m^2}+\frac{a\hbar^2}{\pi m^2}\int_0^\infty(k^2+k_0^2-k\sqrt{k^2+2k_0^2})\left(1+\frac{2k}{\sqrt{k^2+2k_0^2}}\right)dk\\&=\frac{4\hbar^2\pi\rho a}{m^2}\left\{1+\frac{32(\rho a^3)^{1/2}}{3\sqrt{\pi}}\right\},\end{aligned} \tag{7·6}$$

up to the term of the order $(\rho a^3)^{1/2}$. The coefficient of $(\rho a^3)^{1/2}$ is different from that of eq. (7·3) by a factor 2/3.

Further, from eqs. (7·5), (7·6) and (6·2), it follows

$$\frac{E_0}{N}=\frac{2\hbar^2\pi\rho a}{m}\left\{1+\frac{160-48\pi}{3\sqrt{\pi}}(\rho a^3)^{1/2}\right\}. \tag{7·7}$$

The first term of the above expression is identical with the corresponding exact one. However the coefficient of $(\rho a^3)^{1/2}$ is different from the exact value. This coefficient is 4.81 in eq. (7·2), while 1.69 in eq. (7·7). This rather large discrepancy of the coefficients seems to be due to the approximations employed in deriving eq. (2·7), that is, we have made the approximation $g(x)\approx 1$, which is not valid in calculating the second term of eq. (7·6) or (7·7). It is not easy to calculate correctly the term of the order of $(\rho a^3)^{1/2}$ in PSCA, and hence we use tentatively the exact value of c^2 and calculate the coefficient of $(\rho a^3)^{1/2}$ in E_0/N. The calculation yields the value

$$\frac{176-48\pi}{3\sqrt{\pi}}=4.70,$$

which is very close to the exact value 4.81.

Apart from the ambiguities of the coefficients, it should be noted that c^2 and E_0/N are expanded in powers of $(\rho a^3)^{1/2}$ in our approximations. This property of the dilute

hard-sphere system is just the characteristic feature of the Bose system, as Lee, Huang and Yang[6] have pointed out, and is owing to the contributions arising from the particle excitations with small momentum transfers, as was pointed out by Brueckner and Sawada.[9] If these contributions are not carefully taken into account, for instance, if one uses the perturbation procedure, the resulting energy eigenvalue diverges. In order to obtain the correct results, one should employ another procedure; this is performed by the transformation of the particle Hamiltonian to the phonon Hamiltonian in Brueckner and Sawada's theory. In other words, the phonon excitations are more essential than the particle excitations for the Bose system. The appearance of the powers of $(\rho a^3)^{1/2}$ is associated with the importance of the phonon excitations in the Bose system.

From the above considerations, it may follow that our approximations take into account correctly the most remarkable properties of the Bose system.

§ 8. Discussion

In this and the previous papers, we have developed the methods appropriate to the determination of the form factor and the cohesive energy of liquid He^4 at absolute zero. In this section, we wish to discuss the nature and the validity of the approximations we have employed.

First, we discuss the pair approximation. In order to clarify the meaning of this approximation, it is appropriate to consider the system of particles with the potential function which has the definite Fourier component. In this case, the ρ_k-representation for the Bose system developed by Bogolyubov and Zubarev[10] can be applied, and the ground state wave function turns out to be the product of the atomic pairs if the phonon-phonon interactions are neglected. Thus, the pair approximation is exact to the extent that the interactions between the elementary excitations are neglected. Of course, for the potential function of which Fourier component diverges, the ρ_k-representation is no longer valid. However, the structure of the ground state wave function may be seen from the work of Lee, Huang and Yang[6]; they have shown that the wave function has the form consistent with the pair approximation. For the hard-sphere system, it should be noted that the wave function of atomic pairs is the simplest one which can satisfy the boundary conditions imposed on the wave function. Further, in principle, the effects of the phonon-phonon interactions can be taken into account in the pair approximation, since the wave function is determined by the requirement that the expectation value of the total Hamiltonian should be minimum.

Secondly, the superposition approximation should be discussed. Even in the classical theory of liquids, the use of this approximation is not yet fully justified; at present it may finally be judged only in the light of its consequences. The same is also true with the quantum liquids. However, the validity of this approximation can be seen from Feynman and Cohen's[11] work; they have calculated the energy gap for the roton states in liquid He^4 using the superposition approximation and obtained the reasonable value for this gap. They have discussed the validity of the superposition approximation and concluded that

this is a fairly good approximation to the actual wave function of liquid He^4. Further, Mazo and Kirkwood[12] have calculated the radial distribution function at finite temperature under the same approximation, and obtained the results consistent with the experimental data. All these facts seem to confirm the use of the superposition approximation in liquid He^4.

Finally, we should discuss the chain approximation. In principle, this is not needed for the determination of the radial distribution function under the pair and superposition approximations. However, without this approximation, the equations become extremely cumbersome, since they contain two unknown quantities, $u(x)$ and $g(x)$, which couple with each other through the non-linear integro-differential equations. To remove such complexities, it is desirable to eliminate the function $u(x)$ from the equations. This is performed by the chain approximation in our theory. It should be noted that the chain approximation corresponds to an approximate solution of eq. (A·4) which can be solved by the perturbation procedure considering (A·5) as the unperturbed solution. We shall not enter into this problem here. We expect that the correction terms are small.

We conclude this section by noting that we have further employed the approximation (2·3) in order to simplify the actual calculations. Without this, we must deal with the non-linear integro-differential equation (2·2). We hope that the solution of this equation will provide a clear answer for the validity of PSCA.

Acknowledgement

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Appendix. Derivation of eqs. (2·1) and (2·2)

From eq. (2·3a) of I, the expectation value of the kinetic energy in the pair approximation is given by

$$\langle K.\ E.\rangle = (\pi\hbar^2\rho^2 Va/2m)\int_0^\infty g'(x)u'(x)x^2dx \qquad (A\cdot1)$$

in the reduced form, where $g(x)$ is related to $u(x)$ under the superposition approximation :

$$\Gamma_1 g(1,2)/g(1,2) = \Gamma_1 u(1,2) + a^3\rho\int g(1,3)g(2,3)\Gamma_1 u(1,3)dv_3. \qquad (A\cdot2)$$

If we define the function $P(1,2)$:

$$\Gamma_1 u(1,2) = \Gamma_1 g(1,2)/g(1,2) + \Gamma_1 P(1,2), \qquad (A\cdot3)$$

we have

$$\Gamma P(\boldsymbol{x}) = -a^3\rho\int\{g(\boldsymbol{x}-\boldsymbol{x}')-1\}\Gamma' g(\boldsymbol{x}')dv' - a^3\rho\int\{g(\boldsymbol{x}-\boldsymbol{x}')-1\}g(\boldsymbol{x}')\Gamma' P(\boldsymbol{x}')dv'. \qquad (A\cdot4)$$

As can be easily shown, the chain approximation is equivalent to approximate $g(x')$ in the last term in eq. (A·4) by 1. Under such an approximation the integral equation (A·4) is easily solved with the aid of the theory of Fourier transforms to give

$$P(x) = -8\pi^2 \rho a^3 \int_0^\infty f^2(k)/S(k) \cdot \sin kx/kx \cdot dk \qquad (A\cdot 5)$$

where $f(k)$ was defined by eq. (3·7) of I and $S(k)$ is the form factor.

From eqs. (A·1), (A·3), (A·5) and (2·7) of I, the expectation value of the Hamiltonian becomes under PSCA

$$\langle H' \rangle = \langle H \rangle \Big/ \left(\frac{\pi\hbar^2 V \rho^2 a}{2m} \right) = \int_0^\infty \frac{g'^2}{g} x^2 dx + \frac{4ma^2}{\hbar^2} \int_0^\infty V(x) g(x) x^2 dx$$

$$+ 8\pi^2 \rho a^3 \int_0^\infty \int_0^\infty g'(x) f^2(k) (\sin kx - kx \cos kx)/kS(k) \cdot dk\, dx. \qquad (A\cdot 6)$$

Substituting

$$g'(x) = 2\int_0^\infty f(k') (k'x \cos k'x - \sin k'x)/x^2 \cdot dk'$$

in the last term and using

$$\int_0^\infty (k'x \cos k'x - \sin k'x) (\sin kx - kx \cos kx)/x^2 \cdot dx = -(\pi/2) kk' [\delta(k-k') + \delta(k+k')],$$

we obtain eq. (2·1). Further, if we take the variation of $g(x)$, we have eq. (2·2) with the aid of eqs. (3·7) and (3·10) of I.

In deriving eq. (2·2) we must take account of the normalization condition for $g(x)$:

$$\rho^2 V a^3 \int_{V^*} 4\pi x^2 g(x)\, dx = N(N-1).$$

Let the Lagrange multiplier for this condition be μ. Then the right-hand side of eq. (2·2) is not zero but turns out to be $4\mu ma^2 x/\hbar^2$. If we multiply the resulting equation by $xg(x)$ and integrate by x in the reduced volume V^*, we see that the left side is of order 1 and the right side is of order μN. Hence μ is of order $1/N$ and can be neglected in the limit $N \to \infty$. This may be seen also from the following consideration : if there appears the term $\mu x (\mu \neq 0)$ on the right side of eq. (2·2), $xg(x)$ becomes to oscillate or diverges to ∞ exponentially in the limit $x \to \infty$, since the second and the third term on the left vanish in this limit. Hence $g(x)$ cannot satisfy the condition $g(x) \to 1$ $(x \to \infty)$. Hence μ should be zero.

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Nuclear Binding Energies and Low-Lying Energy Levels in the $2s_{1/2}$ and $1d_{3/2}$ Shells

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On the basis of the *j-j* coupling shell model, the nuclear energies in the $2s_{1/2}$ and $1d_{3/2}$ shells are given by linear combinations of the parameters expressing the nuclear and Coulomb interaction energies in two-nucleon states, the single nucleon energies in $2s_{1/2}$ and $1d_{3/2}$ shells and the Coulomb interaction energies of a single proton in $2s_{1/2}$ and $1d_{3/2}$ shells with the closed shells. The best values of these parameters are determined by a least squares fit to the experimental binding energy data. The mean deviations are very small, and this shows the adequacy of the shell model in the ground states of these nuclei. By using these values of parameters the excitation energies are calculated, and it is shown that in some excited states configuration interactions seem to be important. The two-body force required to give the best fit with the interaction energies in two-nucleon states is discussed.

§ 1. Introduction

Recently I. Talmi et al. showed that the potential well of the nucleus is approximately constant for nuclei belonging to the same shell.[1),2),3),4),5)] These investigations are based upon following assumptions:

i) the nucleons in the same unfilled shell move in the some central field,
ii) the spin j of each nucleon is a good quantum number (j-j coupling),
iii) the effect of the configuration interaction on the nuclear binding energies is negligible, and
iv) the interaction between nucleons is a charge independent two-body force plus the Coulomb force.

They used the group theoretical method and expressed the nuclear binding energy as a linear combination of several parameters which characterize the nuclear interaction and are determined by a least squares fit[3),4)] to the experimental data. A good agreement was found between the calculated and the experimental energies, especially in $2s_{1/2}$, $1d_{3/2}$[3)] and $2s_{1/2}$-$1d_{3/2}$[4)] nuclei.

In this paper, the energies of the same $2s_{1/2}$, $1d_{3/2}$ and $2s_{1/2}$-$1d_{3/2}$ nuclei are calculated by the less elegant method of fractional parentage coefficients,[6),7)] which however, makes it possible to express the expectation value of the sum of the interaction energies in a shell model state of a many-nucleon system as a linear combination of the interaction energies of two-nucleon states. The contributions, from $2s_{1/2}$ and $1d_{3/2}$ nucleons outside of ${}_{14}Si_{14}^{28}$ to the total energies of these nuclei are expressed in terms of several parameters; the single-nucleon energies in $2s_{1/2}$ and $1d_{3/2}$ shells, the Coulomb interaction energies of

a single proton in $2s_{1/2}$ and $1d_{3/2}$ shells, and the interaction energies in two-nucleon states. The values of these energy parameters are determined from the experimental binding energies by the method of least squares (§ 2).

The method used by I. Talmi and R. Thieberger[3] gives the same results as ours, so far as the binding energies of the nuclei considered in this paper are concerned. However, in specific cases, their method gives identical expectation values for different states, (for example $d^2_{3/2}$ $(T=0\ J=1)$ and $d^2_{3/2}$ $(T=0\ J=3)$ have the same energies in ref. (3)). Therefore their method gives rather average energies of some states, while the method in this paper can give different expectation values for different states.

Our results are given in Table VI. The agreement is remarkably good. The root mean square deviation is sufficiently small to suggest that the assumptions i)~iv) are adequare for calculation of the binding energies of these nuclei.

Since some of the interaction energies of two-nucleon states always appear in the same combinations in the expressions of the binding energies, it is impossible to determine the values of such parameters from the binding energies alone. The values of these undetermined parameters are found by considering the energies of the excited levels.

In § 3, instead of using the excited states, these undetermined parameters are calculated from the two-body force required to give the best fit with the parameters already determined, making use of the harmonic oscillator potential and its eigenfunctions. This two-body interaction consists of central, tensor and spin-orbit forces, and it is interesting that the depths of the central and tensor forces in the even-state are nearly equal to those of the phenomenological potential fitted in the deuteron and the nucleon-nucleon scattering data. However, this potential is not in accordance with the Serber condition.

In § 4, the energies for some excited states of several nuclei are calculated by using these parameters. It is found that the calculated values agree with the experimental ones for the excited states which are expected to arise from pure configuration. However, the configuration interactions seems to have not negligible effect in some excited states.

§ 2. Determination of parameters

Let us consider a nucleus which has n_1 nucleons in the $2s_{1/2}$ shell and n_2 nucleons in the $1d_{3/2}$ shell. According to the assumptions made in the introduction, the energy of the closed shells is the same for all the nuclei considered and is equal to the binding energy of the preceding closed-shells nucleus ${}_{14}\mathrm{Si}^{28}_{?}$. The binding energy of ${}_{14}\mathrm{Si}^{28}_{14}$ is subtracted from the binding energies of all the nuclei treated in this paper to obtain the contribution of the extra nucleons.

An antisymmetric state of $2s^{n_1}_{1/2}$ can be expressed simply as

$$\psi(2s^{n_1}_{1/2} T_1 J_1 ;\ T_{1z} J_{1z}), \tag{1}$$

where T_1 and J_1 are the resultant isotopic spin and ordinary spin respectively. The possible states of the pure configurations $1d^{n_2}_{3/2}$ have been classified by Flowers by means

of group theory.[8] Edmond and Flowers also have given the fractional parentage coefficients.[7] An antisymmetric state of $1d_{3/2}^{n_2}$ is written as

$$\phi(1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ;\ T_{2z} J_{2z}), \tag{2}$$

where t_2 and s_2 are respectively the reduced isotopic spin and the seniority number. T_2 and J_2 are the resultant isotopic and ordinary spins.

Then a complete set of states of $2s_{1/2}^{n_1}\ 1d_{3/2}^{n_2}$ can be constructed by vector coupling ψ and ϕ to give the resultant T, J and then antisymmetrizing. The vector coupled product is denoted by $\Phi(2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ ;\ T_z J_z)$ which is given by the equation

$$\Phi = \sum_{T_{1z} J_{1z}}{}' (T_1 T_2 T_{1z} T_{2z} | T_1 T_z)\ (J_1 J_2 J_{1z} J_{2z} | J J_z)\, \psi(2s_{1/2}^{n_1} T_1 J_1 ;\ T_{1z} J_{1z}) \times \phi(1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ;\ T_{2z} J_{2z}), \tag{3}$$

where $(ab\alpha_i \beta | c\gamma)$ is a Clebsch-Gordan coefficient. Using this Φ, antisymmetric states of $2s_{1/2}^{n_1}\ 1d_{3/2}^{n_2}$ are denoted by

$$\Psi(2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ ;\ T_z J_z) = \sqrt{\frac{n_1 ! \, n_2 !}{(n_1 + n_2)!}}\, \{1 + \sum_{P_{ij}}{}' (-1) P_{ij}\}\, \Phi(2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ ;\ T_z J_z) \tag{4}$$

where on the right-hand side the group $2s_{1/2}^{n_1}$ consists of the nucleons 1, 2, $\cdots i$, $\cdots n_1$, the group $1d_{3/2}^{n_2}$ consists of the nucleons n_1+1, n_1+2, $\cdots j$, $\cdots n_1+n_2$, and P_{ij} is the permutation operator which exchanges i with j.

The expectation values of the two-body nuclear interaction

$$V = \sum_{i>j} V_{ij} \tag{5}$$

are divided into three parts as follows:

$$\begin{aligned}
&\langle 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ | V | 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ \rangle \\
&= \langle 2s_{1/2}^{n_1} T_1 J_1 | \sum_{i>k} V_{ik} | 2s_{1/2}^{n_1} T_1 J_1 \rangle + \langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 | \sum_{j>l} V_{jl} | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 \rangle \\
&+ \langle 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ | \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_2} V_{ij} | 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ \rangle,
\end{aligned} \tag{6}$$

where i, k stand for the nucleons belonging to $2s_{1/2}^{n_1}$ and j, l for those belonging to $1d_{3/2}^{n_2}$.

In order to express the expectation values of $\sum_{i>j}{}' V_{ij}$ in the j^n configuration in terms of those in j^2, it is convenient to use the relation

$$\begin{aligned}
&E^N(j^n \alpha TJ) \\
&\equiv \langle j^n \alpha TJ | V | j^n \alpha TJ \rangle \\
&= \frac{n}{n-2} \sum_{\alpha_1 \alpha_2 T_1 J_1} (j^n \alpha TJ \{ | j^{n-1}(\alpha_1 T_1 J_1) jTJ)\ (j^{n-1}(\alpha_2 T_1 J_1) jTJ | \} j^n \alpha TJ) \\
&\times \langle j^{n-1} \alpha_1 T_1 J_1 | V | j^{n-1} \alpha_2 T_1 J_1 \rangle
\end{aligned} \tag{7}$$

Table I Table of $E^N(2s^n_{1/2}TJ) \equiv \langle 2s^n_{1/2}TJ|V|2s^n_{1/2}TJ\rangle$

n	T	J	$E^N(2s^n_{1/2}TJ)$
2	1	0	$E^N(2s^n_{1/2}10)$
	0	1	$E^N(2s^2_{1/2}01)$
3	1/2	1/2	$3/2\,E^N(2s^2_{1/2}10)+3/2\,E^N(2s^2_{1/2}01)$
4	0	0	$3\,E^N(2s^2_{1/2}10)+3\,E^N(2s^2_{1/2}10)$

Table II Table of $E^N(1d^n_{3/2}\alpha TJ) \equiv \langle 1d^n_{3/2}\alpha TJ|V|1d^n_{3/2}\alpha TJ\rangle$

n	T	J	$(s\ t)$	
2	0	1	(2, 0)	$E^N(1d^2_{3/2}01)$
	0	3	(2, 0)	$E^N(1d^2_{3/2}03)$
	1	0	(0, 0)	$E^N(1d^2_{3/2}10)$
	1	2	(2, 1)	$E^N(1d^2_{3/2}12)$
3	1/2	3/2	(1, 1/2)	$9/20\,E^N(1d^2_{3/2}01)+21/20\,E^N(1d^2_{3/2}03)-5\,4\,E^N(1d^2_{3/2}10)+1/4\,E^N(1d^2_{3/2}12)$
	1/2	1/2	(3, 1/2)	$3/2\,E^N(1d^2_{3/2}01)+3/2\,E^N(1d^2_{3/2}12)$
	1/2	5/2	(3, 1/2)	$7/10\,E^N(1d^2_{3/2}01)+4/5\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}12)$
	1/2	7/2	(3, 1/2)	$3/2\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}12)$
	3/2	3/2	(1, 1/2)	$1/2\,E^N(1d^2_{3/2}10)+5/2\,E^N(1d^2_{3/2}12)$
4	0	0	(0, 0)	$9/10\,E^N(1d^2_{3/2}01)+21/10\,E^N(1d^2_{3/2}03)+5/2\,E^N(1d^2_{3/2}10)+1/2\,E^N(1d^2_{3/2}12)$
	0	2	(2, 1)	$9/10\,E^N(1d^2_{3/2}01)+21/10\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}10)+3/2\,E^N(1d^2_{3/2}12)$
	0	2	(4, 1)	$9/5\,E^N(1d^2_{3/2}01)+6/5\,E^N(1d^2_{3/2}03)+3\,E^N(1d^2_{3/2}12)$
	0	4	(4, 1)	$2/5\,E^N(1d^2_{3/2}01)+13/5\,E^N(1d^2_{3/2}03)+3\,E^N(1d^2_{3/2}12)$
	1	2	(2, 1)	$3/5\,E^N(1d^2_{3/2}01)+7/5\,E^N(1d^2_{3/2}03)+E^N(1d^2_{3/2}10)+3\,E^N(1d^2_{3/2}12)$
	1	1	(2, 0)	$13/10\,E^N(1d^2_{3/2}01)+7/10\,E^N(1d^2_{3/2}03)+1/2\,E^N(1d^2_{3/2}10)+7/2\,E^N(1d^2_{3/2}12)$
	1	3	(2, 0)	$3/10\,E^N(1d^2_{3/2}01)+17/10\,E^N(1d^2_{3/2}03)+1/2\,E^N(1d^2_{3/2}10)+7/2\,E^N(1d^2_{3/2}12)$
	2	0	(0, 0)	$E^N(1d^2_{3/2}10)+5\,E^N(1d^2_{3/2}12)$
5	1/2	3/2	(1, 1/2)	$6/5\,E^N(1d^2_{3/2}01)+14/5\,E^N(1d^2_{3/2}03)+2\,E^N(1d^2_{3/2}10)+4\,E^N(1d^2_{3/2}12)$
	1/2	1/2	(3, 1/2)	$9/4\,E^N(1d^2_{3/2}01)+7/4\,E^N(1d^2_{3/2}03)+3/4\,E^N(1d^2_{3/2}10)+21/4\,E^N(1d^2_{3/2}12)$
	1/5	5/2	(3, 1/2)	$29/20\,E^N(1d^2_{3/2}01)+51/20\,E^N(1d^2_{3/2}03)+3/4\,E^N(1d^2_{3/2}10)+21/4\,E^N(1d^2_{3/2}12)$
	1/2	7/2	(3, 1/2)	$3/4\,E^N(1d^2_{3/2}01)+13/4\,E^N(1d^2_{3/2}03)+3/4\,E^N(1d^2_{3/2}10)+21/4\,E^N(1d^2_{3/2}12)$
	3/2	3/2	(1, 3/2)	$3/4\,E^N(1d^2_{3/2}01)+7/4\,E^N(1d^2_{3/2}03)+5\,4\,E^N(1d^2_{3/2}10)+25/4\,E^N(1d^2_{3/2}12)$
6	0	1	(2, 0)	$5/2\,E^N(1d^2_{3/2}01)+7/2\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}10)+15/2\,E^N(1d^2_{3/2}12)$
	0	3	(2, 0)	$3/2\,E^N(1d^2_{3/2}01)+9/2\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}10)+15/2\,E^N(1d^2_{3/2}12)$
	1	0	(0, 0)	$3/2\,E^N(1d^2_{3/2}01)+7/2\,E^N(1d^2_{3/2}03)+5/2\,E^N(1d^2_{3/2}10)+15/2\,E^N(1d^2_{3/2}12)$
	1	2	(2, 1)	$3/2\,E^N(1d^2_{3/2}01)+7/2\,E^N(1d^2_{3/2}03)+3/2\,E^N(1d^2_{3/2}10)+17/2\,E^N(1d^2_{3/2}12)$
7	1/2	3/2	(1, 1/2)	$9/4\,E^N(1d^2_{3/2}01)+21/4\,E^N(1d^2_{3/2}03)+9\,4\,E^N(1d^2_{3/2}10)+45/4\,E^N(1d^2_{3/2}12)$
8	0	0	(0, 0)	$3\,E^N(1d^2_{3/2}01)+7\,E^N(1d^2_{3/2}03)+3\,E^N(1d^2_{3/2}10)+15\,E^N(1d^2_{3/2}12)$

Table III Table of $E^N_{\rm int}(2s^{n_1}{}_{1/2}(T_1J_1)1d^{n_2}{}_{3/2}(T_2J_2)TJ) \equiv \langle 2s^{n_1}{}_{1/2}(T_1J_1)1d^{n_2}{}_{3/2}(T_2J_2)TJ| \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} V_{ij}|2s^{n_1}{}_{1/2}(T_1J_1)1d^{n_2}{}_{3/2}(T_1J_1)TJ\rangle$

n_1	n_2	T_1	J_1	T_2	J_2	T	J	$E^N_{\rm int}(2s^{n_1}{}_{1/2}(T_1J_1)1d^{n_2}{}_{3/2}(T_2J_2)TJ)$
1	1	1/2	1/2	1/2	3/2	0	1	$E^N(2s_{1/2}1d_{3/2}01)$
						0	2	$E^N(2s_{1/2}1d_{3/2}02)$
						1	1	$E^N(2s_{1/2}1d_{3/2}11)$
						1	2	$E^N(2s_{1/2}1d_{3/2}12)$
2	1	0	1	1/2	3/2	1/2	1/2	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + + 3/2\,E^N(2s_{1/2}1d_{3/2}11)$
						1/2	3/2	$5/16\,E^N(2s_{1/2}1d_{3/2}01) + 3/16\,E^N(2s_{1/2}1d_{3/2}02) + 15/16\,E^N(2s_{1/2}1d_{3/2}11) + 9/16\,E^N(2s_{1/2}1d_{3/2}12)$
						1/2	5/2	$1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
		1	0	1/2	3/2	1/2	3/2	$9/16\,E^N(2s_{1/2}1d_{3/2}01) + 15/16\,E^N(2s_{1/2}1d_{3/2}02) + 3/16\,E^N(2s_{1/2}1d_{3/2}11) + 5/16\,E^N(2s_{1/2}1d_{3/2}12)$
						3/2	3/2	$3/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
1	2	1/2	1/2	0	1	1/2	1/2	$5/16\,E^N(2s_{1/2}1d_{3/2}01) + 3/16\,E^N(2s_{1/2}1d_{3/2}02) + 15/16\,E^N(2s_{1/2}1d_{3/2}11) + 9/16\,E^N(2s_{1/2}1d_{3/2}12)$
						1/2	3/2	$1/8\,E^N(2s_{1/2}1d_{3/2}01) + 3/8\,E^N(2s_{1/2}1d_{3/2}02) + 3/8\,E^N(2s_{1/2}1d_{3/2}11) + 9/8\,E^N(2s_{1/2}1d_{3/2}12)$
				0	3	1/2	5/2	$7/16\,E^N(2s_{1/2}1d_{3/2}01) + 1/16\,E^N(2s_{1/2}1d_{3/2}02) + 21/16\,E^N(2s_{1/2}1d_{3/2}11) + 3/16\,E^N(2s_{1/2}1d_{3/2}12)$
						1/2	7/2	$1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
				1	0	1/2	1/2	$9/16\,E^N(2s_{1/2}1d_{3/2}01) + 15/16\,E^N(2s_{1/2}1d_{3/2}02) + 3/16\,E^N(2s_{1/2}1d_{3/2}11) + 5/16\,E^N(2s_{1/2}1d_{3/2}12)$
						3/2	1/2	$3/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
				1	2	1/2	3/2	$9/8\,E^N(2s_{1/2}1d_{3/2}01) + 3/8\,E^N(2s_{1/2}1d_{3/2}02) + 3/8\,E^N(2s_{1/2}1d_{3/2}11) + 1/8\,E^N(2s_{1/2}1d_{3/2}12)$
						1/2	5/2	$3/16\,E^N(2s_{1/2}1d_{3/2}01) + 21/16\,E^N(2s_{1/2}1d_{3/2}02) + 1/16\,E^N(2s_{1/2}1d_{3/2}11) + 7/16\,E^N(2s_{1/2}1d_{3/2}12)$
						3/2	3/2	$3/2\,E^N(2s_{1/2}1d_{3/2}11) + 1/2\,E^N(2s_{1/2}1d_{3/2}12)$
						3/2	5/2	$1/4\,E^N(2s_{1/2}1d_{3/2}11) + 7/4\,E^N(2s_{1/2}1d_{3/2}12)$
3	1	1/2	1/2	1/2	3/2	0	1	$1/4\,E^N(2s_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}11)$
						0	2	$3/4\,E^N(2s_{1/2}1d_{3/2}01) + 3/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						1	1	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 7/4\,E^N(2s_{1/2}1d_{3/2}12)$
2	2	0	1	0	1	0	0	$5/8\,E^N(2s_{1/2}1d_{3/2}01) + 3/8\,E^N(2s_{1/2}1d_{3/2}02) + 15/8\,E^N(2s_{1/2}1d_{3/2}11) + 9/8\,E^N(2s_{1/2}1d_{3/2}12)$
						0	1	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + 1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}11) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						0	2	$1/4\,E^N(2s_{1/2}1d_{3/2}01) + 3/4\,E^N(2s_{1/3}1d_{3/2}02) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 9/4\,E^N(2s_{1/2}1d_{3/2}12)$
				0	3	0	2	$7/8\,E^N(2s_{1/2}1d_{3/2}01) + 1/8\,E^N(2s_{1/2}1d_{3/2}02) + 21/8\,E^N(2s_{1/2}1d_{3/2}11) + 3/8\,E^N(2s_{1/2}1d_{3/2}12)$
						0	3	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + 1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}11) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						0	4	$E^N(2s_{1/2}1d_{3/2}02) + 3\,E^N(2s_{1/2}1d_{3/2}12)$

				1	0	1	1	$3/8\,E^N(2s_{1/2}1d_{3/2}01) + 5/8\,E^N(2s_{1/2}1d_{3/2}02) + 9/8\,E^N(2s_{1/2}1d_{3/2}11) + 15/8\,E^N(2s_{1/2}1d_{3/2}12)$
				1	2	1	1	$3/4\,E^N(2s_{1/2}1d_{3/2}01) + 1/4\,E^N(2s_{1/2}1d_{3/2}02) + 9/4\,E^N(2s_{1/2}1d_{3/2}11) + 3/4\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + 1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}11) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						1	3	$1/8\,E^N(2s_{1/2}1d_{3/2}01) + 7/8\,E^N(2s_{1/2}1d_{3/2}02) + 1/8\,E^N(2s_{1/2}1d_{3/2}11) + 7/8\,E^N(2s_{1/2}1d_{3/2}12)$
		1	0	0	1	1	1	$3/8\,E^N(2s_{1/2}1d_{3/2}01) + 5/8\,E^N(2s_{1/2}1d_{3/2}02) + 9/8\,E^N(2s_{1/2}1d_{3/2}11) + 15/8\,E^N(2s_{1/2}1d_{3/2}12)$
				0	3	1	3	$3/8\,E^N(2s_{1/2}1d_{3/2}01) + 5/8\,E^N(2s_{1/2}1d_{3/2}02) + 3/8\,E^N(2s_{1/2}1d_{3/2}11) + 5/8\,E^N(2s_{1/2}1d_{3/2}12)$
				1	0	0	0	$9/8\,E^N(2s_{1/2}1d_{3/2}01) + 15/8\,E^N(2s_{1/2}1d_{3/2}02) + 3/8\,E^N(2s_{1/2}1d_{3/2}11) + 5/8\,E^N(2s_{1/2}1d_{3/2}12)$
						1	0	$3/4\,E^N(2s_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
						2	0	$3/2\,E^N(2s_{1/2}1d_{3/2}11) + 5/2\,E^N(2s_{1/2}1d_{3/2}12)$
				1	2	0	2	$9/8\,E^N(2s_{1/2}1d_{3/2}01) + 15/8\,E^N(2S_{1/2}1d_{3/2}02) + 3/8\,E^N(2s_{1/2}1d_{3/2}11) + 5/8\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$3/4\,E^N(2s_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}11) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
						2	2	$3/2\,E^N(2s_{1/2}1d_{3/2}11) + 5/2\,E^N(2s_{1/2}1d_{3/2}12)$
1	3	1/2	1/2	1/2	3/2	0	1	$3/8\,E^N(2s_{1/2}1d_{3/2}01) + 9/8\,E^N(2s_{1/2}1d_{3/2}02) + 11/8\,E^N(2s_{1/2}1d_{3/2}11) + 1/8\,E^N(2s_{1/2}1d_{3/2}12)$
						0	2	$27/40\,E^N(2s_{1/2}1d_{3/2}01) + 33/40\,E^N(2s_{1/2}1d_{3/2}02) + 3/40\,E^N(2s_{1/2}1d_{3/2}11) + 57/40\,E^N(2s_{1/2}1d_{3/2}12)$
						1	1	$11/24\,E^N(2s_{1/2}1d_{3/2}01) + 1/24\,E^N(2s_{1/2}1d_{3/2}02) + 31/24\,E^N(2S_{1/2}1d_{3/2}11) + 29/24\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$1/40\,E^N(2s_{1/2}1d_{3/2}01) + 19/40\,E^N(2s_{1/2}1d_{3/2}02) + 29/40\,E^N(2s_{1/2}1d_{3/2}11) + 71/40\,E^N(2s_{1/2}1d_{3/2}12)$
				1/2	1/2	0	0	$3/2\,E^N(2s_{1/2}1d_{3/2}01) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						0	1	$1/4\,E^N(2S_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 3/4\,E^N(2s_{1/2}1d_{3/2}12)$
						1	0	$1/2\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}11) + E^N(2s_{1/2}1d_{3/2}12)$
						1	1	$1/4\,E^N(2s_{1/2}1d_{3/2}01) + 1/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/4\,E^N(2s_{1/2}1d_{3/2}11) + 7/4\,E^N(2s_{1/2}1d_{3/2}12)$
				1/2	5/2	0	2	$19/20\,E^N(4s_{1/2}1d_{3/2}01) + 11/20\,E^N(2s_{1/2}1d_{3/2}02) + 21/20\,E^N(2s_{1/2}1d_{3/2}11) + 9/20\,E^N(2s_{1/2}1d_{3/2}12)$
						0	3	$2/7\,E^N(2s_{1/2}1d_{3/2}01) + 17/14\,E^N(2s_{1/2}1d_{3/2}02) + 3/14\,E^N(2s_{1/2}1d_{3/2}11) + 9/7\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$7/20\,E^N(2s_{1/2}1d_{3/2}01) + 3/20\,E^N(2s_{1/2}1d_{3/2}02) + 33/20\,E^N(2s_{1/2}1d_{3/2}11) + 17/20\,E^N(2s_{1/2}1d_{3/2}12)$
						1	3	$7/20\,E^N(2s_{1/2}1d_{3/2}01) + 3/20\,E^N(2s_{1/2}1d_{3/2}02) + 69/70\,E^N(2s_{1/2}1d_{3/2}11) + 53/45\,E^N(2s_{1/2}1d_{3/2}12)$
				1/2	7/2	0	3	$27/28\,E^N(2s_{1/2}1d_{3/2}01) + 15/28\,E^N(2s_{1/2}1d_{3/2}02) + 9/7\,E^N(2s_{1/2}1d_{3/2}11) + 3/14\,E^N(2s_{1/2}1d_{3/2}12)$
						0	4	$1/4\,E^N(2s_{1/2}1d_{3/2}01) + 5/4\,E^N(2s_{1/2}1d_{3/2}02) + 3/2\,E^N(2s_{1/2}1d_{3/2}12)$
						1	3	$3/7\,E^N(2s_{1/2}1d_{3/2}01) + 1/14\,E^N(2s_{1/2}1d_{3/2}02) + 51/28\,E^N(2s_{1/2}1d_{3/2}11) + 19/28\,E^N(2s_{1/2}1d_{3/2}12)$
						1	4	$1,2\,E^N(2s_{1/2}1d_{3/2}02) + 1,4\,E^N(2s_{1/2}1d_{3/2}11) + 9,4\,E^N(2s_{1/2}1d_{3/2}12)$
				3/2	3/2	1	1	$7/6\,E^N(2s_{1/2}1d_{3/2}01) + 5/6\,E^N(2s_{1/2}1d_{3/2}02) + 7/12\,E^N(2s_{1/2}1d_{3/2}11) + 5/12\,E^N(2s_{1/2}1d_{3/2}12)$
						1	2	$1/2\,E^N(2s_{1/2}1d_{3/2}01) + 3/2\,E^N(2s_{1/2}1d_{3/2}02) + 1/4\,E^N(2s_{1/2}1d_{3/2}11) + 3/4\,E^N(2s_{1/2}1d_{3/2}12)$
						2	1	$7/4\,E^N(2s_{1/2}1d_{3/2}11) + 5/4\,E^N(2s_{1/2}1d_{3/2}12)$
						2	2	$3/4\,E^N(2s_{1/2}1d_{3/2}11) + 9,4\,E^N(2s_{1/2}1d_{3/2}12)$

where α's represent additional quantum numbers necessary for the definition of states, and $(j^n\alpha TJ\{|j^{n-1}(\alpha_1 T_1 J_1)jTJ)$ is the fractional parentage coefficient, given in the tables of reference (7) for $j=3/2$, $5/2$ and $7/2$. Thus the matrix elements of V for n particles can be constructed from those for two-particles. The results are shown in Tables I and II.

The last term in (6), which stands for the mutual interaction of n_1 $2s_{1/2}$ nucleons and n_2 $1d_{3/2}$ nucleons, is easily calculated as follows,

$$\begin{aligned}
&\langle 2s_{1/2}^{n_1}(T_1J_1)\,1d_{3/2}^{n_2}(t_2s_2T_2J_2)\,TJ|\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}V_{ij}|\,2s_{1/2}^{n_1}(T_1J_1)\,1d_{3/2}^{n_2}(t_2s_2T_2J_2)\,TJ\rangle\\
&=n_1n_2\sum'(2s_{1/2}^{n_1}T_1J_1\{\ 2s_{1/2}^{n_1-1}(T_{11}J_{11})\,2s_{1/2}TJ)\ (1d_{3/2}^{n_2}t_2s_2T_2J_2\{|1d_{3/2}^{n_2-1}(t_{22}s_{22}T_{22}J_{22})\,1d_{3/2}TJ)\\
&\times\langle(T_{11}J_{11})\,2s_{1/2}](T_1J_1)[(T_{22}J_{22})\,1d_{3/2}](T_2J_2)\,TJ|\,V_{n_1,n_1+n_2}|[(T_{11}J_{11})\,2s_{1/2}](T_1J_1)\\
&\times[(T_{22}J_{22})\,1d_{3/2}](T_2J_2)\,TJ\rangle\\
&\times(2s_{1/2}^{n_1-1}(T_{11}J_{11})\,2s_{1/2}T_1J_1|\}\,2s_{1/2}^{n_1}T_1J_1)\ (1d_{3/2}^{n_2-1}(t_{22}s_{22}T_{22}J_{22})\,1d_{3/2}TJ|\}\,1d_{3/2}^{n_2}t_2s_2T_2J_2),
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
&\langle[(T_{11}J_{11})\,2s_{1/2}](T_1J_1)[(T_{22}J_{22})\,1d_{3/2}](T_2J_2)\,TJ|\,V_{n_1,n_1+n_2}|[(T_{11}J_{11})\,2s_{1/2}](T_1J_1)\\
&\times[(T_{22}J_{22})\,1d_{3/2}](T_2J_2)\,TJ\rangle\\
&=\sum'\langle T_{11}\tfrac{1}{2}(T_1)\,T_{22}\tfrac{1}{2}(T_2)\,T|\,T_{11}T_{22}(T_{12})\,\tfrac{1}{2}\,\tfrac{1}{2}(T)\,T\rangle^2\langle J_{11}\tfrac{1}{2}(J_1)J_{22}\tfrac{3}{2}(J_2)J|\,J_{11}J_{22}(J_{12})\\
&\times\tfrac{1}{2}\,\tfrac{3}{2}(J')J\rangle^2\langle 2s_{1/2}1d_{3/2}T'J'|\,V|2s_{1/2}1d_{3/2}T'J'\rangle,
\end{aligned}\tag{9}$$

where the transformation function is expressible by the U-coefficient (or Wigner's $9j$-symbol),[9] and the wave function $|2s_{1/2}1d_{3/2}T'J'\rangle$ represents the vector-coupled and antisymmetric state of two particles $2s_{1/2}$ and $1d_{3/2}$. The results are given in Table III for the case $n_1+n_2\leqq 4$.

Although the calculation of the Coulomb interaction,

$$V'=\sum_{i>j}'V_{ij}^c=\sum_{i>j}'\frac{1-\tau_{zi}}{2}\cdot\frac{1-\tau_{zj}}{2}\cdot\frac{e^2}{r_{ij}}$$

is complicated, because it is non-scalar in isotopic spin space, the Coulomb energies also can be written as linear combinations of the two-body matrix elements of the form

Table IV. Table of $E^c(2s^n{}_{1/2}TJ;\,T_zJ_z)\equiv\langle 2s^n{}_{1/2}TJ;\,T_zJ_z|V^c|2s^n{}_{1/2}TJ;\,T_zJ_z\rangle$

n	T	J	$E^c(2s^2{}_{1/2}TJ;\,T_zJ_z)$
2	0	1	
	1	0	$\delta(T_z,-1)E^c(2s^2{}_{1/2}10;\,-10)$
3	1/2	1/2	$\delta(T_z,-1/2)E^c(2s^2{}_{1/2}10;\,-10)$
4	0	0	$E^c(2s^2{}_{1/2}10;\,-10)$

Table V. Table of $E^c(1d^{n}_{3/2}\alpha TJ;\, T_zJ_z)\equiv\langle 1d^{n}_{3/2}\alpha TJ;\, T_zJ_z|\,V^c\,|1d^{n}_{3/2}\alpha TJ;\, T_zJ_z\rangle$

n	T	J	$E^c(1d^{n}_{3/2}TJ;\, T_zJ_z)$	
2	0	1 (2, 0)	0	
	0	3 (2, 0)	0	
	1	0 (0, 0)	$\delta(T_z,-1)E^c(1d^2_{3/2}10;\,-10)$	
	1	2 (2, 1)		$\delta(T_z,-1)E^c(1d^2_{3/2}12;\,-1J_z)$
3	1/2	3/2 (1, 1/2)	$1/12(1-2T_z)\{5E^c(1d^2_{3/2}10;\,-10)+$	$E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1/2	1/2 (3, 1/2)		
	1/2	5/2 (3, 1/2)		$1/2(1-2T_z)E^c(1d^2_{3/2}12;\,-1J_z)$
	1/2	7/2 (3, 1/2)		
	3/2	3/2 (1, 1/2)	$1/48(1-2T_z)(3-2T_z)\{E^c(1d^2_{3/2}10;\,-10)+$	$E^c(1d^2_{3/2}12;\,-1J_z)\}$
4	0	0 (0, 0)	$1/6\{5E^c(1d^2_{3/2}10;\,-10)+$	$E^c(1d^2_{3/2}12;\,-1J_z)\}$
	0	2 (2, 1)	$1/2\{E^c(1d^2_{3/2}10;-10)+$	$E^c(1d^2_{3/2}12;\,-1J_z)\}$
	0	2 (4, 1)		$E(1d^2_{3/2}12;\,-1J_z)$
	0	4 (4, 1)		
	1	2 (2, 1)	$1/4(1-T_z)\{(2+T_z)E^c(1d^2_{3/2}10;\,-10)+$	$(2-3T_z)E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1	1 (2, 0)	$-1/4(1-T_z)\{T_zE^c(1d^2_{3/2}10;\,-110)+$	$(4-T_z)E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1	3 (2, 0)		
	2	0 (0, 0)	$1/12(2-T_z)(1-T_z)\{E^c(1d^2_{3/2}10;-10)+$	$5E^c(1d^2_{3/2}12;\,-1J_z)\}$
5	1/2	3/2 (1, 1/2)	$1/3\{(2+T_z)E^c(1d^2_{3/2}10;\,-10)+$	$(4-7T_z)E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1/2	1/2 (3, 1/2)		
	1/2	5/2 (3, 1/2)	$1/4\{(1-2T_z)E^c(1d^2_{3/2}10;\,-10)+$	$(7-6T_z)E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1/2	7/2 (3, 1/2)		
	3/2	3/2 (1, 3/2)	$1/48(3-2T_z)(5-2T_z)\{E^c(1d^2_{3/2}10;\,-10)+$	$5E^c(1d^2_{3/2}12;\,-1J_z)\}$
6	0	1 (2, 0)	$1/2E^c(1d^2_{3/2}10;\,-10)+$	$5E^c(1d^2_{3/2}12;\,-1J_z)$
	0	3 (2, 0)		
	1	0 (0, 0)	$1/2\{(1+T_z^2)E^c(1d^2_{3/2}10;\,-10)+$	$(1-T_z)5E^c(1d^2_{3/2}12;\,-1J_z)\}$
	1	2 (2, 1)	$1/2\{(1-T_z)E^c(1d^2_{3/2}10;\,-10)+(5-4T_z+T^2_z)E^c(1d^2_{3/2}12;\,-1J_z)\}$	
7	1/2	3/2 (1, 1/2)	$1/4(3-2T_z)\{E^c(1d^2_{3/2}10;\,-10)+$	$5E^c(1d^2_{3/2}12;\,-1J_z)\}$
8	0	0 (0, 0)	$E^c(1d^2_{3/2}10;\,-10)+$	$5E^c(1d^2_{3/2}12;\,-1J_z)$

$\langle j_1j_2T=1J;\, T_z=-1J_z|\,V^c\,|\,j_1j_2T=1J;\, T_z=-1J_z\rangle$. The results for $\langle 2s^{n_1}_{1/2}TJ;\, T_zJ_z|\sum V^c|\,2s^{n_1}_{1/2}TJ;\, T_zJ_z\rangle$ and $\langle 1d^{n_2}_{3/2}ts\,TJ;\, T_zJ_z|\sum V^c|\,1d^{n_2}_{3/2}ts\,TJ;\, T_zJ_z\rangle$ are written in Tables IV and V. However, the

$$\langle 2s^{n_1}_{1/2}(T_1J_1)\,1d^{n_2}_{3/2}(t_2s_2T_2J_2)TJ;\, T_zJ_z|\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}V^c_{ij}|\,2s^{n_1}_{1/2}(T_1J_1)\,1d^{n_2}_{3/2}(t_2s_2T_2J_2)TJ;\, T_zJ_z\rangle$$

are omitted because they are complicated except in the case of $n_1=4$. They vanish in

the ground states of the $2s_{1/2}$ and the $2s_{1/2}-1d_{3/2}$-nuclei, and in the ground states of the $1d_{3/2}$-nuclei they are very simple since $n_1=4$, and they reduce to

$$\langle 2s_{1/2}^4(00)\,1d_{3/2}^{n_2}(tsTJ)\,TJ\,;\,T_zJ_z|\sum_{i=1}^{4}\sum_{j=5}^{n_2+4}V_{ij}^c|2s_{1/2}^4(00)\,1d_{3/2}^{n_2}(tsTJ)\,TJ\,;\,T_zJ_z\rangle$$

$$=z_2\langle 2s_{1/2}^4(00)\,1d_{3/2}\,T=\tfrac{1}{2},J=\tfrac{3}{2}\,;\,T_z=-\tfrac{1}{2}J_z|\sum_{i=1}^{4}V_{i5}^c|2s_{1/2}^4(00)\,1d_{3/2},T=\tfrac{1}{2},J=\tfrac{3}{2}\,;\,T_z=-\tfrac{1}{2}J_z\rangle \tag{10}$$

with $z_2=-2T_z$,

Table VI. Experimental and calculated energies in Mev

i) $2s_{1/2}$-nuclei

Nucleus	Spin of ground state	Binding energies minus that of Si^{28}	
		Experimental	Calculated
$_{14}Si_{15}^{29}$	1/2	− 8.47	− 8.48
$_{14}Si_{16}^{30}$	0	−19.09	−19.09
$_{15}P_{14}^{29}$	1/2	− 2.79	− 2.80
$_{15}P_{16}^{31}$	1/2	−26.38	−26.37
$_{16}S_{15}^{31}$	1/2	−20.16	−20.15
$_{16}S_{16}^{32}$	0	−35.24	−35.25

ii) $1d_{3/2}$-nuclei

Nucleus	Spin of ground state	Binding energies minus that of S^{32}	
		Experimental	Calculated
$_{16}S_{17}^{33}$	3/2	− 8.65	− 8.64
$_{16}S_{18}^{34}$	0	−20.05	−20.07
$_{16}S_{19}^{35}$	3/2	−27.07	−27.09
$_{16}S_{20}^{36}$	0	−36.97	−36.91
$_{17}Cl_{16}^{33}$	3/2	− 2.42	− 2.39
$_{17}Cl_{17}^{34}$	0	−13.75	−13.83
$_{17}Cl_{18}^{35}$	3/2	−26.46	−26.37
$_{17}Cl_{19}^{36}$	2	−35.03	−35.14
$_{17}Cl_{20}^{37}$	3/2	−45.39	−45.27
$_{18}A_{17}^{35}$	3/2	−19.69	−19.49
$_{18}A_{18}^{36}$	0	−34.96	−34.83
$_{18}A_{19}^{37}$	3/2	−43.79	−43.91
$_{18}A_{20}^{38}$	0	−55.54	−55.73
$_{19}K_{18}^{37}$	3/2	−36.88	−37.04
$_{19}K_{19}^{38}$	0	−48.88	−48.92
$_{19}K_{20}^{39}$	3/2	−61.96	−61.90
$_{20}Ca_{19}^{39}$	3/2	−54.32	−54.40
$_{20}Ca_{20}^{40}$	0	−70.29	−70.18

iii) $2s_{1/2}$-$1d_{3/2}$-nuclei

Nucleus	Spin of ground state	Binding energy minus that of Si^{28}	Equation (22)		The right hand side of (22)
			Experimental	Calculated	
$_{14}Si_{17}^{31}$	3/2	−25.68	2.05	2.06	$-3/8\{E_N(2s_{1/2}\,1d_{3/2}\,01)+E^N(2s_{1/2}\,1d_{3/2}\,11)\}$
					$-5/8\{E^N(2s_{1/2}\,1d_{3/2}\,02)+E^N(2s_{1/2}\,1d_{3/2}\,12)\}$
$_{14}Si_{18}^{32}$	0	−35.00	4.17	4.12	$-3/4\{E^N(2s_{1/2}\,1d_{3/2}\,01)+E^N(2s_{1/2}\,1d_{3/2}\,11)\}$
					$-5/4\{E^N(2s_{1/2}\,1d_{3/2}\,02)+E^N(2s_{1/2}\,1d_{3/2}\,12)\}$
$_{15}P_{18}^{33}$	1/2	−44.42	2.02	2.06	$-3/8\{E^N(2s_{1/2}\,1d_{3/2}\,01)+E^N(2s_{1/2}\,1d_{3/2}\,11)\}$
					$-5/8\{E^N(2s_{1/2}\,1d_{3/2}\,01)+E^N(2s_{1/2}\,1d_{3/2}\,12)\}$
$_{15}P_{17}^{32}$	1	−34.31	0.70		$-1/8\{E^N(2s_{1/2}\,1d_{3/2}\,01)+E^N(2s_{1/2}\,1d_{3/2}\,11)\}$
					$+5/8\{E^N(2s_{1/2}\,1d_{3/2}\,02)+E^N(2s_{1/2}\,1d_{3/2}\,12)\}$

Table VII. Energy Parameters of the Present Model (in Mev) determined by the binding energies.

i)

$A^N(2s_{1/2})$	−8.478
$E^N(2s^2_{1/2}\,10)$	−2.134
$E^N(2^2s_{1/2}\,01)$	−2.278
$A^c(2s_{1/2})$	+5.680
$E^c(2s^2_{1/2}\,10)$	+0.540

ii)

$A'^N(1d_{3/2})$	−8.639
$E^N(1d^2_{3/2}\,10)$	−2.796
$E^N(1d^2_{3/2}\,12)$	+0.089
$1/7(3E^N(1d^2_{3/2}\,01)+7E^N(1d^2_{3/2}\,03)$	−3.074
$A'^c(1d_{3/2})$	+6.246
$E^c(1d^2_{3/2}\,10)$	+0.693
$E^c(1d^2_{3/2}\,12)$	+0.365

iii)

$1/8\{3E^N(2s_{1/2}\,1d_{3/2}\,01)+5E^N(2s_{1/2}\,1d_{3/2}\,02)+3E^N(2s_{1/2}\,1d_{3/2}\,11)+5E^N(2s_{1/2}\,1d_{3/2}\,12)\}$	−2.062

and

$$\langle 2s^4_{1/2}(00)\,1d_{3/2}T=\tfrac{1}{2}\,J=\tfrac{3}{2}\,;\,T_z=-\tfrac{1}{2}\,J_z|\sum_{i=1}^{4}V^c_{ik}|2s^4_{1/2}(00)\,1d_{3/2}\,T=\tfrac{1}{2}\,J=\tfrac{3}{2}\,;\,T_z=-\tfrac{1}{2}\,J_z\rangle$$

$$=1/4\cdot\sum(2J+1)\langle 2s_{1/2}\,1d_{3/2}T=1J\,;\,T_z=-1J_z|V^c|2s_{1/2}\,1d_{3/2}T=1J\,;\,T_z=-1J_z\rangle. \tag{11}$$

Consequently the contribution of the extra nucleons to the binding energy is expressed as

$$
\begin{aligned}
&\text{(Binding Energy of nucleus)} - \text{(Binding Energy of } Si^{28}) \\
&= n_1 A^N(2s_{1/2}) + n_2 A^N(1d_{3/2}) + \langle 2s_{1/2}^{n_1} T_1 J_1 | V | 2s_{1/2}^{n_1} T_1 J_1 \rangle \\
&+ \langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 | V | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 \rangle + \langle 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ | \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_2} V_{ij} | \\
&\times 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ \rangle \\
&+ z_1 A^c(2s_{1/2}) + z_2 A^c(1d_{3/2}) + \langle 2s_{1/2}^{n_1} T_1 J_1 ;\, T_{1z} J_{1z} | V^c | 2s_{1/2}^{n_1} T_1 J_1 ;\, T_{1z} J_{1z} \rangle \\
&+ \langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ;\, T_{2z} J_{2z} | V^c | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ;\, T_{2z} J_{2z} \rangle \\
&+ \langle 2s_{1/2}^{n_1}(T_1 J_1)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ ;\, T_z J_z | \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_2} V_{ij}^c | 2s_{1/2}^{n_1}(T_1 J_1) \\
&\times 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ ;\, T_z J_z \rangle
\end{aligned}
\tag{12}
$$

where $A^N(j)$ is the single-nucleon energy in the j-shell without Coulomb energy, (i. e. the kinetic energy of a nucleon in the j-shell and its nuclear interaction with the closed shells), $A^c(j)$ is the energy of the Coulomb interaction of a single proton in the j-shell with the closed shells, and z_1 and z_2 are the numbers of protons in the $2s_{1/2}$ and $1d_{3/2}$ shells respectively. The other terms are written as linear combinations of the two-body matrix elements which are given in Tables I—V. For simplicity, in the following the two-body matrix elements are denoted as

$$E^N(j_1 j_2 TJ) = \langle j_1 j_2 TJ | V | j_1 j_2 TJ \rangle \tag{13}$$

and

$$E^c(j_1 j_2 T=1J ;\, T_z=-1J_z) = \langle j_1 j_2 T=1J ;\, T_z=-1J_z | V^c | j_1 j_2 T=1J ;\, T_z=-1J_z \rangle. \tag{14}$$

These A^N, A^C, E^N and E^C can be calculated, if the two-body interaction and the wave function are known. They, however, are not well known at present, and so in this section no specific assumptions are made on the two-body interaction and radial dependence of the wave function. These energy parameters are determined by a least squares fit to the observed binding energies.

The nuclei considered in this paper are classified into three types ; (i) the protons and neutrons occupy the $2s_{1/2}$ shell in the ground state ($2s_{1/2}$-nucleus), (ii) they occupy the $1d_{3/2}$-shell ($1d_{3/2}$-nucleus), and (iii) the protons occupy the $2s_{1/2}$-shell and the neutrons the $1d_{3/2}$-shell ($2s_{1/2}$-$1d_{3/2}$ nucleus).

i) $2s_{1/2}$-nucleus

The experimental[10] and the calculated binding energies are shown in Table VI. The values of the parameters are summarized in Table VII. P^{30} also belongs to this group and usually is considered to correspond to the configuration $2s_{1/2}^2(01)$. However, $|E^N(2s_{1/2}\,1d_{3/2}01)|$ seems to be quite large, so that in the ground state of P^{30} two configurations $2s_{1/2}^2(01)$ and $2s_{1/2}\,1d_{3/2}(01)$ may interact with each other. For this reason, this nucleus is not employed for determination of the parameters. The binding energy

of P^{30} will be discussed in § 4.*

ii) $1d_{3/2}$-nucleus

For the nucleus belonging to this type, the equation (12) reduces to

$$\begin{aligned}&(\text{Binding Energy of } 1d_{3/2}\text{-nucleus}) - (\text{Binding Energy of } S^{32})\\&= n_2 A'^N(1d_{3/2}) + \langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 | V | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 \rangle\\&+ z_2 A'^c(1d_{3/2}) + \langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ; T_{2z} J_{2z} | V^c | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ; T_{2z} J_{2z} \rangle, \qquad (15)\end{aligned}$$

where $$z_2 = -2T_{2z},$$

and

$$A'^N(1d_{3/2}) = A^N(1d_{3/2}) + \langle 2s_{1/2}^4(00)\, 1d_{3/2}\ 1/2\ 3/2 \,|\, V | 2s_{1/2}^4(00)\, 1d_{3/2}\ 1/2\ 3/2 \rangle \qquad (16)$$

and

$$\begin{aligned}&A'^c(1d_{3/2}) = A^c(1d_{3/2}) + \langle 2s_{1/2}^4(00)\, 1d_{3/2}\ 1/2\ 3/2\ ;\ -1/2\ J_z | V^c | 2s_{1/2}^4(00)\\&\times 1d_{3/2}\ 1/2\ 3/2\ ;\ -1/2\ J_z \rangle. \qquad (17)\end{aligned}$$

Here (10) and the following relations are used,

$$\begin{aligned}&(\text{Binding Energy of } S^{32})\\&= (\text{Binding Energy of } Si^{28}) + 4A^N(2s_{1/2}) + \langle 2s_{1/2}^1 00 | V | 2s_{1/2}^4 00 \rangle\\&+ 2A^c(2s_{1/2}) + \langle 2s_{1/2}^4 00 | V^c | 2s_{1/2}^4 00 \rangle \qquad (18)\end{aligned}$$

and

$$\begin{aligned}&\langle 2s_{1/2}^4(00)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ | V | 2s_{1/2}^4(00)\, 1d_{3/2}^{n_2}(t_2 s_2 T_2 J_2) TJ \rangle\\&= n_2 \langle 2s_{1/2}^4(00)\, 1d_{3/2}\ 1/2\ 3/2 \,|\, V | 2s_{1/2}^4(00)\, 1d_{3/2}\ 1/2\ 3/2 \rangle. \qquad (19)\end{aligned}$$

Therefore in Table VI (ii) are given the binding energies of $1d_{3/2}$-nuclei minus that of S^{32}. In (15), $\langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 | V | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 \rangle$ and $\langle 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ; T_{2z} J_{2z} | V^c | 1d_{3/2}^{n_2} t_2 s_2 T_2 J_2 ; T_{2z} J_{2z} \rangle$ are expressed as linear combinations of $E^N(1d_{3/2}^2 TJ)$ and $E^c(1d_{3/2}^2 T=1J;\ T_z=-1J_z)$, respectively. The parameters

$$A'^N,\ A'^C,\ E^N(1d_{3/2}^2 TJ), \text{ and } E^C(1d_{3/2}^2 T=1J;\ T_z=-1J_z)$$

are determined by a least squares fit. These values of parameters are shown in Table VII ii), and the calculated binding energies are given in Table VI (ii). The agreement is remarkably good, the mean deviation being 0.13 Mev.

The expectation value of the two-body interaction in the ground state involves

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* There are the following relations between Talmi's parameters and ours in $2s_{1/2}$-nuclei,

$$A = A^N(2s_{1/2}),$$
$$a = 1/4\{3E^N(2s^2{}_{1/2}\,10) + E^N(2s^2{}_{2/1}\,01)\},$$
$$b = 1/2\{E^N(2s^2{}_{1/2}\,10) - E^N(2s^2{}_{1/2}\,01)\}$$

and $$c = 0$$

where A, a, b and c are Talmi's parameters in ref. 3).

$E^N(1d^2_{3/2}01)$ and $E^N(1d^2_{3/2}03)$ always in the same linear combination, i. e. $3E^N(1d^2_{3/2}01)+7E^N(1d^2_{3/2}03)$. Therefore their values cannot be obtained separately.

The ground state of Cl^{34} has spin 0 and isotopic spin 1, while the 0.142 Mev level probably has spin 3 and isotopic spin 0, so that these two states correspond to $1d^2_{3/2}(10)$ and $1d^2_{3/2}(03)$ respectively.[11] If the effect of configuration interaction on the energies of these states can be neglected, it follows that

$$E^N(1d^2_{3/2}03)=E^N(1d^2_{3/2}10)+0.142=-2.654 \text{ Mev}. \tag{20}$$

$E^N(1d^2_{3/2}01)$ is also obtained from (20) and

$$1/7\cdot\{3E^N(1d^2_{3/2}01)+7E^N(1d^2_{3/2}03)\}=-3.074 \text{ Mev}, \tag{21}$$

which is determined by the binding energies.*

Table VIII. Energy Parameters of the Present Model (in Mev) determined by the excited levels.

$A(1d_{3/2})$	-7.178	First excited state of Si^{29} which energy we assume 1.300 Mev
$1/8\{3E^N(2s_{1/2}1d_{3/2}01)+9E^N(11)+5E^N(02)+15E^N(12)\}$	-1.461	$A'(1d_{3/2})-A(1d_{3/2})$
$E^N(1d^2_{3/2}03)$	-2.654	First excited state of Cl^{34} which energy is 0.14 Mev
$E^N(1d^2_{3/2}01)$	-0.980	

iii) $2s_{1/2}$-$1d_{3/2}$-nucleus

As discussed in the next section, the expectation values of the Coulomb interaction show that the potential well is almost the same for $2s_{1/2}$-nuclei and $1d_{3/2}$-nuclei. Thus the parameters determined in (i) and (ii) can be used for the expressions of the binding energies of the $2s_{1/2}$-$1d_{3/2}$ nuclei. Using equations (16) and (17), $A^N(1d_{3/2})$ and $A^C(1d_{3/2})$ can be eliminated from those expressions. Accordingly they reduce to

$$\begin{aligned}
&(\text{Binding Energy of } 2s_{1/2}\text{-}1d_{3/2} \text{ nucleus})-(\text{Binding Energy of } {}_{14}Si^{28}_{14})\\
&-n_1A^N(2s_{1/2})-n_2A'^N(1d_{3/2})-\langle 2s^{n_1}_{1/2}T_1J_1|V|2s^{n_1}_{1/2}T_1J_1\rangle-\langle 1d^{n_2}_{3/2}t_2s_2T_2J_2|V|1d^{n_2}_{3/2}t_2s_2T_2J_2\rangle\\
&-z_1A^C(2s_{1/2})-z_2A'^C(1d_{3/2})-\langle 2s^{n_1}_{1/2}T_1J_1\,;\,T_{1z}J_{1z}|V^C|2s^{n_1}_{1/2}T_1J_1\,;\,T_{1z}J_{1z}\rangle\\
&-\langle 1d^{n_2}_{3/2}t_2s_2T_2J_2\,;\,T_{2z}J_{2z}|V^C|1d^{n_2}_{3/2}t_2s_2T_2J_2\,;\,T_{2z}J_{2z}\rangle\\
&=\langle 2s^{n_1}_{1/2}(T_1J_1)\,1d^{n_2}_{3/2}(t_2s_2T_2J_2)\,TJ|\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}V_{ij}|2s^{n_1}_{1/2}(T_1J_1)\,1d^{n_2}_{3/2}(t_2s_2T_2J_2)\,TJ\rangle\\
&-n_2\langle 2s^4_{1/2}(00)\,1d_{3/2}\;1/2\;3/2|\sum_{i=1}^{4}V_{i5}|2s^4_{1/2}(00)\,1d_{3/2}\;1/2\;3/2\rangle
\end{aligned}$$

* There are the following relations between Talmi's parameters and ours in $1d_{3/2}$-nuclei.

$A=A'^N(1d_{3/2})$,

$a=1/40\cdot\{3E^N(1d^2_{3/2}01)+7E^N(1d^2_{3/2}03)\}-1/8\cdot E^N(1d^2_{3/2}10)+7/8\cdot E^N(1d^2_{3/2}12)$,

$b=-1/20\cdot\{3E^N(1d^2_{3/2}01)+7E^N(1d^2_{3/2}03)\}-1/4\cdot E^N(1d^2_{3/2}10)+3/4\cdot E^N(1d^2_{3/2}12)$,

and $c=-1/8\cdot E^N(1d^2_{3/2}10)+1/8\cdot E^N(1d^2_{3/2}12)$,

where A, a, b and c are Talmi's parameters in ref. 3).

$$+\langle 2s_{1/2}^{n_1}(T_1J_1)\,1d_{3/2}^{n_2}(t_2s_2T_2J_2)TJ,\ T_zJ_z|\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}V_{ij}^{c}|2s_{1/2}^{n_1}(T_1J_1)\,1d_{3/2}^{n_2}(t_2s_2T_2J_2)TJ\,;\ T_zJ_z\rangle$$

$$-z_2\langle 2s_{1/2}^{4}(00)\,1d_{3/2}\ 1/2\ 3/2\,;\ -1/2\ J_z|\sum_{i=1}^{4}V_{i5}^{c}|2s_{1/2}^{4}(00)\,1d_{3/2}\ 1/2\ 3/2\,;\ -1/2\ J_z\rangle. \tag{22}$$

The left-hand side of (22) is evaluated by using the observed binding energy and results obtained in (i) and (ii). The right-hand side is given as a linear combination of $E^N(2s_{1/2}1d_{3/2}TJ)$ and $E^C(2s_{1/2}1d_{3/2}1J\,;\,-1J_z)$. The nuclei in Table VI (iii) have $z_2=0$ in the ground state, so that the Coulomb terms in the right-hand side of (22) vanish. Therefore they cannot be determined. These Coulomb energy-parameters become necessary when the energies of the excited states are calculated. However, they may be small and will be neglected.

For the nuclei in Table VI (iii) except ${}_{15}P_{17}^{32}$, each linear combination is always a multiple of

$$1/8\cdot\{3E^N(2s_{1/2}1d_{3/2}01)+5E^N(2s_{1/2}1d_{3/2}02)+3E^N(2s_{1/2}1d_{3/2}11)+5E^N(2s_{1/2}1d_{3/2}12)\}$$
$$=-2.062\ \text{Mev}, \tag{23}$$

so that this value -2.062 Mev seems quite certain.

The first and third excited levels of Si^{29} are classified as $(3/2+)$, and the capture probabilities for neutrons in the Si^{28} $(d,\,p)$ Si^{29} reaction indicate the single-particle property for the first excited state and the complex structure for the third one. In order to obtain the numerical value of the parameter $A^N(1d_{3/2})$, it is assumed that the first excited state is a pure single particle state which has a $1d_{3/2}$ neutron. In this paper this excitation energy is taken as 1.30 Mev, compared with 1.28 Mev from experiment. Therefore,

$$A^N(1d_{3/2})=A^N(2s_{1/2})+1.300=-7.178\ \text{Mev}, \tag{24a}$$

and using (16) and Table III,

$$\begin{aligned}&A'^N(1d_{3/2})-A^N(1d_{3/2})\\&=\langle 2s_{1/2}^{4}(00)\,1d_{3/2}\ 1/2\ 3/2\,|\,V|2s_{1/2}^{4}(00)\,1d_{3/2}\ 1/2\ 3/2\rangle\\&=1/8\cdot\sum(2T+1)\,(2J+1)E^N(2s_{1/2}1d_{3/2}TJ)\\&=-1.461\ \text{Mev}.\end{aligned} \tag{24b}$$

Equations (23) and (24b) are two conditions to determine the four parameters $E^N(2s_{1/2}1d_{3/2}TJ)$. If two more independent conditions are found, all the $E^N(2s_{1/2}1d_{3/2}TJ)$ can be determined, and all the parameters that are necessary to calculate the expectation value of the two-body interaction in the configuration $2s_{1/2}^{n_1}\ 1d_{3/2}^{n_2}$ are known except $E^C(2s_{1/2}1d_{3/2}1J\,;\,-1J_z)$, which can be neglected. To obtain two more conditions, it is very desirable to have more experimental data about spins and parities of the excited states of the nuclei belonging to these types.

In the following, two methods are introduced to determine $E^N(2s_{1/2}1d_{3/2}TJ)$:

1) the two more conditions are derived from the binding energy of F^{32}, and the

excitation energy of Si^{30}. P^{32} gives the condition

$$-1/8\{E^N(2s_{1/2}1d_{3/2}01)+E^N(2s_{1/2}1d_{3/2}11)\}+5/8\{E^N(2s_{1/2}1d_{3/2}02)$$
$$+E^N(2s_{1/2}1d_{3/2}12)\}=-0.70 \text{ Mev.} \tag{25}$$

Si^{30} has only two neutrons outside the closed shells, and has a simple low energy spectrum. The ground state corresponds to $2s^2_{1/2}(10)$, and the $2s_{1/2}1d_{3/2}(12)$ and/or $2s_{1/2}1d_{3/2}(11)$ states probably correspond to the first, second or third excited state. In Table IX, the parameters $E^N(2s_{1/2}1d_{3/2}TJ)$ are shown, for three identification.*

Table IX. The Values of $E^N(2s_{1/2}1d_{3/2}TJ)$ determined used P^{32} and Si^{30}

	Identification		
	(a)	(b)	(c)
$E^N(2s_{1/2}1d_{3/2}01)$	−3.403	−1.525	−2.805
$E^N(4s_{1/2}1d_{3/2}02)$	−1.737	−2.864	−2.096
$E^N(2s_{1/2}1d_{3/2}11)$	0.674	−1.204	0.076
$E^N(2s_{1/2}1d_{3/2}12)$	0.076	1.203	0.435

a) the state $2s_{1/2}1d_{3/2}(T=1\ J=2)$ is the second excited state at 3.51 Mev,
b) the state $2s_{1/2}1d_{3/2}(T=1\ J=1)$ is the first excited state at 2.23 Mev,
c) the state $2s_{1/2}1d_{3/2}(T=1\ J=1)$ is the second excited state at 3.51 Mev.
The identifications b) and c) give the results

$$E^N(2s_{1/2}1d_{3/2}11)\rangle E^N(2s_{1/2}1d_{3/2}12) \tag{26a}$$

and/or

$$E^N(2s_{1/2}1d_{3/2}01)\langle E^N(2s_{1/2}1d_{3/2}02). \tag{26b}$$

However, these values of the parameters do not seem to be admissible, probably because the short-range central force dominates in the two-body interaction, and this force cannot give the relations (26a) and (26b). Hence, only the set of values of the parameters in the identification a), which is called set 1, is used to calculate the excitation energy in § 4.

2) If the configuration interaction occurs in the ground state of P^{32}, condition (25) is not valid. Accordingly, in the next section, the two-body force is taken into consideration to find another set of values of $E^N(2s_{1/2}1d_{3/2}TJ)$, called set 2.

§ 3. Parameters and two-body force

In this section besides the four assumptions given in the introduction, assumptions are made about the wave function and the form of the nuclear two body interaction.

* $2s_{1/2}1d_{3/2}(12)$ cannot be identified with the first excited state, because this identification gives $|E^N(2s_{1/2}1d_{3/2}01)|$ such a large value that condition (36) derived from the binding energy P^{30} is violated.

For simplicity, the harmonic oscillator wave function is chosen.[12),13)]

$$R_{nl}(r) = N_{nl} \exp(-\nu r^2/2) r^l v_{nl}(r), \tag{27}$$

where N_{nl} is a normalization constant and $v_{nl}(r)$ is an associated Laguerre polynomial

$$v_{nl}(r) = L_{n+l+1/2}^{l+1/2}(\nu r^2). \tag{28}$$

The shape of the Coulomb interaction is well known, and its expectation values are functions of ν. By comparing them with the experimental values determined in § 2, the numerical values of ν are found, and are shown in Table X. These values of ν can be compared with those which are given by the nuclear radius in terms of ν;

Table X. Determination of ν

Coulomb energy parameter	Mev	$\nu \times 10^{-26}$ cm^{-2}
$A^c(2s_{1/2})$	5.680	0.300
$A^c(1d_{3/2})$	6.246	0.295
$E^c(2s^2_{1/2}10; -10)$	0.540	0.478
$E^c(1d^2_{3/2}10; -10)$	0.693	0.737
$E^c(1d^2_{3/2}2; -1M_z)$	0.365	0.299

r_0 cm	$\nu \times 10^{-26}$cm^{-2}		Relation between ν and nuclear radius
	$A=28$	$A=40$	
1.2×10^{-13}	0.439	0.346	$3/5\, r_0^2 A^{2/3} = \dfrac{2l+3+4(n-1)}{2\nu}$
1.4×10^{-13}	0.323	0.254	

$$\frac{3}{5} R^2 = r^2 = N_{nl}^2 \int_0^\infty e^{-\nu r^2} r^{2l+4} v_{nl}^2(r)\, dr = \frac{2l+3+4(n-1)}{2\nu} \tag{29}$$

where $R = 1.2\, A^{1/3} \times 10^{-13}$ cm. The value of ν fitted to $A^C(2s_{1/2})$ is very near that fitted to $A^C(1d_{3/2})$, and in this paper ν is fixed to be 0.300. The values obtained from $E^C(2s_{1/2}10)$, $E^C(1d_{3/2}10)$ and $E^C(1d_{3/2}12)$ show a considerable fluctuation, because of relatively large errors included in the values of E^C.

The interaction is (—in addition to Coulomb potential—) assumed as

$$\begin{aligned} V(12) = &\left(\omega - m\frac{1+\sigma_1\sigma_2}{2}\cdot\frac{1+\tau_1\tau_2}{2} + b\frac{1+\sigma_1\sigma_2}{2} + h\frac{1+\tau_1\tau_2}{2}\right) e^{-r_{12}^2/r_c^2} \\ &+ \left(\omega' + h'\frac{1+\tau_1\tau_2}{2}\right) S_{12}\, e^{-r_{12}^2/r_t^2} \\ &+ \left(\omega'' + h''\frac{1+\tau_1\tau_2}{2}\right) O_s\, e^{-r_{12}^2/r_{sp}^2}, \end{aligned} \tag{30a}$$

where

$$S_{12}=\frac{3(\sigma_1\cdot\boldsymbol{r}_{12})(\sigma_2\cdot\boldsymbol{r}_{12})}{r_{12}^2}-(\sigma_1\cdot\sigma_2) \tag{30b}$$

and

$$O_s=1/\hbar((\sigma_1+\sigma_2)\cdot[\boldsymbol{r}_{12}\times(\boldsymbol{P}_1-\boldsymbol{P}_2)]) \tag{30c}$$

which expresses the phenomenological spin-orbit interaction.

Since $E^N(2s_{1/2}^2 10)$ and $E^N(2s_{1/2}^2 01)$ depend only on the central interaction, they give the numerical values $w+m+b-h(=^3V_c^{\text{even}})$ and $w+m-b+h(=^1V_c^{\text{even}})$, which are the depths of central part of nuclear interaction in ^1E and $^2E^*$ states, respectively, in nucleus. According to Kalos, Biedenharn and Blatt,[14] the nuclear potential fitted to the data of two nucleon scattering in ^1E state is given by

$$-46.1\ e^{-r^2\times 10^{26}/2.2613}\ \text{Mev.} \tag{31}$$

The expectation value of the interaction energy depends on the quantity $\lambda_\alpha=r_\alpha\sqrt{\nu}/2$ (α : c, t or sp.), which is the ratio of the measure of the linear extention of the wave function to the range of nuclear force. By assuming the range of the central potential in the nucleus is the same as that in (31), the following relations,

$$^1V_c^{\text{even}}=w+m-b+h=-39.185\ \text{Mev}, \tag{32}$$

are obtained from $E^N(2s_{1/2}^2 01)$, and

$$^3V_c^{\text{even}}=w+m+b-h=-41.829\ \text{Mev} \tag{33}$$

from $E^N(2s_{1/2}^2 10)$. $^1V_c^{\text{even}}$ is slightly shallower than -46.1 Mev in (31).

The depth of the central potential in the ^3E state ($^3V_c^{\text{even}}$) is nearly equal to that in the ^1E state. This suggests the existence of the tensor force, unless the interaction in ^3E state in nucleus is diminished.

The two linear combinations of parameters $3E^N(2s_{1/2}1d_{3/2}11)+5E^N(2s_{1/2}1d_{3/2}12)$ and $5E^N(1d_{3/2}^2 12)-E^N(1d_{3/2}^2 10)$ do not depend on tensor force. Using $w+m$ and $b-h$ determined above, one can eliminate w and b from the expressions of these two combinations, and obtain the expressions involving only a parameter $m-h$ for the central potential part. If there is no spin-orbit interaction ($w''=h''=0$), each of the two expressions gives a different value for $m-h$ beyond the range of the uncertainties of the parameters. Further, this contradiction cannot be eliminated even if the Yukawa-potential is employed instead of the Gaussian one. There must be some other kind of interaction than the central and tensor interaction. In the present work, the phenomenological spin-orbit interaction is adopted.[15]

According to Kalos et al.,[14] if the central potential in ^3E is taken as same as that in ^1E (31), the following numerical values are obtained for the tensor force,

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* The two nucleon system is devided into four states triplet even (^3E), triplet odd (^3O), singlet even (^1E), and singlet odd (^1O).

$$^3V_t^{\text{even}} = w' - h' = 25 \sim 23 \text{ Mev} \tag{34}$$

and $r_t = 2.00 \sim 2.12$, which corresponds to the magnitude of $\lambda_t = \gamma_t \sqrt{\nu}/2$ changing from 0.77 to 0.82. Accordingly λ_t is taken as 0.80 and λ_{sp} is changed from 0.8 to $\sqrt{3/2}$.

There are eight unknowns and eight linearly independent conditions;

$$E^N(2s_{1/2}^2 10) = -2.134 \text{ Mev} = 0.05446(w + m - h + b)$$

$$E^N(2s_{1/2}^2 01) = -2.278 \text{ Mev} = 0.05446(w + m + h - b)$$

$$3E^N(2s_{1/2}1d_{3/2}01) + 5E^N(2s_{1/2}1d_{3/2}02) = -18.894 \text{ Mev}$$
$$= 0.3276(w - h) + 0.2324(m + b) + 3E_{sp}^N(2s_{1/2}1d_{3/2}01) + 5E_{sp}^N(2s_{1/2}1d_{3/2}02)$$

$$3E^N(2s_{1/2}1d_{3/2}11) + 5E^N(2s_{1/2}1d_{3/2}12) = 2.402 \text{ Mev}$$
$$= 0.2360(w + h) - 0.0493(m - b) + 3E_{sp}^N(2s_{1/2}1d_{3/2}11) + 5E_{sp}^N(2s_{1/2}1d_{5/2}12)$$

$$E^N(1d_{3/2}^2 10) = -2.796 \text{ Mev}$$
$$= 0.06445(w + h) + 0.01711(m - b) + 0.11883(w' + h') + E_{sp}^N(1d_{3/2}^2 10)$$

$$E^N(1d_{3/2}^2 12) = 0.089 \text{ Mev}$$
$$= 0.02439(w + h) - 0.00232(m - b) + 0.02388(w' + h') + E_{sp}^N(1d_{3/2}^2 12)$$

$$E^N(1d_{3/2}^2 01) = -0.980 \text{ Mev}$$
$$= 0.04442(w - h) + 0.01601(m + b) - 0.03399(w' - h') + E_{sp}^N(1d_{3/2}^2 01)$$

$$E^N(1d_{3/2}^2 03) = -2.654 \text{ Mev}$$
$$= 0.04442(w - h) + 0.03896(m + b) - 0.01938(w' - h') + E_{sp}^N(1d_{3/2}^2 03) \tag{35}$$

where the E_{sp}^N are given in Table XI for several λ_{sp}. These equations determine all eight parameters w, m and so on for each λ_{sp}, which are shown in Table XII and Fig. 1. By using the eight parameters obtained above, $E^N(2s_{1/3}1d_{3/2}TJ)$ are calculated, and are shown as functions of λ_{sp} in Table XIII.

The ground state of P^{30} is characterized experimentally as $T=0$ $J=1$. And theoretically low lying states of $T=0$ $J=1$ are considered to correspond to the configurations $2s_{1/2}^2(01)$ or $2s_{1/2}1d_{5/2}(01)$. If $|E^N(2s_{1/2}1d_{3/2}01)|$ should be larger than $+3.9$

Table XI. Matrix elements of spin-orbit force

λ_{sp}^2	$E_{sp}^N(1d_{3/2}^2 10) \div (w''+h'')$	$E_{sp}^N(1d_{3/2}^2 12) \div (w''+h'')$	$E_{sp}^N(1d_{3/2}^2 01) \div (w''-h'')$	$E_{sp}^N(1d_{3/2}^2 03) \div (w''-h'')$	$E^N(2s_{1/2}1d_{3/2}01) \div (w''-h'')$	$E^N(2s_{1/2}1d_{3/2}02) \div (w''-h'')$	$E^N(2s_{1/2}1d_{3/2}11) \div (w''+h'')$	$E^N(2s_{1/2}1d_{3/2}12) \div (w''+h'')$
0.64	−0.2376	−0.1045	−0.1342	−0.1085	−0.2818	−0.0564	−0.1261	−0.0252
1.00	−0.3978	−0.2811	−0.3032	−0.3120	−0.5690	−0.1138	−0.2983	−0.0597
1.20	−0.4783	−0.4015	−0.4041	−0.4584	−0.7278	−0.1456	−0.4172	−0.0834
1.354	−0.5364	−0.4998	−0.4817	−0.5814	−0.8467	−0.1693	−0.5158	−0.1032
1.5	−0.5889	−0.5961	−0.5547	−0.7046	−0.9571	−0.1914	−0.6135	−0.1227

Table XII. Depths of the two-body forces obtained in § 3

λ_{sp}^2	$w+m+b-h$	$w+m-b+h$	$w-m+b+h$	$w-m-b-h$	$w''+h''$	$w''-h''$	$w'+h'$	$w'-h'$
0.64	−41.829	−39.185	−10.305	−129.19	−14.922	11.822	−37.874	−1.026
0.81	−41.829	−39.185	10.157	101.403	−5.7806	7.2129	−27.324	−10.627
1.00	−41.829	−39.185	16.973	76.453	−3.0489	4.7526	−23.670	−18.787
1.20	−41.829	−39.185	19.953	56.017	−1.9254	3.3821	−21.808	−25.141
1.354	−41.829	−39.185	21.173	43.529	−1.4732	2.7318	−20.951	−28.867
1.50	−41.829	−39.185	21.909	33.657	−1.1957	2.2944	−20.374	−31.717

The radial dependences of the central, tensor and spin-orbit forces are assumed as Gaussian type. $r_c=1.50\times10^{-13}$ cm and $r_t=2.07\times10^{-13}$ cm.

Table XIII. Energy parameter $E^N(2s_{1/2}1d_{3/2}TJ)$ calculated by using the potential given in Table XII (in Mev)

λ_{sp}^2	$E^N(2s_{1/2}1d_{3/2}11)$	$E^N(2s_{1/2}1d_{3/2}12)$	$E^N(2s_{1/2}1d_{3/2}01)$	$E^N(2s_{1/2}1d_{3/2}02)$
0.64	4.215	−2.049	−5.264	−0.620
0.81	3.254	−1.472	−4.756	−0.926
1.00	2.937	−1.272	−4.346	−1.202
1.20	2.762	−1.177	−3.947	−1.411
1.354	2.689	−1.133	−3.729	−1.542
1.50	2.641	−1.104	−3.559	−1.644
∞	2.397	−0.958	−2.341	−2.314

Mev, the absolute value of the binding energy of P^{30} must be larger than the experimental value. Therefore the condition for $E^N(2s_{1/2}1d_{3/2}01)$ can be obtained as follows :

$$E^N(2s_{1/2}1d_{3/2}01)\geq-3.9 \text{ Mev}. \tag{36}$$

In Si^{30} the configuration $2s_{1/2}1d_{3/2}(12)$ is considered to correspond to the first or second excited state. Experimentally the first excited level has the excitation energy 2.25 Mev, and if it is assumed that it is a $2s_{1/2}1d_{3/2}(12)$ state, the following condition is obtained :

$$E^N(2s_{1/2}1d_{3/2}12)=-1.18 \text{ Mev}.^* \tag{37}$$

From Table XIII it follows that these two conditions are approximately satisfied when λ_{sp}^2 is 1.354, i.e. the range of the spin-orbit interaction is twice as large as that of the central interaction. In this case,

$$^3V_T^{\text{even}}=w'-h'=-28.72 \text{ Mev} \tag{38}$$

is obtained, and this value is comparable with the value in (18), which is fitted to the two-body data.**

* If the condition (25) obtained from the binding energy of P^{32} is used, (37) contradicts (36).

** The spin-orbit interaction has little effect on the binding energy of the deuteron, because the D-state probability is very small and in the deuteron the non-diagonal matrix elements of the spin-orbit coupling between the states having different L vanishes. Therefore the results obtained by Kalos et al are available, although they did not take into account the spin-orbit interaction.

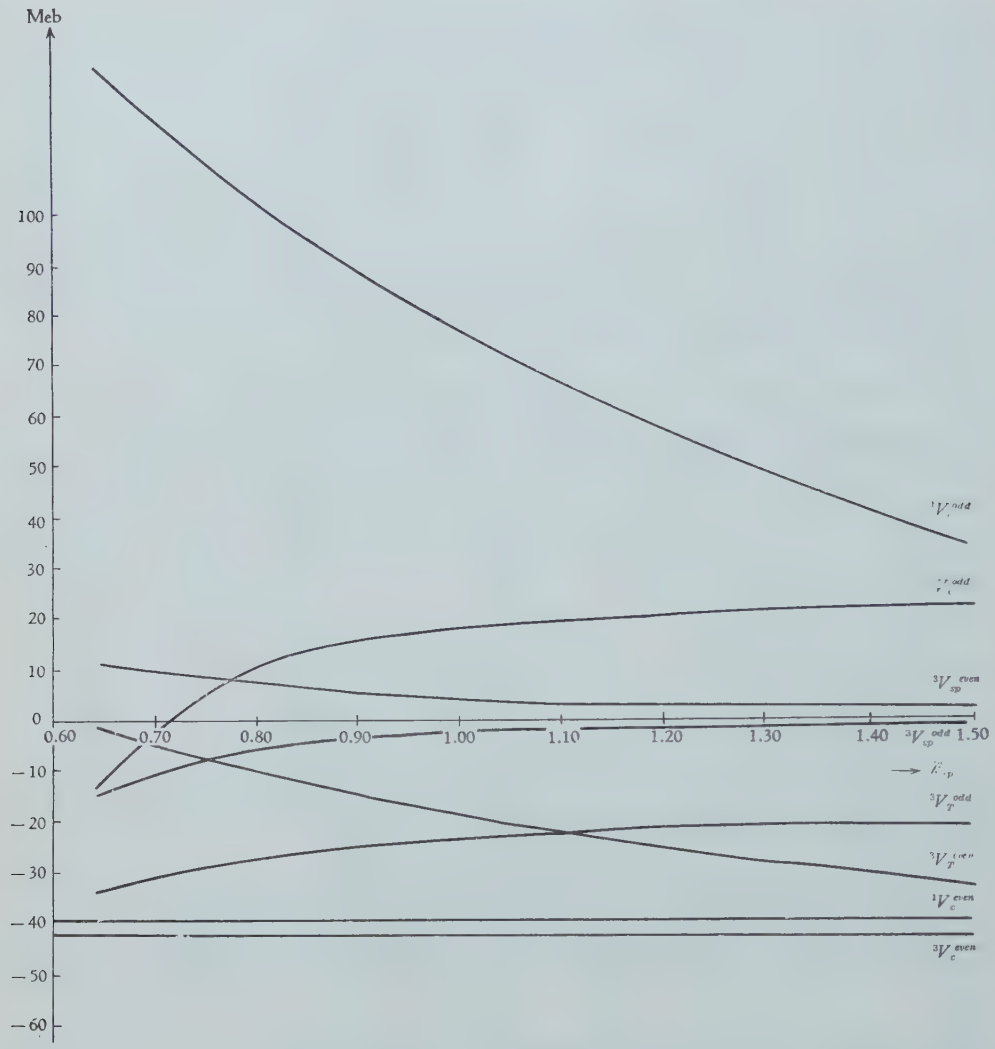

Fig. 1. The strengths of potentials fitted to the values of the energy parameters of two-nucleon states which are determined by the nuclear binding energies. $^3V_\alpha^{\text{even}}$, for example, shows the strength of α-interaction (α; central, tensor and spin-orbit) in the triplet even state.

Consequently λ_{sp}^2 is taken to be 1.354 which corresponds to the value $r_{sp}=3.00\times 10^{-13}$ cm. The spin-orbit force thus determined is a weak but long range interaction. R. Hüper has obtained qualitatively the same conclusion about the phenomenological spin-orbit force by the analysis of the energy levels of the light nuclei, and also quantitatively our numerical values w'' and h'' are nearly equal to his results.[17]

The two-body potential obtained above does not satisfy the Serber condition. However the meson theoretical potential also shows such behaviour.[18] Probably the effects of the repulsive central potential in ^3O and ^1O states cancel the attractive tensor potential

in ^3O state. Elliott and Flowers[19] and Redlich[20] have found that, if they use the Serber force, the $T=1$ group falls below $T=0$ in F^{18}, and this is clearly inadmissible unless it can be shown that F^{18} has $T=1$ in its ground state.

It is interesting to note that the forces within the outer shell do not appear to be much weaker than they are between free particles.

The values of the parameters $E^N(2s_{1/2}1d_{3/2}TJ)$ (set 2) determined by this potential are different from the values determined in § 2 by using the binding energy of P^{32} and the excited state Si^{30} (set 1). This discrepancy cannot be eliminated even if the Yukawa type radial dependence is assumed instead of the Gaussian one for the nuclear potential. In the next section, the excitation energies are calculated, and those results show that set 2 gives rather good agreement with experiment, but fails to explain the binding energy of P^{32}. Set 1 succeeds in giving the binding energy of P^{32} but does not agree so well with experiment on the spectra of the other nuclei. This shows that neither set is correct. Consequently in order to obtain the values of $E^N(2s_{1/2}1d_{3/2}TJ)$, more experimental data are much desired.

§ 4. Low-lying energy levels

In this section are treated the low lying states, which are considered to arise from simple $2s_{1/2}^{n_1}1d_{3/2}^{n_2}$ configurations. The expectation values of the energies in these configurations can be calculated by using the energy parameters determined in § 2 and § 3. For $E^N(2s_{1/2}1d_{3/2}TJ)$ the values of the set 1 in § 2 and set 2 in § 3 are used. In the figures Exp. shows the experimental levels and Cal I and Cal II show the calculated levels using set 1 and set 2 respectively.

$_{14}Si_{16}^{30}$ Our calculation shows that the configuration $1d_{3/2}^2(10)$ lies quite low, and the excitation energy is about 2 Mev. A 2.23 Mev excited state is found experimentally, but gamma-transition from this level to

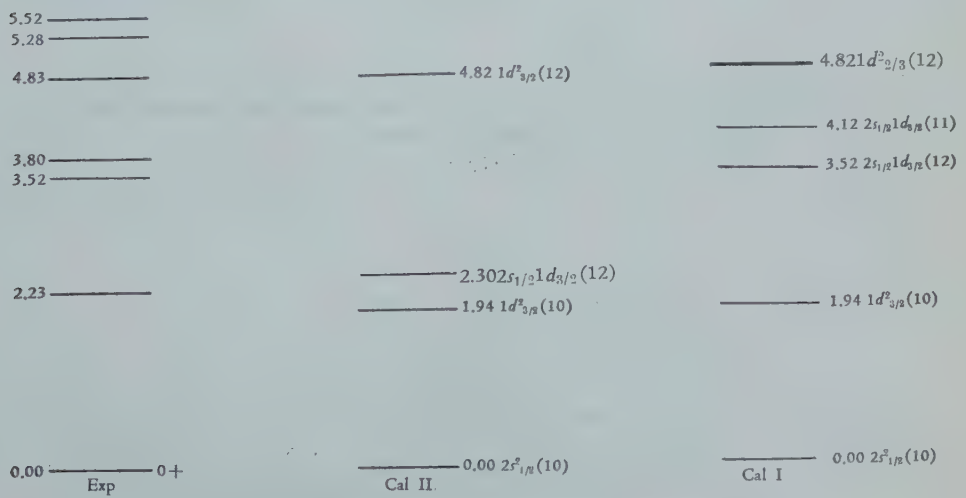

Fig. 2. Energy levels of $_{14}Si_{16}^{30}$ in Mev. $[j_1j_2](TJ)$ show the states.

the ground state is also observed. Thus it is important to know whether a level corresponding to $1d^2_{3/2}(10)$ has been missed in the experiments or not.

$_{15}P_{15}^{30}$ The calculation shows that $E^N(2s_{1/2}\,1d_{3/2}\,01)$ of the set 2 is so large that the state $2s_{1/2}\,1d_{3/2}(01)$ competes with $2^2s_{1/2}(01)$, and they interact with each other. Therefore the wave function of the ground state is expressed as follows:

$$\Psi = a\Psi(2s_{1/2}{}^2\,01) + b\Psi(2s_{1/2}\,1d_{3/2}\,01), \tag{39}$$

where $a^2+b^2=1$. a, b and the energy E are determined by the secular equation

$$\begin{vmatrix} 2A^N(2s_{1/2}) + E^N(2s^2_{1/2}\,01) + A^c(2s_{1/2}) - E & \langle 2s_{1/2}\,1d_{3/2}\,01 \,|\, V| \,2s^2_{1/2}\,01\rangle \\ \langle 2s_{1/2}\,1d_{3/2}\,01 \,|\, V| \,2s^2_{1/2}\,01\rangle & A^N(2s_{1/2}) + A^N(1d_{3/2}) + A^c(2s_{1/2}) + E^N(2s_{1/2}\,1d_{3/2}\,01) - E \end{vmatrix} = 0. \tag{40}$$

If $\langle 2s_{1/2}\,1d_{3/2}\,01 \,|\, V| \,2s^2_{1/2}\,01\rangle = \pm 0.320$ Mev, the smaller solution gives the experimental binding energy $= -13.97$ Mev (relative to that of Si^{28}), and the larger one is -13.31 Mev. If the wave-function of § 3 and the potential in Table XII with $\lambda^2_{sp}=1.354$, are used, $\langle 2s_{1/2}\,1d_{3/2}\,01 \,|\, V| \,2s^2_{1/2}\,01\rangle$ becomes -0.215 Mev.

Corresponding to these two values of E, the wave functions,

$$\Psi_{gn} = 0.624\Psi(2s^2_{1/2}\,01) - 0.782\Psi(2s_{1/2}\,1d_{3/2}\,01) \tag{41}$$

and

$$\Psi_{ex} = 0.782\Psi(2s^2_{1/2}\,01) - 0.624\Psi(2s_{1/2}\,1d_{3/2}\,01), \tag{42}$$

are obtained, where Ψ_{gn} is the wave function of the ground state and Ψ_{ex} is that of the excited state. They show that in the ground state the admixture of $\Psi(2s_{1/2}\,1d_{3/2}\,01)$ is greater than that of $\Psi(2s^2_{1/2}\,01)$. It is clear why the beta transition to the $_{14}Si_{16}^{30}$ ground state $(2s^2_{1/2}(10))$ is somewhat hindered.

In Fig. 3, the calculations I and II show that there are two low-lying excited states. The levels of P^{30} have been investigated by several workers, using the Si^{29} (d, n) P^{30} and Si^{29} (p, γ) P^{30} reactions. A level at

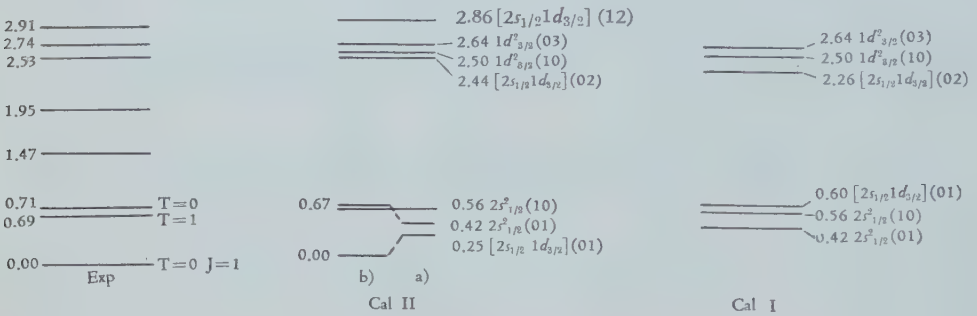

Fig. 3. Energy levels of $_{15}P_{15}^{30}$ in Mev. In Cal. II., a) shows zeroth-order energy scheme, and b) shows the results taking into account the configuration interation.

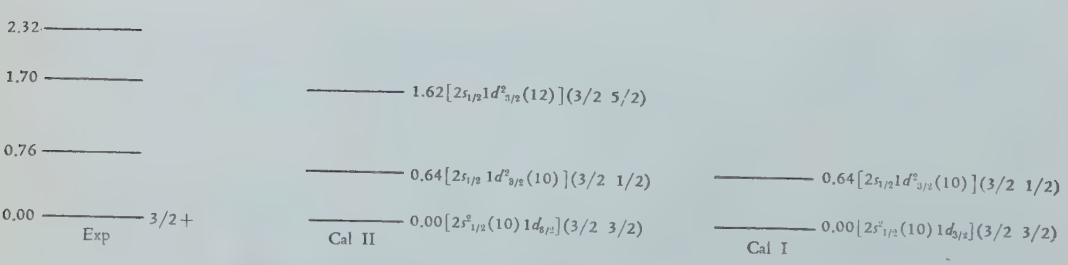

Fig. 4. Energy levels of $_{14}Si_{17}^{31}$ in Mev.

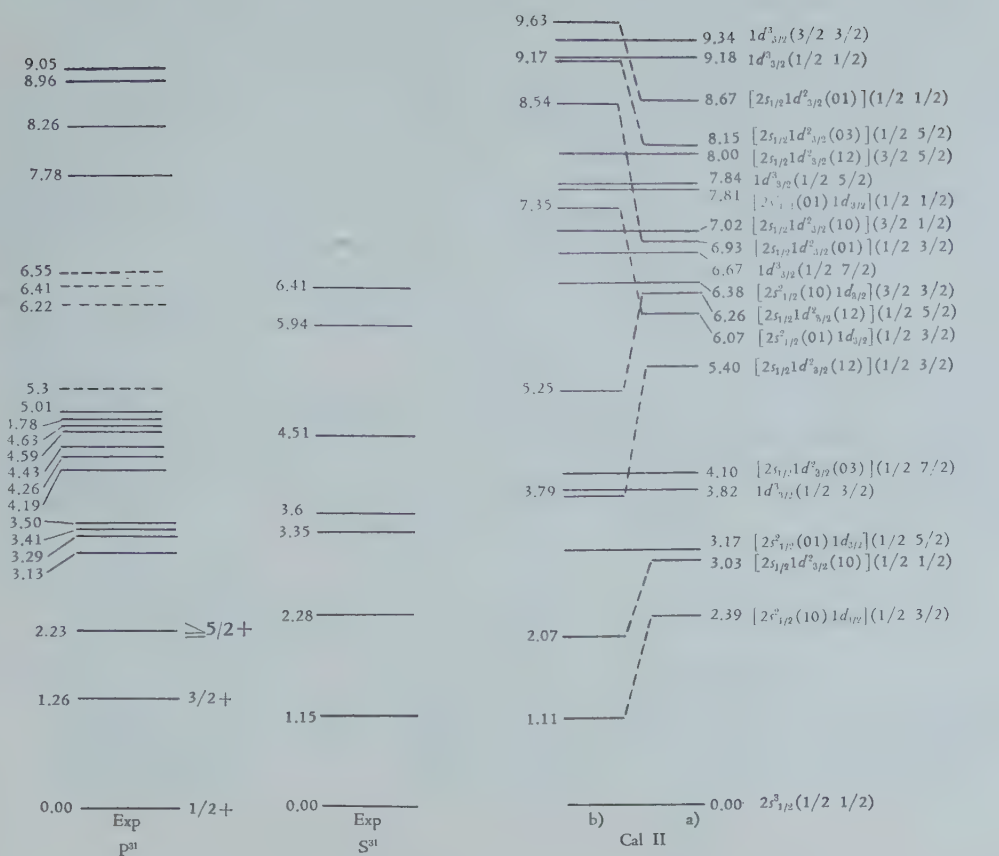

Fig. 5. Energy levels of $_{15}P_{16}^{31}$ and $_{16}S_{15}^{31}$ in Mev.

about 700 Kev excitation has been observed, to which $T=1$ and $J=0$ have been assigned on the basis of excitation energy and gamma-ray angular distribution. Recently the S^{32} (d, α) P^{30} reaction has been studied by magnetic analyser, and it is known that, if isotopic spin selection rules may be considered valid, the prominence of the 0.71 Mev level in this reaction indicates that it is probably $T=0$. According to our calculation II this state corresponds to the configuration of Ψ_{ex}, which has theoretically 0.66 Mev excitation energy. A level at 0.69 Mev was found in the same reaction, while the low intensity of the alpha-particle leading to this level can be explained by assuming that this level is the expected low-lying $T=1$ level, which

Fig. 6. Energy levels of $_{15}P_{17}^{32}$ in Mev.

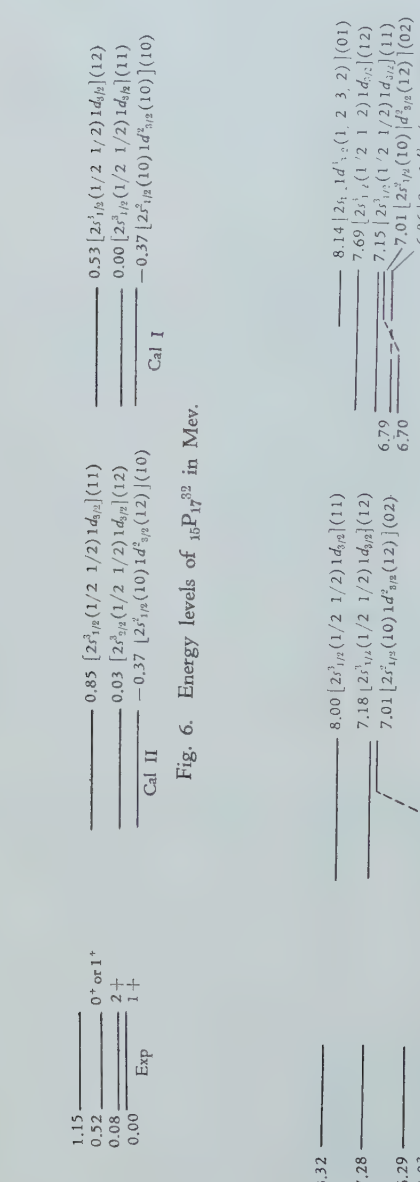

Fig. 7. Energy levels of $_{16}S_{16}^{32}$ in Mev.

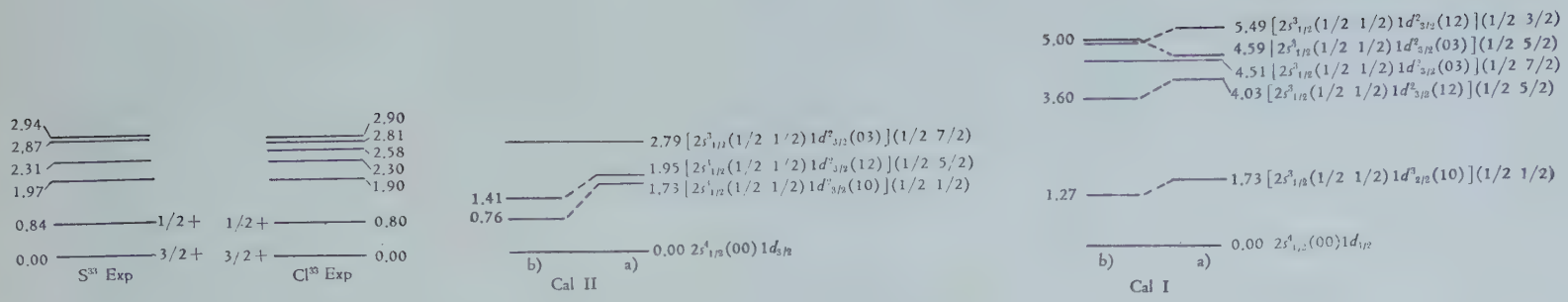

Fig. 8. Energy levels of $_{16}S_{17}^{33}$ and $_{17}Cl_{16}^{33}$ in Mev.

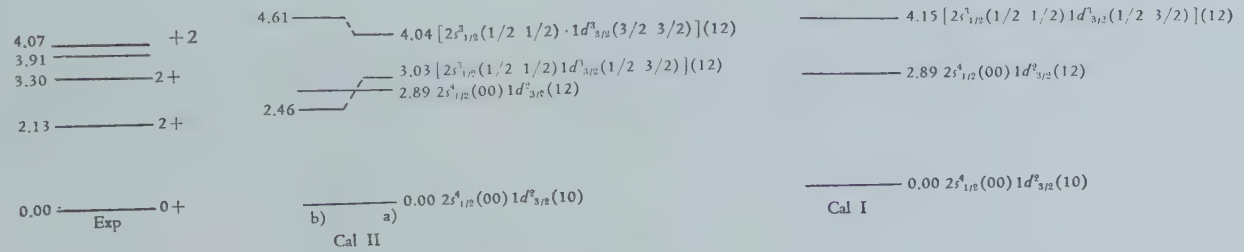

Fig. 9. Energy levels of $_{16}S_{18}^{34}$ in Mev.

corresponds to the configuration $2s^2_{1/2}(10)$ having the theoretical excitation energy of 0.56 Mev. This experiment also shows that the intensity leading to the 2.94 Mev level was about 30% of the average intensity of the other levels, so that this level probably has again $T=1$.[21]

$_{14}Si_{17}^{31}$ The calculated value agrees fairly well with the experiment.

$_{15}P_{16}^{31}$ and $_{16}S_{15}^{31}$ Theoretically the low-lying states of $T=1/2$ and $J=3/2$ are $[s^2_{1/2}(10)d_{3/2}]$ (1/2 3/2), $[s^2_{1/2}(01)d_{3/2}]$ (1/2 3/2), $[s_{1/2}d^2_{3/2}(01)]$ (1/2 3/2) and $[s_{1/2}d_{3/2}(12)]$ (1/2 3/2), and in zeroth order these configurations have the high excitation energies shown in Fig. 5 Cal Ia) and IIa). There may be configuration interactions between them. Generally, the Hamiltonian has large non-diagonal elements between states having similar configurations such as $[s^2_{1/2}(10)d_{3/2}]$ (1/2 3/2) and $[s^2_{1/2}(01)d_{3/2}]$ (1/2 3/2), and small non-diagonal elements between states having different configurations, for example $[s^2_{1/2}(10)d_{3/2}]$ (1/2 3/2) and $[s_{1/2}d^2_{3/2}(01)]$ (1/2 3/2). For simplicity, only the interactions in the first case are taken into consideration, and the non-diagonal elements in such cases are expressed as linear combinations of the four parameters $E^N(2s_{1/2}1d_{3/2}TJ)$, for example

$$\langle 2s^2_{1/2}(10)1d_{3/2}\,1/2\,3/2\,|V|\,2s^2_{1/2}(01)\times 1d_{3/2}\,1/2\,3/2\rangle = 3\sqrt{5}/16\cdot[E^N(2s_{1/2}1d_{3/2}01) - E^N(2s_{1/2}1d_{3/2}11) - E^N(2s_{1/2}1d_{3/2}02) + E^N(2s_{1/2}1d_{3/2}12)]. \quad (45)$$

Fig. 5. Cal Ia) and IIa) show the excitation energies in the zeroth order and Cal Ib) and IIb) show the results where such configuration interactions are considered. In Cal IIb), the theoretical energy of the first excited state results in 1.10 Mev, and agrees with the experimental value fairly well.

$_{15}P_{17}^{32}$ Set 2 cannot give the ground state of $J=1$, and set 1 is fitted to the observed binding energy of this nucleus, but neither can succeed in explaining the level structure. It seems that the configuration interactions have a large effect in this nucleus. Actually if they have appropriate strength, set 2 can also give the binding energy.

$_{16}S_{16}^{32}$ The configuration $[2s^3_{1/2}$

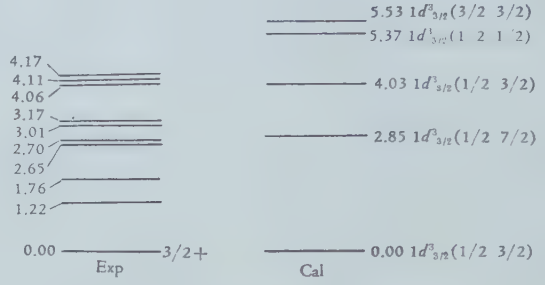

Fig. 10. Energy levels of $_{17}Cl_{18}^{35}$ in Mev.

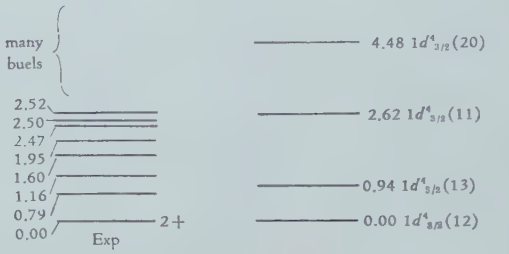

Fig. 11. Energy levels of $_{16}Cl_{19}^{36}$ in Mev.

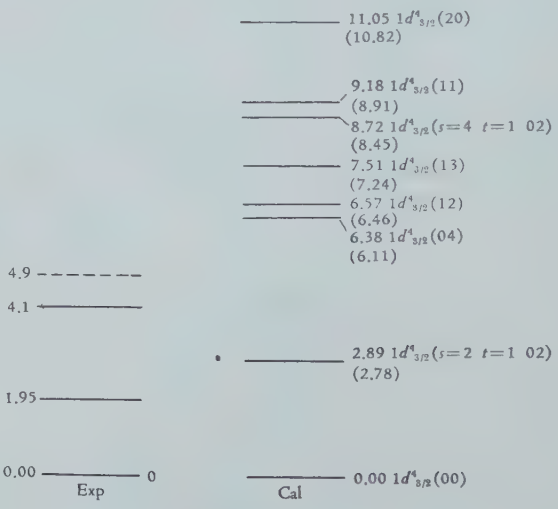

Fig. 12. Energy levels of $_{18}A_{18}^{36}$ in Mev.
The value in the parenthesis was submitted to correction owing to the Coulomb interaction.

$(1/2\ 1/2)1d_{3/2}]$ (02) has 2.27 Mev excitation in Cal II and may correspond to the first excited state which has experimentally 2.25 Mev excitation. However, the other $[2s^3{}_{1/2}\,(1/2\ 1/2)\,1d_{3/2}]$ (TJ) states have larger energies than those of the second and other low-lying levels, and then the second and other low-lying states have more complex configurations. If it is experimentally sure that there are no low-lying 0+levels in P^{32} and S^{32}, the theoretical low-lying 0+levels in these nuclei show the inadequacy of the values of the parameters.

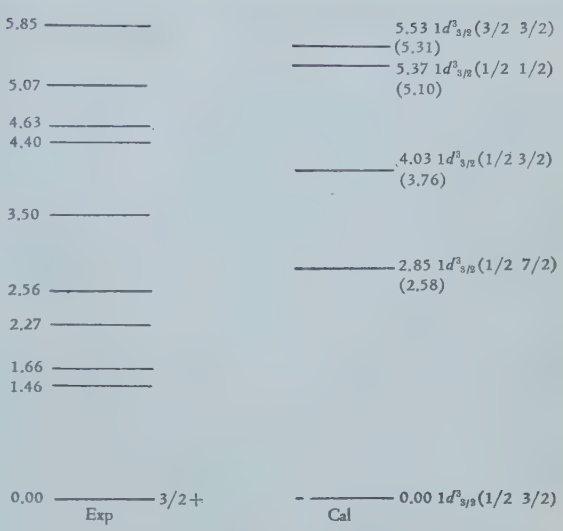

Fig. 13. Energy levels of $_{18}A_{19}{}^{37}$ in Mev

The value in the parenthesis was submitted to correction owing to the Coulomb interaction.

$_{16}S_{17}{}^{33}$ As in the first excited state of P^{31}, the large effect of configuration mixing in the first excited state can be expected. Really in the zeroth order the first excited state has too large an excitation energy, but taking into account configuration interaction, in Cal II b) the result becomes 0.76 Mev, which is comparable with the experimental value 0.85. The spin of this state is 1/2+theoretically and experimentally.

$_{16}S_{18}{}^{34}$ The excitation energy of the configuration $2s^4{}_{1/2}(00)\,1d^2{}_{3/2}(12)$ is higher than the experimental first excited state. However, Fig. 9 shows that there are other configurations of $T=1$ $J=2$ which compete with $1d_{3/2}{}^2(12)$. Hence, configuration interaction probably can explain the level scheme of this nucleus. In Fig. 9 Cal II b) simply the interaction between $[2s^3{}_{1/2}(1/2\ 1/2)\ 1d^3{}_{3/2}(1/2\ 3/2)]$ (12) and $[2s^3{}_{1/2}(1/2\ 1/2)\ 1d^3{}_{3/2}(3/2\ 3/2)]$ (12) is considered. This result shows that the first excited state consists of the configuration

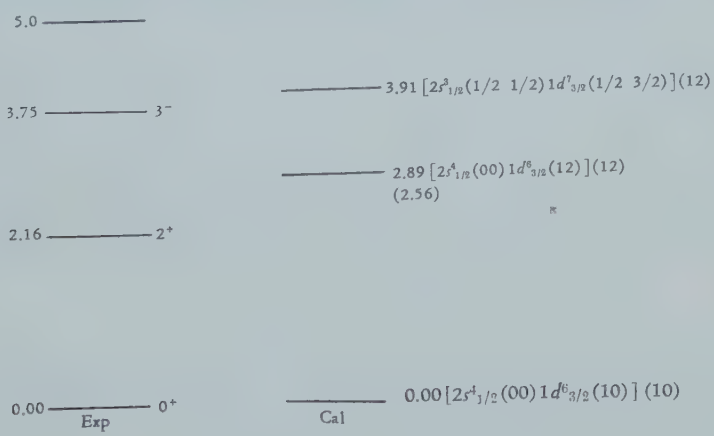

Fig. 14. Energy levels of $_{18}A_{20}{}^{38}$ in Mev probably two configuration which are of $J=2$ interact each other. If $\langle 1s_{1/2}{}^4\,100)\,1d_{3/2}{}^6(12)12\,|\,V|1s_{1/2}{}^3(1/2\ 1/2)\,1d_{3/2}{}^7$ $(1/2\ 3/2)12\rangle \sim \pm 0.8$ Mev, theoretically first excited state appears at 2.16 Mev. However, this value of 0.8 Mev is rather larger than the expected.

The value in the parenthesis was submitted to correction owing to the Coulomb interaction.

$2s^3_{1/2}1d^3_{3/2}$ mainly, and the second excited state is mainly $2s^4_{1/2}(00)\ 1d^2_{3/2}(12)$.

The large value of log ft (6.1) from the first excited state of Cl^{34} to the first excited state of S^{34} at 2.10 Mev shows that the first excited state of S^{34} has the configuration $2s^3_{1/2}1d^3_{3/2}$ rather than $2s^4_{1/2}1d^2_{3/2}$ if the first excited state of Cl^{34} is $1d^2_{3/2}(03)$. Also the beta transition from the first excited state of Cl^{34} to the second of S^{34} at 3.24 Mev shows that the state at 3.24 Mev is mainly the configuration $2s^4_{1/2}(00)1d^2_{3/2}(12)$, because the value of logft=4.8 would be expected in an allowed transition.[22]

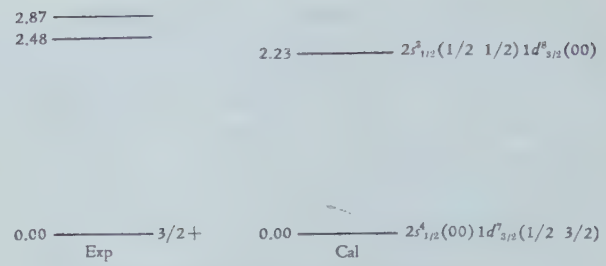

Fig. 15. Energy levels of $_{19}K_{20}^{39}$ in Mev

$39 > A \geqslant 35$ Figures 10 to 14 show the energies of states of the configurations $1d^n_{3/2}$ (TJ), and the level schemes of the nuclei $39 > A \geqslant 35$. From these figures, it can be seen that energies of the states, which arise from the reorientation of the ground state nucleons, are large. Thus it seems that the configurations in which $2s_{1/2}$ or $1d_{5/2}$ nucleons are excited appear in the low-lying states.

$_{19}K_{20}^{39}$ This nucleus has low excited states consisting again of very simple configurations, say $2s_{1/2}{}^{-1}$ $(1/2\ 1/2)1d^6_{3/2}(00)$ and $d_{5/2}{}^{-1}(1/2\ 5/2)\ 2s^4_{1/2}(00)1d^8_{3/2}(00)$. The theoretical excitation energy of the $2s_{1/2}{}^{-1}$-configuration is 2.23 Mev compared with the experimental value of 2.48 Mev, the spin of which, however, is not observed. It is desired to determine spins of these low-lying states.

§ 5. Discussion and conclusion

The binding energies of $2s_{1/2}$, $1d_{3/2}$ and $2s_{1/2}$-$1d_{3/2}$ nuclei are given by several parameters which can be determined by a least squares fit. The agreement with the experimental data indicates the adequacy of the shell model in describing the ground states of these nuclei. However, when the energies of the excited states are calculated, it seems frequently to be necessary to take into account the effect of configuration mixing.

H. Horie and the present author showed that configuration mixing has considerable effect on the magnetic[23] and quadrupole moments[24] in the ground states and on the transition probabilities.[25),26] The coefficients of admixture of the wave functions of the excited states to that of the shell model wave function were estimated and their values were found to be of the order 0.1. Usually the zeroth-order energy difference is about 5 Mev or more, and the non-diagonal matrix elements of interaction are about 0.5 Mev. First order perturbation theory gives the coefficient of admixture,

$$\alpha \lessapprox 0.5/5 = 0.1,$$

and this configuration mixing gives rise to large deviations of the expetation values of the nuclear moments from their shell model values. However, energetically this configuration mixing does not cause considerable variation, because

$$\Delta W \lessapprox (0.5)^2/5 = 0.05 \text{ Mev}.$$

The states having the same parity and spin as the ground states usually have quite large excitation energies, so that in most ground states the effect of the configuration mixing on the binding energies can be neglected. In the excited states, and in the ground

states of some nuclei, there are states with the same parity and spin which compete and interact strongly with each other.

According to Levinson and Ford[27] the effective nuclear interaction potential in Ca^{42} is represented by a Gaussian which is about one-half as deep and two-times longer-ranged. However, they only took into consideration the central force, the range of which is extraordinarily long. Taking into consideration the tensor force, there remains the probability that the effective potential in Ca^{42} does not differ from the potential needed to fit the low-energy scattering and deuteron data. Indeed Elliott and Flowers concluded in their analysis of the structure of the nuclei of mass 18 and 19 that the forces within the outer-shell do not appear to be much weaker than they are between free-particles.[19] The potential calculated in § 3 is approximately equal to the phenomenological potential which describes the low-energy two-body problem in the even state, and this potential leads to the same conclusion as Elliott and Flowers. The analysis in § 3 shows the necessity of the spin-orbit interaction and it has the right sign and intensity to produce the one-body spin-orbit forces needed in the *j-j* coupling shell model. However, it has a rather longer range than the spin-orbit interaction of meson theory. Recently, it is recognised that the two-body spin orbit interaction is necessary to explain the polarization in high energy nucleon-nucleon scattering[28),29] and, also, to interpret the spin-orbit coupling in the *j-j* coupling shell model.[30]

Finally, it is very desirable to get more accurate experimental information in order to test the validity of the model developed in this work.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Quantization of Spin Wave Field

Isao Mannari

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December 25, 1957

A few supplemental notes are presented as to the quantization technique of spin wave field.[1] And this note includes some remarks on Frank's treatment[2] of spin wave. A crystal of N atoms is considered and each atom is assumed to have a valence-electron in its s-orbit. Hence the Hamiltonian of the system is

$$\mathcal{H}=\sum_{\mu=1}^{N}H(\mu)+\sum_{\mu<\nu}G(\mu,\nu) \qquad (1)$$

where

$$H(\mu)=-\frac{1}{2}\Delta_{\mu}-\frac{1}{r_{n\mu}},$$

$$G(\mu,\nu)=\frac{1}{R_{nm}}+\frac{1}{r_{\mu\nu}}-\frac{1}{r_{n\nu}}-\frac{1}{r_{m\mu}} \qquad (2)$$

and the other notations used here have usual meanings. Assuming the orthonormality of the atomic orbitals $\psi_n(\mu)$ the Hamiltonian is in the second quantized scheme written as[3]

$$\mathcal{H}=\sum_{n\sigma}E_0\,a^*_{n\sigma}a_{n\sigma}+\frac{1}{2}\sum_{\substack{n_1\cdots n_4\\ \sigma_1\cdots\sigma_4}}G_{n_1\sigma_1 n_2\sigma_2,\,n_3\sigma_3 n_4\sigma_4}\times a^*_{n_1\sigma_1}a^*_{n_2\sigma_2}a_{n_4\sigma_4}a_{n_3\sigma_3} \qquad (3)$$

where

$$E_0=\int\psi_n^*(1)H(1)\psi_n(1)\,dv_1,$$

$$G_{n_1\sigma_1\cdots n_4\sigma_4}=\int\psi^*_{n_1}(1)\psi^*_{n_2}(2)G(1,2)\times\psi_{n_3}(1)\psi_{n_4}(2)\chi^*_{\sigma_1}(1)\chi^*_{\sigma_2}(2)\chi_{\sigma_3}(1)\times\chi_{\sigma_4}(2)\,d\tau_1 d\tau_2, \qquad (4)$$

$\chi_\sigma(\mu)$ being a spin function, and $a_{n\sigma}$, $a^*_{n\sigma}$ are the annihilation and creation operators satisfying that

$$[a_{n\sigma},\ a^*_{m\rho}]_+=\delta_{nm}\delta_{\sigma\rho},\ \text{otherwise}\ [\quad]_+=0. \qquad (5)$$

Neglecting the ionic configurations we have that[2]

$$\sum_{\sigma}a^*_{n\sigma}a_{n\sigma}=1, \qquad (6)$$

and the Hamiltonian is

$$\mathcal{H}=NE_0+\sum_{n<m}C_{nm}-\sum_{n<m}J_{nm}\sum_{\sigma\rho}a^*_{n\sigma}a_{n\rho}a^*_{m\rho}a_{m\sigma}, \qquad (7)$$

where use has been made of eq. (5) and of the orthonormality of $\chi_\sigma(\mu)$. C and J are the usual Coulomb and exchange integrals of Heitler-London type,[4] respectively. The first term is the energy of N isolated atoms and the second the usual

Coulomb interaction energy of N atoms. Although the orthonormality of the atomic orbitals has been assumed in the derivation of the Hamiltonian, it is rather reasonable to take account of the non-orthogonality in the evaluation of J.[5] And J is different from Frank's one.[2]

Now we introduce the vector operator $\boldsymbol{S}_n$ as follows:

$$S_n{}^x + iS_n{}^y = a_{n+}^* a_{n-}, \quad S_n{}^x - iS_n{}^y = a_{n-}^* a_{n+},$$
$$S_n{}^z = \tfrac{1}{2}(a_{n+}^* a_{n+} - a_{n-}^* a_{n-}). \tag{8}$$

Then $\boldsymbol{S}$'s are easily verified to be the usual spin operator by the use of eqs. (5), (6) and (8), and the last term (say H_{ex}) of eq. (7) becomes

$$-\sum_{n<m} J_{nm}(\tfrac{1}{2} + 2\boldsymbol{S}_n \cdot \boldsymbol{S}_m). \tag{9}$$

Of course this result is easily anticipated from the first because the last term of (7) consists of the terms exchanging the spin states of the n- and m-th sites. We define the spin deviation operators c_n, $c_n{}^*$ with the substitutions[1]

$$a_{n+}^* a_{n-} \to \alpha_n, \quad a_{n-}^* a_{n+} \to \alpha_n{}^*,$$
$$\alpha_n \nu_n \to c_n, \quad \nu_n \alpha_n{}^* \to c_n{}^*, \tag{10}$$

where $\nu_n = \prod_{p=1}^{n} (1-2n_p)$ is the so-called sign-function, and n_p ($=0$ or 1) is the number of spin deviations on the p-th lattice site and is the eigenvalue of the operator $c_p{}^* c_p (= \alpha_p{}^* \alpha_p = a_{p-}^* a_{p-})$. Eq. (10) is slightly different from Frank's substitutions and is much more straightforward. Finally we arrive at

$$\mathcal{H}_{ex} = 2\sum_{n<m} J_{nm}(-\tfrac{1}{2} + c_n{}^* c_n - F c_n{}^* c_m - c_n{}^* c_n c_m{}^* c_m),$$
$$[c_n, c_m{}^*]_+ = \delta_{nm},$$
$$[c_n, c_m]_+ = [c_n{}^*, c_m{}^*]_+ = 0, \tag{11}$$

$$F = \prod_{p=n+1}^{m-1} (1-2n_p).$$

These are the same as Frank's results except the form of J's.

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Note on Energy Levels of Polyatomic Molecules

S. Datta Majumdar

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January 21, 1958

The calculation of the energy levels of polyatomic molecules[1] has been simplified by introducing a set of one-variable operators for the components of angular momentum P_x, P_y, P_z. These operators, however, assume different forms for different types of functions. For functions of the type $u^+ = \sum_p a_{j-2p} e^{i(j-2p)\varphi}$, $(P_x \pm iP_y)^+ = \pm e^{\pm i\varphi}(F + D^2 \pm iD)$, $P_z = iD$, where $F \equiv j(j+1)$, $D \equiv d/d\varphi$, and for functions of the type $u^- = \sum_p a_{j-2p-1} e^{i(j-2p-1)\varphi}$,

$$(P_x \pm iP_y)^- = \mp e^{\pm i\varphi}, \quad P_z = iD.$$

By the use of these operators the wave equation of the asymmetrical top is reduced to the following two adjoint differential equations:

$$(L+\mu)u^+ \equiv [g(F+D^2) + \tfrac{1}{2}g'D + \mu]u^+ = 0 \tag{1}$$

$$(L^*+\mu)u^- \equiv [g(F+D^2)+\tfrac{3}{2}g'D - 2g+2+\mu]u^- = 0 \qquad (2)$$

where

$$g=1-k\cos 2\varphi,\ \ k=(b-a)/(2c-a-b),$$

$$\mu=(E-cF)/[c-(a+b)/2].$$

The operators and the differential equation (2) of the author's previous work[2] on triatomic molecules are thus obtained.

If the eigenfunctions of eq. (1) are denoted by v and those of eq. (2) by w, then it is seen that, $v=g^{1/2}w$, and $g^{-1/2}Lg^{1/2}=L^*$. By a change of the independent variable these equations are transformed into Lamé's equation. The solution of these equations has also been obtained in a compact form which facilitates the calculation of the matrix elements necessary for a perturbation calculation.

To determine the corrections to energy arising from centrifugal stretching and the coupling of rotation and vibration by a perturbation procedure the basic rotational functions are taken to be the eigenfunctions v^+ and w^- of eqs. (1) and (2), which form a linearly independent but non-orthogonal set.

An alternative reduction of the problem is possible in which the operators have the same form for both types of functions, namely,

$$P_x \pm iP_y = -e^{\pm i\varphi}(j \pm iD),\ \ P_z = iD.$$

From the solution of the one-variable problem the expansion in terms of symmetric rotor functions can be easily obtained.

Complete calculations will be published afterwards.

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Natures of Nuclear Force Indicated by the Photodisintegration of the Deuteron, III

S. H. Hsieh

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February 11, 1958

The detection of the sign of the phase shift of the ${}^3P_1$ state is very important, because we can test the applicability of the pion potential by it.[1] Here we shall indicate that we can discriminate the sign of δ_1^{2} [2] from the measurement of the total cross section (σ_T) of $d(\gamma, p)n$ at $E_\gamma \simeq 10$ Mev.[2]

The reason is as follows. The triplet odd phase shifts which can fit the $p-p$ scattering data and the angular distribution of $d(\gamma, p)n$ of the lower energies are characterized by[2),3)]

a) $\delta_1^0 > 0$, $\delta_1^1 < 0$, and small δ_1^2, or

b) $\delta_1^0 < 0$, $\delta_1^1 > 0$, and small δ_1^2,

which means $|V_t| \gg |V_c|$ at such energies. (Here the phase shifts are denoted by δ_L^J.) Now if we make use of these phase shifts for the computation of $d(\gamma, p)n$ process by the method described in the previous papers,[2] we can easily see from the formulas of σ_{ed} that, 1) the contribution of ${}^3P_0$ to σ_{ed} is very small, 2) the contribution from ${}^3P_1$ is large and much dependent on the different sets of the phase shifts, and 3) ${}^3P_2$ gives a contribution which is not so

different by the different sets of the phase shifts owing to the smallness of δ_1^2. Because at lower energies σ_T is almost due to σ_{ed} arising from ${}^3S_1+{}^3D_1\to{}^3P$, we find that the magnitude of σ_T is determined by δ_1^1 at lower energies.

For example, if we use the following phase shifts given by B−C−V[3] which can account for 18.2 Mev $p-p$ scattering and the angular distribution of $d(\gamma, p)n$ at $E_\gamma=11.3$ Mev,

A) $\delta_0^0=43.7°$, $\delta_1^0=19.4°$,
$\delta_1^1=-7.4°$, $\delta_1^2=2.8°$,

B) $\delta_0^0=45°$, $\delta_1^0=-21°$,
$\delta_1^1=7.5°$, $\delta_1^2=1.5°$,

and the wave function of the deuteron given by I−O−T−W[4] to the computation of σ_T of $d(\gamma, p)n$ at 11.3 Mev, (by the method previously used) we get

A) $\sigma_T=\sigma_{ed}+\sigma_{md}=109.3+3.3=112.6$,

B) $\sigma_T=\sigma_{ed}+\sigma_{md}=119.2+3.2=122.4$

in unit of 10^{-29} cm^2.

Here the error in the computation of σ_{ed} is estimated to be within 2%, and the lower limit of σ_{md} is adopted, because σ_{md} is near to its lower limit at such energy.

We see the difference of σ_T between A) and B) is not small. From such calculation, it seems that, if the D-state probability is between 4 and 9%,[5] σ_T would be larger than 1.20 mb or would be smaller than 1.15 mb at $E_\gamma\simeq 11.3$ Mev.

Furthermore it should be noted that, even if we adopt phase shifts whose values are much smaller than A) and B), (such sets fail to account for the photodisintegration of the deuteron) the difference of σ_T between the sets a) and b) is still remained, but somewhat smaller.

Thus we might conclude that, it would be possible to reveal the rough shape of the triplet odd potential by the accurate measurement of σ_T of $d(\gamma, p)n$ at lower energies. At $E_\gamma\simeq 11$ Mev it seems that, if σ_T is smaller than 1.15 mb, the sign of δ_1^1 is negative and $V_t>0$, and if σ_T is larger than 1.20 mb, that of δ_1^1 is positive and $V_t<0$. (To discern it about 90% of accuracy is demanded in the measurement of σ_T.)

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An Attempt of a Nonlinear Field Theory

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March 3, 1958

The purpose of this note is to propose a new unified field theory. It is assumed that there exists only a basic spinor field $\psi(x)$ which satisfies the following *nonlinear* equation,

$$\gamma^{\mu}(\partial/\partial x_{\mu}-\Gamma_{\mu})\psi=0, \qquad (1)$$

where matrix-functions $\gamma^{\mu}(x)$ and $\Gamma_{\mu}(x)$ are *certain functions of* ψ. It is further required that eq. (1) should be invariant under general coordinate-transformations. At a glance, eq. (1) bears a resemblance to the general Dirac equation in a given curved space-time.[1] The former is, however essentially different from the latter in that our metric tensor $g_{\mu\nu}$ is a certain function of ψ itself. Namely our $g_{\mu\nu}$ *is composed of* ψ *fusion-theoretically.* γ^{μ} and Γ_{μ} are defined *in terms of these* $g_{\mu\nu}$ in an analogous way to the usual one.[1] The dependences of $g_{\mu\nu}$, γ^{μ} and Γ_{μ} on ψ will be given in the later part of this letter (eqs. (4), (5), (6)). Thus *the spinor field* ψ *plays a fundamental role not only as the matter field but also as the metric field.*

Now we can define the energy-momentum tensor $T_{\mu\nu}$ in the following way. First construct, by using our $g_{\mu\nu}$, the symmetric tensor

$$-(4\pi\kappa)^{-1}(R_{\mu\nu}-1/2\cdot Rg_{\mu\nu}), \qquad (2)$$

corresponding to the left-hand side of the Einstein equation, and then reduce this expression, by using the fundamental equation (1), to an expression which includes as low derivatives of ψ as possible. We call the symmetric tensor thus obtained the energy-momentum tensor of matter and denote it as $T_{\mu\nu}$. The form of $T_{\mu\nu}$ as a function of ψ is determined obviously by the fusion-theoretical specification of $g_{\mu\nu}$. At the same time we get the Einstein equation

$$R_{\mu\nu}-1/2\cdot Rg_{\mu\nu}=-4\pi\kappa T_{\mu\nu}. \qquad (3)$$

Thus our formalism will be a unitary generalization of the Einstein theory of gravitation. It should be remarked that the fundamental equation (1) with the fusion-theoretical specification of $g_{\mu\nu}$ suffices to determine the metric of space-time without use of eq. (3). The *conservation law of energy-momentum* is also valid with the only difference from the usual one that *in our case the gravitational parts of the total energy-momentum* P_{μ} *are also composed of the basic spinor field* ψ.

The dependences of $g_{\mu\nu}$, γ^{μ} and Γ_{μ} on ψ are given by the following relations

$$g_{\mu\nu}=1/4\cdot tr(\gamma_{\mu}\gamma_{\nu})$$
$$=(\bar{\psi}\gamma_{[\mu\rho]}\psi)(\bar{\psi}\gamma_{[\nu\sigma]}\psi)g^{\rho\sigma}, \qquad (4)$$

$$\partial\gamma_{\mu}/\partial x_{\nu}=\Gamma^{\rho}_{\mu\nu}\gamma_{\rho}+\Gamma_{\nu}\gamma_{\mu}-\gamma_{\mu}\Gamma_{\nu}, \qquad (5)$$

$$i/4\cdot tr(\partial\Gamma_{\mu}/\partial x_{\nu}-\partial\Gamma_{\nu}/\partial x_{\mu})$$
$$=l^{-2}(\bar{\psi}\gamma_{[\mu\nu}\psi), \qquad (6)$$

where $\gamma_{[\mu\nu]}=i/2\cdot(\gamma_{\mu}\gamma_{\nu}-\gamma_{\nu}\gamma_{\mu})$ and $\bar{\psi}=\psi^{*}\gamma_{4}$. Eqs. (4) determine the dependences of $g_{\mu\nu}$ and γ^{μ} on ψ. Here $(\bar{\psi}\gamma_{[\mu\rho]}\psi)(\bar{\psi}\gamma_{[\nu\sigma]}\psi)$ is, except a numerical factor, the expansion coefficient of $\psi\bar{\psi}\times\psi\bar{\psi}$ —a direct product of a four-four matrix-function $\psi\bar{\psi}$ with itself—for a basis element $\gamma^{[\mu\rho]}\times\gamma^{[\nu\sigma]}$ when $\psi\bar{\psi}\times\psi\bar{\psi}$ is expanded by a set of 16^2 independent basis elements. The ψ-dependence of Γ_{μ} is determined apart from its trace by eq. (5), where $\Gamma^{\rho}_{\mu\nu}$ is the Christoffel symbol. That of tr Γ_{μ} is determined by eq. (6), where l is a universal constant with dimension of length which is obtained by combining the gravitational constant κ and the light velocity c with the elementary charge e; $l=\kappa^{1/2}c^{-2}e$. Also $(\bar{\psi}\gamma_{[\mu\nu]}\psi)$ is, except a numerical factor, the expansion coefficient of $\psi\bar{\psi}$ for a basis element $\gamma^{[\mu\nu]}$ when $\psi\bar{\psi}$ is expanded.

The quantity $e/(4i)\cdot\mathrm{tr}\Gamma_{\mu}$ plays a role of the electromagnetic potential. Therefore we can define the *charge-current density* as

$$j^{\nu}=el^{-2}\,\partial\{\sqrt{-g}\;g^{\mu\rho}\;g^{\nu\sigma}(\bar{\psi}\gamma_{[\rho\sigma]}\psi)\}/\partial x_{\mu}. \tag{7}$$

The charge conservation law is obviously valid from eq. (7). Further it is of interest in connection with the so-called two-component neutrino theory[2] to investigate the *properties of* eq. (1) *for the* γ_5*-transformation.*

It is necessary to quantize the basic spinor field ψ in order to obtain a unified field theory of the elementary particles[3] in this formalism. As Deser pointed out, the usual rule of quantization

$$\partial\psi/\partial x_{\mu}=i[\psi,\ P_{\mu}],$$

which is an origin of the divergence difficulties in the conventional quantum theory of fields, has no longer proper meaning in generally covariant theories. It is being intended to investigate the problem of quantization and further implications of the present formalism.

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Yours sincerely,

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Progress of Theoretical Physics, Vol. 19, No. 5, May 1958

The Conservation Laws in the Lepton Process and Two Kinds of Neutrinos

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(Received November 14, 1957)

An attempt is made to forbid the unwanted processes involving leptons by assuming the conservation law of lepton number and that of neutrino charge. To do this, suitable assignment of these quantum numbers to each lepton is necessary in addition to the assumption of the existence of two kinds of neutrinos (a right circularly polarized neutrino and a left polarized neutrino). The neutrinoless lepton processes are successfully forbidden, except for the lepton pair processes, and the theoretical predictions for the $\pi-\mu-e$-decay and the β-decay agree well with experiment. In the $\pi-\mu-e$-decay and the β-decay, the vector and axial vector couplings are preferable if the interaction Hamiltonians are written in the form of charge exchange theory. The other sort of unwanted processes than neutrinoless ones, $\pi\to e+\nu$ and $\pi\to e+\nu+\gamma$, are relatively (or dynamically) forbidden from the view-point of the universal Fermi interaction.

§ 1. Introduction

It has long been known that the lepton processes caused by the weak interaction are always associated with the neutrinos. Therefore we must answer the question why the processes without involving the neutrino do not occur in nature.

The problem was investigated along several lines. Especially Konopinski and Mahmoud[1] succeeded in forbidding such processes as

$$\begin{aligned} \mu^{\pm}+n &\to e^{\pm}+n \\ \mu^{\pm}+p &\to e^{\pm}+p \\ \mu^{\pm} &\to e^{\pm}+e^{+}+e^{-} \\ &\to e^{\pm}+\gamma \end{aligned} \tag{1}$$

by requiring the conservation law of Fermion number. In order to forbid these processes, it was necessary and sufficient to assume μ^{+} as well as e^{-} was particle, and μ^{-} as well as e^{+} was antiparticle. As it is well known that the conservation law of baryon number holds in elementary processes, their assumption is regarded as that of the conservation law of lepton number.

In the course of the development of experiments, another sort of unwanted processes has been recognized. One of them is

$$K^{+}\to\mu^{+}+e^{+}+\pi^{-}. \tag{2}$$

This process cannot be forbidden by the use of Konopinski and Mahmoud's rule.

Recently, Lee and Yang[2] proposed an important question of possible violation of the conservation law of parity in weak interactions. Being motivated by their proposal, an experiment of β-decay of oriented nuclei was made by Wu et al.[3] By their experiments the violation of the conservation law of parity and the non-invariance under charge conjugation were confirmed. Under these circumstances, Lee and Yang,[4] Landau[5] and Salam[6] have proposed the two component theory of the neutrino, expecting that the violation of the laws comes from the exclusively unfamiliar nature of the neutrino whose mass is exactly zero. According to their theory, the neutrino is described with the two component spinor, and it has spin parallel (right circularly polarized) or antiparallel (left circularly polarized) to its momentum. Lee and Yang[5] assigned the particle to the right polarized neutrino and the antiparticle to the left polarized neutrino. Their theory predicted that in the μ-decay the Michel parameter[7] would be zero if two identical neutrinos were emitted, and 3/4 if one neutrino and one antineutrino were emitted. Thus, in order to reconcile the two component theory of the neutrino with the experimental value of Michel parameter $\rho \lesssim 3/4$, under the assumption of the conservation law of lepton number, they had to assign that μ^- as well as e^- was particle and μ^+ as well as e^+ was antiparticle. Their assignment could forbid the process (2), but could not forbid the processes (1) which Konopinski and Mahmoud succeeded in forbidding.*

In connection with the discovery of the nonconservation of parity in weak interactions, Umezawa and Visconti,[8] Touschek[9] and Pauli[10] proposed the invariance of the weak interaction Hamiltonian under the gauge transformation of a sort,

$$\psi_\nu \to e^{\pm i\tau_3\theta}\psi_\nu, \tag{3}$$

$$\psi_e \to e^{\pm i\theta}\psi_e, \tag{4}$$

$$\psi_\mu \to e^{\pm i\theta}\psi_\mu \tag{5}$$

which leaves the free Hamiltonian of respective lepton invariant. Analogously to the conservation law of electric charge, the invariance property under the gauge transformation (3)—(5) gives us the conservation law of neutrino charge (after Umezawa and Visconti's nomenclature). Konuma[11] investigated the observed processes in general, assuming the conservation law of the lepton charge and that of the neutrino charge.

We make an attempt to forbid the lepton processes which do not involve the neutrinos (except lepton pair processes) by assuming the conservation law of the lepton number and that of the neutrino charge. To do this, we must assign a lepton number and a neutrino charge to each lepton (§ 2). The ensuing selection rules are discussed, and it is shown that the neutrinoless lepton processes (except lepton pair processes) are forbidden (§ 3). Some theoretical predictions and comparison with experiment are made for the $\pi-\mu-e$-decay and the β-decay (§ 4). Discussions are made on other possibilities of assignment (§ 5—1), and on the universal Fermi interaction (§ 5—2), and on branching

* These arguments show that it is impossible to forbid both of the unwanted processes (1) and (2) by the conservation law of lepton number only.

ratio of $\pi\to\mu+\nu$ to $\pi\to e+\nu$ and to $\pi\to e+\nu+\gamma$ (§ 5—3). Finally, it is shown that, by the conservation law of lepton number or by that of neutrino charge, the lepton pair processes cannot be forbidden (§ 5—4).

§ 2. The conservation laws in the lepton process

Throughout this paper, we assume that the two conservation laws hold in the lepton processes caused by weak interactions.

(I) *The conservation law of the lepton number*

The quantity

$$L=n(\mu^+)-n(\mu^-)+n(e^-)-n(e^+)$$
$$+n(\nu_R)-n(\nu_R^c)+n(\nu_L)-n(\nu_L^c) \tag{6}$$

is conserved in the lepton processes caused by weak interactions, where $n(a)$ refers to a number of particle a.

The eq. (6) means that μ^+, e^-, ν_R and ν_L are particles, and μ^-, e^+, ν_R^c and ν_L^c are antiparticles, respectively. This assignment is identical with that of Konopinski and Mahmoud[2] except for the introduction of two kinds of neutrinos (and their antiparticles). They are an R-neutrino ν_R and an L-neutrino ν_L (and respective antiparticle, an anti-R-neutrino ν_R^c and an anti-L-neutrino ν_L^c), whose wave functions are defined, from the four component Dirac neutrino field ψ_ν, as

$$\psi_{\nu R}=\tfrac{1}{2}(1-\gamma_5)\psi_\nu,$$
$$\psi_{\nu L}=\tfrac{1}{2}(1+\gamma_5)\psi_\nu, \tag{7}$$

where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$.

Under charge conjugation $\psi_{\nu R}$ and $\psi_{\nu L}$ are transformed into anti-R-neutrino field $\psi_{\nu R}^c$ and anti-L-neutrino field $\psi_{\nu L}^c$, respectively, defined by

$$\psi_{\nu R}^c=C\bar{\psi}_{\nu R}^T=\tfrac{1}{2}(1+\gamma_5)\psi_\nu^c,$$
$$\psi_{\nu L}^c=C\bar{\psi}_{\nu L}^T=\tfrac{1}{2}(1-\gamma_5)\psi_\nu^c \tag{8}$$

where $C^+C=-C^{-1}C^T=+1$, and $\psi^c=C\bar{\psi}^T$. It is easy to see from eqs. (7) and (8) that the R-neutrino field and the anti-L-neutrino field are right circularly polarized, while the L-neutrino field and the anti-R-neutrino field are left circularly polarized. Therefore, after second quantization, $\psi_{\nu R}$ has an annihilation operator of R-neutrino ν_R (right circularly polarized neutrino) and a creation operator of anti-R-neutrino ν_R^c (left circularly polarized antineutrino) etc.

Formally speaking, the conservation law of the lepton number imposes on the interaction Hamiltonian invariance under the simultaneous gauge transformations

$$\psi\to e^{+i\varphi}\psi \tag{9a}$$

for fields and

$$\psi^c\to e^{-i\varphi}\psi^c \tag{9b}$$

for anti-fields.

(II) *The conservation law of neutrino charge*

The quantity

$$N=n(\mu^+)-n(\mu^-)-n(e^-)+n(e^+)$$
$$+n(\nu_R)-n(\nu_R^c)-n(\nu_L)+n(\nu_L^c) \qquad (10)^{11)}$$

is conserved in the lepton processes caused by weak interactions.

This is consequence of the invariance requirement of the interaction Hamiltonian under the gauge transformation

$$\psi \to e^{i\theta}\psi \qquad (11)$$

for the muon field, the antielectron field, the R-neutrino field and the anti-L-neutrino field, and

$$\psi \to e^{-i\theta}\psi \qquad (12)$$

for the antimuon field, the electron field, the anti-R-neutrino field and the L-neutrino field. It is easily verified from the identity $(1\mp\gamma_5)e^{-i\gamma_5\theta}=e^{\pm i\theta}(1\mp\gamma_5)$ that the transformation (11) and (12) for the polarized neutrino fields are identical with the transformation for the four component Dirac neutrino field

$$\psi_\nu \to e^{-i\gamma_5\theta}\psi_\nu. \qquad (13)$$

§ 3. Ensuing selection rules

In order to investigate the selection rules ensuing from the conservation law of the lepton number and that of the neutrino charge, we are enough to consider leptons only since other particles involved in the lepton processes have no connection with these laws.

From the conservation law of lepton number, μ^+ cannot be transmuted into e^+, and μ^- into e^-, and vice versa. Hence such processes as

$$\begin{aligned} \mu^\pm + n &\to e^\pm + n \\ \mu^\pm + p &\to e^\pm + p \\ \mu^\pm &\to e^\pm + e^- + e^+ \\ &\to e^\pm + \gamma \end{aligned} \qquad (14)$$

are absolutely forbidden. Further, μ^+ and e^- cannot be created or annihilated together, and μ^- and e^+ cannot be annihilated or created together. Hence such processes as

$$K^\pm \to \mu^\pm + e^\mp + \pi^\pm \qquad (15)$$

are absolutely forbidden.

From the conservation law of neutrino charge, μ^+ cannot be transmuted into e^-, and vice versa, and μ^- cannot be transmuted into e^+, and vice versa. Further, μ^+ and e^+ cannot be created or annihilated together, and μ^- and e^- cannot be created or annihilated together. Hence such processes as

$$K^{\pm} \to \mu^{\pm} + e^{\pm} + \pi^{\mp} \tag{16}$$

are absolutely forbidden.

From any one of the laws, μ^+ cannot be transmuted into μ^-, and vice versa, and e^+ cannot be transmuted into e^-, and vice versa.

Summarizing the results, we can conclude that the lepton processes without involving the neutrino are absolutely forbidden by the conservation law of lepton number and/or that of neutrino charge, except for the lepton pair processes such as

$$\begin{aligned} K^{\pm} &\to \mu^+ + \mu^- + \pi^{\pm} \\ &\to e^+ + e^- + \pi^{\pm} \\ &(\to \nu_R + \nu_R^c \text{ or } \nu_L + \nu_L^c + \pi^{\pm}). \end{aligned} \tag{17}$$

The other sort of unwanted process, such as

$$\begin{aligned} \pi^+ &\to e^+ + \nu_L \\ \pi^- &\to e^- + \nu_L^c \\ K^+ &\to e^+ + \nu_L \\ K^- &\to e^- + \nu_L^c \end{aligned} \tag{18}$$

and

$$\begin{aligned} \pi^+ &\to e^+ + \nu_L + \gamma \\ \pi^- &\to e^- + \nu_L^c + \gamma \\ K^+ &\to e^+ + \nu_L + \pi^0 \\ K^- &\to e^- + \nu_L^c + \pi^0 \end{aligned} \tag{19}$$

which are not absolutely forbidden, are relatively (or dynamically) forbidden (see § 5—3).

The observed processes are all allowed, and may be written as

$$\pi^+ \to \mu^+ + \nu_R^c \tag{20a}$$

$$\pi^- \to \mu^- + \nu_R \tag{20b}$$

$$\mu^+ \to e^+ + \nu_R + \nu_L \tag{21a}$$

$$\mu^- \to e^- + \nu_R^c + \nu_L^c \tag{21b}$$

$$n \to p + e^- + \nu_L^c \tag{22}$$

$$\mu^+ + n \to p + \nu_R$$

$$\mu^- + p \to n + \nu_R^c \tag{23}$$

$$K^+ \to \mu^+ + \nu_R^c (+\pi^0)$$

$$K^- \to \mu^- + \nu_R (+\pi^0). \tag{24}$$

From the eqs. (20)—(24), we may say that the muon field is always coupled with

the R-neutrino field, while the electron field is always coupled with the L-neutrino field. This may be important consequence of our theory from the view-point of the universal Fermi interaction.

§ 4. Some theoretical predictions and comparisons with experiment

In this section some theoretical predictions and comparisons with experiment are made for the observed processes, the $\pi-\mu-e$-decay and the β-decay.

1. *$\pi-\mu-e$-decay.* Let us first take up the decay of the positive pion (20a). Since the neutrino is left circularly polarized, and since the pion has spin zero, the μ^+ emitted must be left circularly polarized to conserve total angular momentum. Next consider the subsequent decay of the positive muon (21a). Since the positons are emitted predominantly in the backward direction with respect to the muon momentum, we consider only the high energy positon coming out in the backward direction. As the polarizations in the two neutrinos cancel each other, conservation of angular momentum tells us that the positon has the same spin component as the muon. Thus the e^+ emitted in μ^+ decay is right circularly polarized. On the other hand, in the $\pi^-\to\mu^-\to e^-$-decay, the μ^-, and therefore the e^- has the polarizations opposite to those of the μ^+ and the e^+ in the $\pi^+\to\mu^+\to e^+$-decay. This conclusion for the polarization is quite opposite to the prediction made by Lee and Yang's two component theory.*

The muon decay (21) may be derived from the interaction Hamiltonian

$$H_\mu=f_V(\bar{\psi}_\mu\gamma_\mu\psi_e^c)(\psi_{\nu R}C^{-1}\gamma_\mu\psi_{\nu L}) \\ +f_A(\bar{\psi}_\mu i\gamma_\mu\gamma_5\psi_e^c)(\psi_{\nu R}C^{-1}i\gamma_\mu\gamma_5\psi_{\nu L})+H.C. \tag{25}$$

if the four Fermions are written in the form of charge retention theory. The normalized electron distribution derived from this Hamiltonian is

$$dN=2x^2[(3-2x)\pm\tilde{\xi}\cos\theta(1-2x)]dxd\Omega_e(4\pi)^{-1} \tag{26}$$

where

$$\tilde{\xi}=\frac{2\mathrm{Re}(f_Vf_A^*)}{|f_V|^2+|f_A|^2} \tag{27}$$

and the upper sign refers to the e^- and the lower one refers to the e^+. The other notations are the same as those of Lee and Yang.[4]

We wish to look for further implications. Let us write the Hamiltonian (25) in the form of charge exchange theory, then we have

$$H_\mu=\Sigma_i g_i(\bar{\psi}_\mu O_i\psi_{\nu R})(\bar{\psi}_e O_i\psi_{\nu L})+H.C. \tag{28}$$

---

* They assumed the existence of ν_R (and $\nu_R{}^c$) only and the conservation law of lepton number where the μ^- was particle. The same conclusion as ours for the polarization may be drawn from the assumption of the existence of ν_L (and $\nu_L{}^c$) only plus the conservation law of lepton number. However, the unwanted processes (1) cannot be forbidden. Another possibility from which the same conclusion as ours can be drawn is to assume the existence of ν_R (and $\nu_R{}^c$) only plus the conservation law of lepton number (6). In this case, the theoretical value of the Michel parameter is zero, in contradicion with experiment. See also § 5—1.

and $$=\Sigma_i g_i' (\bar{\psi}_\mu O_i \psi_{\nu L}) (\bar{\psi}_e O_i \psi_{\nu R}) + H.C. \quad (29)$$

Now we assume the argument concerning with the polarization of the electron in $\pi-\mu-e$-decay is correct. Then ψ_e may be replaced by $1/2(1+\gamma_5)\psi_e$. Thus we must take $g_S=g_T=g_P=g_V'=g_A'=g_T'=0$, and so we have

$$2f_V=2f_A=-2g_V=-2g_A=g_S'=-g_P'^*, \quad (30)$$

$$\text{Other coupling constants}=0.$$

This relation between coupling constants may be very important from the view-point of the universal Fermi interaction.

Our theory predicts that three parameters characterizing the electron distribution[14),15)] are

$$\rho(\text{Michel parameter})=\delta \text{ (asymmetry Michel parameter)}=3/4, \quad (31)$$

$$\xi=1.$$

Experimental values seem to be somewhat smaller than the values (31), $\rho\sim 0.7$ and $\xi\sim 0.8$.[16)] The discrepancy may be interpreted by some corrections which are not taken into account here, for example, radiative corrections.[15)]

2. *β-decay.* The β-decay schemes allowed in our theory are

$$N(A, Z)\to N(A, Z+1)+e^-+\nu_L^c \quad (32)$$

$$N(A, Z)\to N(A, Z-1)+e^++\nu_L \quad (33)$$

$$n\to p+e^-+\nu_L^c \quad (22)$$

where $N(A, Z)$ refers to a nucleus with mass number A and atomic number Z. Double β-decay is, of course, forbidden.

The decay (32), (33) and (22) are caused by the interaction Hamiltonian

$$H_\beta=2\Sigma_i C_i(\bar{\psi}_p O_i \psi_n)(\bar{\psi}_e O_i \psi_{\nu L})+H.C. \quad (34)$$

in Lee and Yang's notations. Note that $C_i'=-C_i$ in their theory, while $C_i'=+C_i$ in our theory. In the following we neglect the pseudoscalar coupling and Coulomb correction which seem to be very small (or zero).

Before the discovery of parity non-conservation in β-decay, it was generally accepted that the vector and axial vector coupling constants were small compared with the scalar and tensor coupling constants.[17)] However, the most precise experiment of this sort, that of $Na^{22}(3^+\to 2^+)$ by Sherr and Miller,[18)] shows only that Gamow-Teller's part of the Fierz interference term is small,

$$\mathrm{Re}(C_T C_A^* + C_T' C_A'^*)=2\mathrm{Re}(C_T C_A^*)\approx 0. \quad (35)$$

Recent experiments[19)] on oriented nuclei, $Co^{60}(5^+\to 4^+, \beta^-)$ and $Co^{58}(2^+\to 2^+$, perhaps Gamow-Teller's matrix elements are dominant, $\beta^+)$, seem to show

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* In the calculation of eq. (30), we have taken account of the anticommutation relations between fermions. See reference 13).

$$\frac{2\mathrm{Re}(C_T C_T'^* - C_A C_A'^*)}{|C_T|^2+|C_T'|^2+|C_A|^2+|C_A'|^2} = \frac{|C_T|^2-|C_A|^2}{|C_T|^2+|C_A|^2} \approx -1. \tag{36}$$

And further, the experiments on oriented nuclei, $Ga^{66}(0^+ \to 0^+, \beta^+)$[20),21)] and $Cl^{34}(0 \to 0)$,[21)] seem to show

$$\frac{2\mathrm{Re}(C_S C_S'^* - C_V C_V'^*)}{|C_S|^2+|C_S'|^2+|C_V|^2+|C_V'|^2} = \frac{|C_S|^2-|C_V|^2}{|C_S|^2+|C_V|^2} \lesssim 0. \tag{37}$$

From eqs. (35)—(37), we may conclude that the vector and axial vector coupling constants are large compared with the scalar and tensor coupling constants.

As far as we are concerned with the coupling type, three theories, i. e., parity conserving Fermi interaction, the two component neutrino theory and our theory, give us the same result from the electron-neutrino angular correlation experiments. However, the experimental results[22)] are not definite at present, so we shall not enter into details of these experiments.

In conclusion, we may say that the vector and axial vector coupling constants are large compared with the scalar and tensor coupling constants in β-decay. The difference between our theory and the two component neutrino theory comes from the fact that the L-neutrino field is associated in the former, while the R-neutrino field is associated in the latter.

§ 5. Discussions and conclusion

1. We first remark on the other possibilities of assignment of lepton number and of neutrino charge to the leptons. For the discussion of the problem, it is to be noted that if the sum of lepton numbers L is conserved, then L is conserved, and the same is true of the sum of neutrino charge N. Therefore we need not consider the case where the signs of lepton number and/or those of neutrino charge of all the leptons are reversed. In the following, we shall assume e^- has the lepton number $+1$ (therefore it is particle) and the neutrino charge -1.

Let us first assume that μ^+ is particle. In this case, two antineutrinos can be emitted in the μ^-decay. If two ν_L^c's or two ν_R^c's are emitted, the predicted value of the Michel parameter is zero in contradiction with experiment.* If ν_L^c and ν_R^c are emitted, there are two ways of the assignments of the neutrino charge to neutrinos. The one is identical with (10) and the other is that the signs of the neutrino charges of all the neutrino are reversed in (10). The latter assignment leads us to the result that the scalar and tensor (and pseudoscalar) coupling constants are large compared with the vector and axial vector coupling constants in the β-decay interaction Hamiltonian. This assignment seems to be less attractive since the unwanted processes (18) and (19) cannot be forbidden by the idea of the universal Fermi interaction (see § 5—2, 3).

Next consider a case where μ^- is particle. In this case, one neutrino and one anti-

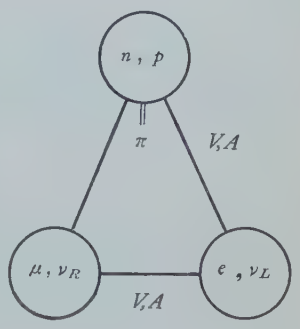
* This assignment cannot forbid the unwanted processes (16).

neutrino are emitted in the μ^- decay. If ν_R and ν_R^c, or ν_L and ν_L^c are emitted, the neutrino charges of neutrino and antineutrino cancel each other. Hence the unwanted processes (14) cannot be forbidden since μ^- must have the same neutrino charge as e^-. If ν_R and ν_L^c are emitted in the μ^- decay, experimental values of ξ and ρ are incompatible with the relation $\xi=3-(16/3)\rho$. If ν_L and ν_R^c are emitted, e^- must have the same neutrino charge as ν_R. In this case, the scalar and tensor (and pseudoscalar) coupling constants are large compared with the vector and axial vector coupling constants in the β-decay interaction Hamiltonian as well as in the muon decay interaction Hamiltonian.* Hence the unwanted processes (18) and (19) cannot be forbidden by the idea of the universal Fermi interaction (see § 5—2, 3).

It is obvious from the above arguments that two kinds of neutrino are necessary to forbid the unwanted processes and to give the predictions consistent with the experiment for the observed processes.

From the above argument, we conclude that there exist no assignments of lepton number and neutrino charge to the leptons consistent with experiments other than ours, (6) and (10).**

2. As we have seen in § 3, the muon field is always coupled with the *R*-neutrino field, and the electron field with the *L*-neutrino field. Let us draw the well-known triangle representing the interactions, in which ν_R may be put into the same circle as μ, and ν_L into the same circle as e. As we have seen in § 4, in the interaction Hamiltonians for the μ-decay and the β-decay, (28) and (34), only the vector and axial couplings are present or predominant. This is a very attractive feature of our theory from the viewpoint of the universal Fermi interaction. To investigate the third leg of the triangle, it is necessary to investigate the capture of polarized muon,[12]



$$\mu^-+p\rightarrow n+\nu_R^c(+\gamma). \tag{38}$$

3. It is known experimentally that, relative to the normal pion decay mode $\pi\rightarrow\mu+\nu$, branching ratios ρ and ρ_τ for the alternative decay modes $\pi\rightarrow e+\nu$ and $\pi\rightarrow e+\nu+\gamma$ are very small (ρ, $\rho_\tau\lesssim 5\times10^{-5}$). Treiman and Wyld[13] investigated the question of whether these limits on branching ratios are consistent with the idea that pion decay occurs through the universal Fermi interaction via disintegration into a virtual nucleon pair. The result was that a value for ρ consistent with the experiments could be obtained only if the pseudoscalar coupling constant in the universal Fermi interaction was small compared with the axial vector constant. As for the photodecay, the process could occur

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* In this case, the vector and axial vector couplings are forbidden in the muon decay.

** If we give up to forbid the unwanted processes, and if we wish to get the same theoretical predictions as ours for the observed processes, the $\pi-\mu-e$-decay and the β-decay, the assumption of the existence of ν_L (and $\nu_L{}^c$) only is sufficient.

through the axial vector and tensor (forbidden for $\pi \to \mu + \nu$ and $\pi \to e + \nu$) interactions. The branching ratio ρ_T was 4×10^{-8} for the axial vector interaction and 7×10^{-5} (C_T/C_A) for the tensor interaction. Treiman and Wyld concluded that the experimental value of ρ_T could not be explained by the idea of the universal Fermi interaction since the C_T/C_A was very large $(C_T/C_A \gtrsim 50)$ as could be seen from the β-decay experiment.[18] However, since in our theory β-decay experiment seems to show that $|C_T| \ll |C_A|$, the idea of the universal Fermi interaction may be very attractive.

4. As we have seen, our theory can forbid almost all of the unobserved processes associated with leptons absolutely or relatively. However, as remarked in § 3, the processes involving lepton pair (17) cannot be forbidden. This consequence comes from the conservation laws where both lepton number and neutrino charge have opposite signs for particle and antiparticle. Therefore, in order to forbid the lepton pair processes some other sorts of laws are necessary. At any rate, further investigations for the law governing in the world of elementary particles may be necessary to solve the question why the unwanted processes do not occur in nature.

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Cepheid Vibration

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In this paper Milne's idea has been used to compute the luminosity variation in a star under various conditions.

§ 1. Introduction

In a paper Milne[1] has suggested that the luminosity of a star should be properly understood as being equal to the rate of diminution of total energy of the star in preference to the prevailing idea that the luminosity being proportional to the fourth power of the surface temperature of the star. This simple hypothesis has brought about a marked change in the nature of variation of the luminosity of a variable star. The situation in a sense has turned upside down, for under the existing hypothesis it came out that the variation in luminosity should be in phase with the surface displacement of the star whereas under Milne's hypothesis the variation in luminosity is found to be in phase with the surface velocity of the star, i.e. a quarter-period out of the phase with the surface displacement. In his paper Milne found out the expression for δL as,

$$\delta L=\frac{\dot{r}_0}{3(\gamma-1)r_0}[\sigma_4{}^2 I_0-(3\gamma-4)\Omega_0]. \tag{1}$$

Where we should note γ is the true ratio of specific heats and not any sort of average. But then he puts the expression in parenthesis equal to zero by using a formula due to Ledoux and Pekiris[2] where it has been shown that

$$\sigma^2=\frac{(3\bar{\gamma}-4)}{I_0}\Omega_0. \tag{2}$$

Here $\bar{\gamma}$ is not γ, the true value of the ratio of specific heats but some average value of γ, which in general differ from γ by a numerical factor different from one. Hence in Milne's expression, eq. (1) we cannot always put $\delta L=0$, for the formula of Ledoux and Pekiris is not applicable in general. However under the very restricted assumption which Milne has made in his paper, namely that

$$\delta r/r=\delta r_0/r_0$$

every where inside the star, it may be shown that the Ledoux-Pekiris average of γ is in this particular case equal to the true ratio of specific heats i.e. $\bar{\gamma}=\gamma$ and hence the

expression on the right-hand side of equation (1) vanishes.

Again if the right-hand side of equation (1) does not vanish, (1) would explain what is really observed; namely that δL is roughly in phase with the surface velocity of the Star. But this is not the case for adiabatic pulsation. In fact Milne's result $\delta L=0$ under adiabatic condition may be proved quite generally without any restriction of the type or any other approximations from the equation of pulsation alone so that Milne's conclusion is entirely valid as along as the vibration are adiabatic.

§ 2. Calculation of δL

We shall here consider the case of pulsations in general and not merely to radial ones. In other words if along with the radial pulsation non-radial ones be also present we shall consider that and calculate δL. For that we write the equation of motion, equation of continuity and the equation of state;

$$\frac{d\boldsymbol{v}}{dt}=-\frac{1}{\rho}\nabla p-\nabla\Omega \tag{3}$$

$$\frac{d\rho}{dt}+\rho\, div\,\boldsymbol{v}=0 \tag{4}$$

$$p=K\rho^{\gamma}. \tag{5}$$

Where Ω is defined as $\Omega=-GM_r/r$, and it is assumed that in course of pulsation the star might have non-radial component of vibration but it keeps spherical symmetry.

Multiplying (3) by $\rho\boldsymbol{v}$ scalarly and integrating throughout the volume bounded by a surface S drawn inside the star and the external boundary of the star, we get

$$\int\rho\frac{d}{dt}(1/2\cdot v^2)\,d\tau=-\int\boldsymbol{v}\cdot\nabla p d\tau-\int\rho\boldsymbol{v}\cdot\nabla\Omega d\tau$$

or,

$$\int\frac{d}{dt}(1/2\cdot v^2)\,dm=-\int p\boldsymbol{v}\cdot d\boldsymbol{S}+\int p\nabla\cdot\boldsymbol{v}d\tau-\int\boldsymbol{v}\cdot\nabla\Omega\,dm. \tag{6}$$

Now we assume that spherical symmetry is preserved in the course of pulsation and

$$\therefore\quad \frac{d\Omega}{dt}=\boldsymbol{v}\cdot\nabla\Omega. \tag{7}$$

Using (4) and (7), eq. (6) becomes,

$$\int\frac{d}{dt}(1/2\cdot v^2)\,dm=-\int p\boldsymbol{v}\cdot d\boldsymbol{S}-\int\frac{p}{\rho}\frac{d\rho}{dt}d\tau-\int\frac{d\Omega}{dt}dm. \tag{8}$$

Using (5) the second term on the right-hand side of (8) this equation becomes:

$$\int\frac{d}{dt}(1/2\cdot v^2)\,dm=-\int p\boldsymbol{v}\cdot d\boldsymbol{S}-\int\frac{d}{dt}\left(\frac{p}{(\gamma-1)\rho}\right)dm-\int\frac{d\Omega}{dt}dm$$

or,

$$\frac{d}{dt}\int\left[\frac{1}{2}v^2dm+\Omega dm+\frac{p}{(\gamma-1)\rho}dm\right]=-\int pv\cdot d\mathbf{S}. \tag{9}$$

We now show two very important consequences of the above.

(1) If we suppose that the portion of the stellar mass inside S is neither loosing nor gaining any energy i.e. the rate of change of total energy inside S remains constant, in that case δL is solely due to the energy in the mass outside S and consequently equation (9) gives :

$$\delta L=\int p\boldsymbol{v}\cdot d\mathbf{S} \tag{10}$$

which now shows that if ξ (the displacement) be proportional to $\sin nt$ then δL is necessarily proportional to $\cos nt$.

(2) Let $S\to 0$ so that equation (9) might be written as

$$\frac{d}{dt}\int\left[1/2\cdot v^2+\Omega+\frac{p}{(\gamma-1)\rho}\right]dm=0. \tag{11}$$

Let us now assume that the changes in pressure, density etc. are non-adiabatic. Let the equation of state be again given by equation (5) but now γ is not the ratio of specific heats but some other constant : So that we presume that the changes are polytropic instead of adiabatic. In this case the ratio of specific heats is designated by γ_0. We now write equation (11) as follows :

$$\begin{aligned}\frac{d}{dt}\int\left[\frac{1}{2}v^2+\Omega+\frac{p}{(\gamma_0-1)\rho}\right]dm&=\frac{d}{dt}\int\frac{p}{\rho}\left[\frac{1}{\gamma_0-1}-\frac{1}{\gamma-1}\right]dm\\&=\left(\frac{\gamma-1}{\gamma_0-1}-1\right)\int\frac{p}{\rho}\frac{d\rho}{dt}d\tau\\&=\frac{\gamma_0-\gamma}{\gamma_0-1}\int p\,\mathrm{div}\,\boldsymbol{v}\,d\tau.\end{aligned} \tag{12}$$

Equation (12) therefore gives according to Milne's idea,

$$\delta L=\frac{\gamma-\gamma_0}{\gamma_0-1}\int p\,\mathrm{div}\,\boldsymbol{v}\,d\tau. \tag{13}$$

If now again ξ is proportional to $\sin nt$ we easily see from equation (13) that $\delta L\propto\cos nt$. A quarter period phase-lag therefore must always happen in such cases. In particular it might be remarked that $\delta L=0$ in the case when $\gamma=\gamma_0$.

If now viscosity be also taken into account, but the changes in pressure, density be assumed to be polytropic i.e. $p=K\rho^\gamma$ then the modified equation of motion is,

$$\frac{d\boldsymbol{v}}{dt}=-\frac{1}{\rho}\nabla p-\nabla\Omega-\frac{1}{\rho}\boldsymbol{f}. \tag{14}$$

Where $\boldsymbol{f}$ is the force of viscosity and is derived from a viscous tensor T ; by taking

divergence

$$\boldsymbol{f} = -\operatorname{div} T.$$

The components of T are given by

$$T_{ik} = \mu\left(\frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i}\right) - \frac{2}{3}\mu \operatorname{div} \boldsymbol{v}\,\delta_{ik}.$$

Following the same sort of integration as before we obtain from (14)

$$\frac{d}{dt}\int\left[\frac{1}{2}r^2 + \Omega + \frac{p}{(\gamma-1)\rho}\right]dm$$

$$= -\int \boldsymbol{v}\cdot\boldsymbol{f}\,d\tau. \tag{15}$$

So that,

$$\delta L = \frac{\gamma_0 - \gamma}{\gamma_0 - 1}\int p \operatorname{div}\boldsymbol{v}\,d\tau + \int \boldsymbol{v}\cdot\boldsymbol{f}\,d\tau. \tag{16}$$

If however we assume that vibrations are adiabatic then $\gamma = \gamma_0$ and (16) becomes,

$$\delta L = \int \boldsymbol{v}\cdot\boldsymbol{f}\,d\tau. \tag{17}$$

Equation (17) means that δL is caused purely by the presence of viscous force $\boldsymbol{f}$ and is equal to the rate of dissipation of energy. Further if first order approximation is concerned we see that the right hand side of equation (17) is a small quantity of 2nd order ; consequently to this order of approximation δL is again zero. Hence in the case of adiabatic pulsations even when the viscous forces are present, there cannot be any luminosity fluctuations to the first order of approximation.

§ 3. Application to a stellar model

Let us for simplicity take the homogeneous model for consideration. Here[3] the displacement $\zeta = \xi(r)\sin nt$ say, $\xi(r)$ for the fundamental mode turns out to be,

$$\xi(r) = \alpha r$$

$$\therefore \quad \boldsymbol{v} = \alpha n \boldsymbol{r}\cos nt$$

and

$$\operatorname{div}\boldsymbol{v} = 3\alpha n \cos nt.$$

In homogeneous model,

$$p = \rho/2 \cdot g_0/R \cdot (R^2 - r^2)$$

where,

$$g_0 = g \quad \text{at} \quad r = R.$$

Hence equation (13) for the determination of δL becomes

$$\delta L=\frac{\gamma-\gamma_0}{\gamma_0-1}\int\left(\frac{\rho}{2}\frac{g_0}{R}\right)(R^2-r^2)\,3\alpha n\cos nt\,4\pi r^2 dr$$

$$=\left(\frac{6\pi\alpha n\rho g_0}{R}\right)\left(\frac{\gamma-\gamma_0}{\gamma_0-1}\right)\int_0^R r^2(R^2-r^2)\,dr\cos nt$$

$$=\frac{4\pi}{5}\left(\frac{\gamma-\gamma_0}{\gamma_0-1}\right)(\alpha n\rho g_0 R^4)\cos nt. \tag{18}$$

We thus find that for this simple model if we associate the polytropic law for the changes in state, the quarter period phase lag turns out in a normal way following Milne's idea, rather than that luminosity variation is caused by the surface temperature variation.

§ 4. Conclusion

From the previous discussion it is clear that if a star undergoes radial oscillations, no matter whether large or small following adiabatic law and the generation of energy remains steady, there cannot be any change in luminosity. A general proof is given of Milne's result. An extension of this result is easily made to cases of vibration allowing cross-radial displacement, provided matter on a shell remains always on the same shell. If the adiabatic law is replaced by a polytropic relation between pressure and temperature it has been shown that δL, the fluctuation in luminosity is not zero but is in phase with the velocity.

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The Phenomenological Model of the Interaction of Elementary Particles*

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The correlations between the invariant character under the space inversion and the strength of the interactions are displayed by the simple principles, i. e., the composite property of strange particles and the individual Γ_5 transformation invariance. The applications of these principles to lepton processes are satisfactory. The utility of these principles to pull out the future theory is expected.

§ 1. Introduction

Although many phenomenological aspects about the elementary particles and their interactions have been made clearer, we have no unified theory yet to comprehend them. But it is certain that clues to such a theory are, to some extent, in our hands. Now, we will try to extract some clues from those various facts available to date.

(I) First, as to the variety of the particles, we know that the particles are classified into fairly well distinguishable families such as lepton family, baryon family and so forth. (See Table I).[1]

(II) Secondly, concerning the interactions of particles, we can say with certainty that the values of the coupling constants are curiously bunched into sharply separated

Table I. M_a means the mass of the particle 'a'. Weak and strong interactions are expressed simply as WI and SI.

Name of family	Lepton	Boson	Baryon
Statistics	Fermi	Bose	Fermi
Mass	$M_\nu \sim M_\mu$	$M_\pi \sim M_K$	$M_{\mathfrak{N}} \sim M_\Xi$
Participating interaction	WI	WI and SI	WI and SI

regions in unit of $\hbar=1$, $c=1$, $r_0(\simeq 10^{-13}\,\mathrm{cm})=1$.[2] And there seems to be some correlations between the strength of an interaction and the number of conservation laws. Particularly important ones would be as follows.

a) Whether the strangeness is conserved or not gives the measure of the strength of the interaction.

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* A preliminary account of the present paper was given by K. Fujii and K. Iwata in Prog. Theor. Phys. **18** (1957), 666.

b) Parity conservation law and charge conjugation invariance seem to hold in the strong interactions but these are violated in cases of weak interactions containing neutrino.[3]*

We have not yet found any logical correlations between above mentioned noticeable facts, (I) and (II). We believe it an urgent task to find the phenomenological laws or the models of the interactions correlating at least a) and b) of (II) to each other. It is expected that such law or model may give us instructive suggestions about the forthcoming theory.

On the fact b), Lee and Yang have made a speculation that the nonconservation of parity is due to the peculiarity of neutrino, and hit upon two component neutrino theory.[4] Their theory explained rather skillfully the asymmetrical angular distribution of decay processes, but left the $\theta-\tau$ dilemma as it is. However, it seems to us that the parity nonconservation is common in all weak interactions participating in decays,* and the parity conservation law is true in all strong interactions.

§ 2. Parity, strangeness and the strength of interactions

We now discuss a model according to which the parity conservation law is violated necessarily in all weak interactions.

We first assume *the six species, ν, e, μ, p, n and Λ as elementary or different particles. Other stronglies are considered to be composite of p, n, Λ and their antiparticles.*[6]** *All interactions are, therefore, of Fermi type.*

We next assume that *the Fermi type interaction*

$$(\textit{coupling constant})\quad (\bar{\psi}_a O_i \psi_b)\quad (\bar{\psi}_c O_i' \psi_d)$$

among four fields a, b, c and d, is strong if $a=b$ and $c=d$ (coupling constant $g\simeq 10^{-42}$ erg·cm^3), and weak if otherwise (coupling constant $f\simeq 10^{-49}$ erg·cm^3). However, we should admit that such combinations as $a=b$ and $c=d$, if they contain leptons, are missing or much weaker than the weak interactions for some unknown reasons.

We lastly assume that *all interactions are invariant under the individual Γ_5 transformation* (abbreviated as $I\Gamma_5 T$).[7] $I\Gamma_5 T$ of 'a'-field is defined as

$$\psi_a \to \eta_a^a \gamma_5 \psi_a,\quad \psi_b \to \eta_b^a \psi_b,\quad \psi_c \to \eta_c^a \psi_c,\quad \cdots$$

$$m_a \to -m_a,\qquad m_b \to m_b,\qquad m_c \to m_c,\quad \cdots$$

where η's are phase factors : $|\eta|^2=1$. Therefore, if a and a' are different, the spinor pair $(\bar{\psi}_a O \psi_{a'})$ should contain a factor $(1\pm\gamma_5)$ because of the invariance assumed, while if a and a' are the same, $(\bar{\psi}_a O \psi_{a'})$ need not contain $(1\pm\gamma_5)$.

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* After this work has been accomplished, the parity nonconservation in hyperon decay is confirmed.[5]

** If we postulate the $I\Gamma_5 T$ invariance (our third assumption), the Sakata model[6] (where n, p and Λ are regarded as fundamental particles), and "Ξ^0, Ξ^-, Λ composite model" (where Ξ^0, Ξ^- and Λ are taken as fundamental) are permissible models without any violation of parity conservation in strong interactions. The latter is not reasonable by analogy with nuclear structure. Therefore, the Sakata model is almost only possible model from the standpoint of $I\Gamma_5 T$ invariance.

These assumptions immediately lead to the results that in all weak interactions the parity and the strangeness cannot be conserved and the charge conjugation invariance is violated, while in strong interactions, where the strangeness is conserved, the parity can be conserved.

Thus a correlation seems to be obtained between the properties under the space inversion and the two kinds of interactions, i. e., strong and weak interactions.

For example,

A) Pion-nucleon interaction is expressed as

$$g(\overline{\mathfrak{N}}\,\mathfrak{N})\,(\overline{\mathfrak{N}}\,\mathfrak{N}),$$

where $\mathfrak{N}$ denotes nucleon. This is the case of $a=b$, $c=d$, so this interaction is strong and the parity can be conserved.

B) $\pi(\Lambda\,\mathfrak{N})$ interaction participating in Λ-decay is written as

$$f(\overline{\mathfrak{N}}\,\mathfrak{N})\,(\overline{\Lambda}\,\mathfrak{N}).$$

Therefore, this is weak and the parity is not conserved. This is the important character of our model. Whenever the transition occurs from Λ-particles to nucleon or vice versa as in the case of the K-meson decay process, the interaction is necessarily weak and the parity is not conserved. So it may be said that we find one way to resolve the $\theta-\tau$ dilemma.

C) β-decay, expressed as

$$f(\overline{\mathfrak{N}}\,\mathfrak{N})\,(\bar{e}\,\nu),$$

must be weak and the parity is not conserved. The results obtained from such interaction are given in § 3. It will be shown that the results are consistent with experimental data of Co^{60} [8] and Co^{58} [9].

D) $\mu-e$-decay interaction is also weak and the parity conservation law does not hold.[10] By the postulation of the II'_5T invariance, this interaction must have the form

$$f_i(\bar{\psi}_e(1\pm\gamma_5)O_i\psi_\mu)\,(\bar{\psi}_\nu O_i\psi_\nu),$$

which differs from that of Lee-Yang's two component neutrino theory. As we shall see in § 3, this interaction imposes rather stronger restrictions than that of Lee-Yang's theory on the distribution of electrons emitted by μ-meson.

§ 3. Application of II'_5T invariance to lepton processes

We apply these assumptions to lepton processes such as the β-decay and $\mu-e$-decay, paying attention to the different results in two cases, that is, ours and Lee-Yang's.[4] In the following discussion, we assume the conservation law of lepton number[4),11] to be valid.

A) β-decay

The general form of the β-decay interaction is taken as the same one as Lee-Yang's. Our third assumption puts restriction on the coupling constants, (see Appendix). If the experimental result of He^6 is certain,[15] two possibilities remain, i. e.,

$$C_S = \pm C_S' = C_P = \pm C_P', \text{ and } C_T = \pm C_T'. \tag{3·1}$$

We denote the case corresponding to the upper (lower) signs by *Case* I_β (*Case* II_β) and this nomenclature is applied to all double signs appearing in the following paragraphs concerning the β-decay. The restricted interaction Hamiltonian density is

$$H_{\text{int}} = \{C_S(\bar{\psi}_p(1\pm\gamma_5)\psi_n)(\bar{\psi}_e(1\pm\gamma_5)\psi_\nu) + C_T/2(\bar{\psi}_p\sigma_{[\lambda\mu]}\psi_n)(\bar{\psi}_e\sigma_{[\lambda\mu]}(1\pm\gamma_5)\psi_\nu)\} + H.C.,$$

where $\sigma_{[\lambda\mu]} = (\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda)/(2i)$.

For allowed transitions, the angular distribution of e^- (e^+) emitted from a polarized nucleus is

$$I(\theta)\,d\theta = \text{const.}\,(1+\alpha\cos\theta)\,d\theta,$$

where

$$\alpha = \pm(\mp)(v_e/c)\langle J_z\rangle/J, \qquad \text{for } J\to J-1 \ (no),$$

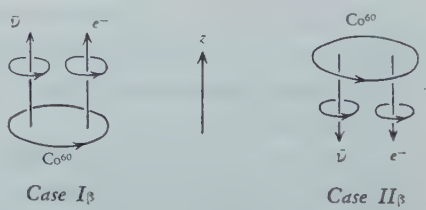
$$\alpha = \pm(\mp)(v_e/c)\langle J_z\rangle/J\left\{\frac{1}{J+1} + (-)\left(\frac{J}{J+1}\right)^{1/2}\cdot\frac{2Re(C_T C_S^*)}{|C_T|^2}\cdot\frac{|M_F|}{|M_{GT}|}\right\}\cdot\frac{1}{\dfrac{|C_S|^2\cdot|M_F|^2}{|C_T|^2\cdot|M_{GT}|^2}+1}, \qquad \text{for } J\to J \ (no),$$

$$\alpha = \mp(\pm)(v_e/c)\langle J_z\rangle/(J+1), \qquad \text{for } J\to J+1 \ (no),$$

and θ is the angle between the direction of motion of e^- (e^+) and the spin of polarized nucleus, i. e., $\theta = \measuredangle(\vec{J}, \boldsymbol{p}_e)$. For positron emitters the signs in the bracket are used.[9),12)]

On the other hand, if the electron mass is negligible compared with its momentum, $(1-(+)\gamma_5)/2$ is the projection operator into the parallel (antiparallel) spin state of electron relative to its direction of motion. Therefore, the emitted electron is almost completely polarized (antipolarized) in *Case* I_β (*Case* II_β).

Combining the above discussions of angular distribution and polarization of electron, we can find the strong correlation between them. For example, in the case of $Co^{60}\to Ni^{60}+e^-+\bar{\nu}$ ($J=5\to J'=4$ (no)), if the electron and antineutrino are emitted along z-axis, here its positive axis being chosen as the direction of nuclear spin, then the nuclear spin, the direction of motion e^-, $\bar{\nu}$ and their spiralities are correlated as shown in Fig. 1. Now we can compare our prediction with the recently available experimental results. We know that the electrons are emitted preferentially backwards with respect to the spin of Co^{60}.[8)] This experiment allows us to prefer *Case* II_β in Fig. 1. Hence the polarization of e^- is uniquely determined as shown in Fig. I, *Case* II_β. We can test this prediction by measuring the polarization



Case I_β *Case* II_β

Fig. 1

of electron. According to the results of Frauenfelder et al.,[8] polarization of electron is almost complete and antiparallel to the direction of motion of electron. Therefore, our results are consistent with the available data on Co^{60}.

In the case of $Co^{58} \to Fe^{58} + e^+ + \nu (J \to J' = J = 2 \ (no))$, *Case* II_3, which was accepted in the decay of Co^{60}, predicts

$$\alpha = + \frac{v_e}{c} \frac{\langle J_z \rangle}{J} \left\{ \frac{1}{J+1} - \left(\frac{J}{J+1} \right)^{1/2} \frac{2Re(C_T C_S^*)}{|C_T|^2} \frac{|M_F|}{|M_{GT}|} \right\} \cdot \frac{1}{\frac{|C_S|^2 \cdot |M_F|^2}{|C_T|^2 \cdot |M_{GT}|^2} + 1} .$$

We have no unambiguous knowledge about the magnitude of the second term in the bracket, but the experiments[9] on α suggest

$$\alpha \simeq + \frac{v_e}{c} \cdot \frac{\langle J_z \rangle}{J} \cdot \frac{1}{3} ,$$

which implies the smallness of the second term in brackets. Then, it is necessary to examine whether consistent determination of the two kinds of parameters, i. e., the nuclear matrix elements and β-decay coupling constants, is possible or not.[9),13] At present, we can say that although there is a substantial difference between the predictions of our assumptions and those of two component neutrino theory, both fit equally well to the experimental data of Co^{60} and Co^{58}. The urgent confirmation of the data of other beta-emitters is desirable.

Assuming that proton and neutron field operators individually suffer $I\,\Gamma_5\,T$, we get the restriction $(3 \cdot 1)$. This will be checked by the investigation of the first forbidden processes of the β-decay.

B) $\mu - e$-dacay

In the $\mu - e$-decay process, the interaction term is taken as*

$$(\bar{\psi}_\mu O_i \psi_e)\ (\bar{\psi}_\nu O_i' \psi_\nu),$$

where (VA)- and (STP)-coupling terms cannot coexist, because of the postulated $I\,\Gamma_5\,T$ invariance. We may calculate the cases of (VA)- and (STP)-coupling separately.

i) (VA)-coupling :

Interaction Hamiltonian density is taken as

$$H_{\text{int}} = \{ f_V (\bar{\psi}_\mu \gamma_\rho \psi_e)\ (\bar{\psi}_\nu \gamma_\rho \psi_\nu) + f_V' (\bar{\psi}_\mu \gamma_\rho \psi_e)\ (\bar{\psi}_\nu \gamma_5 \gamma_\rho \psi_\nu)$$
$$+ f_A (\bar{\psi}_\mu \gamma_5 \gamma_\rho \psi_e)\ (\bar{\psi}_\nu \gamma_5 \gamma_\rho \psi_\nu) + f_A' (\bar{\psi}_\mu \gamma_5 \gamma_\rho \psi_e)\ (\bar{\psi}_\nu \gamma_\rho \psi_\nu) + H.\,C.^{**}$$

* It may be possible to take $(\bar{\psi}_\mu O_i \psi_e)\ (\bar{\psi}_\nu O_i' \psi_{\bar{\nu}})$ as the $\mu - e$-decay interaction Hamiltonian. The results of this case is obtained by setting $f_T = f_T' = 0$, $f_V = f_A' = 0$ in the following calculations. By this replacements, there is no formal change of results for (VA), whereas, since $\rho = 0$ for (SP), we have no experimentally consistent result. Therefore, only allowable choice is the case of (VA).

** ψ_μ and ψ_e are the annihilation operators of μ^- and e^-.

where, according to the invariance assumed, $f_V = \pm f_A'$, $f_A = \pm f_V'$, double signs being chosen in the same order. We denote *Case* I_μ (*Case* II_μ) for the upper (lower) signs. The spectrum of $e^-(e^+)$ is

$$dN = \frac{2}{3(2\pi)^4} x^2 \cdot dx \cdot d\Omega \cdot W^5 [(3-2x) \mp (\pm) r \cdot (1-2x) \cos\theta],$$

$$\rho = 3/4,$$

and

$$(\textit{angular distribution of } e^-(e^+)) = \text{const.}\,(1 \pm (\mp)\,(r/3)\cos\theta),$$

where $W \equiv (\textit{maximum energy of } e^\pm) \simeq m_\mu/2$,

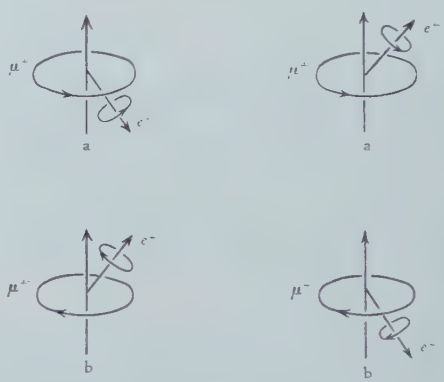
$x \equiv (\textit{energy of } e^\pm)/W \simeq p_e/(\textit{maximum value of } p_e)$,

$r \equiv (\textit{the longitudinal polarization degree of } \mu^\pm)$.

The similar argument as in the β-decay can also be applied. Thus, $e^-(e^+)$ is almost completely polarized (antipolarized) along the direction of its motion for *Case* I_μ. For *Case* II_μ, e^- and e^+ are interchanged. Combining this argument with the angular distribution of emitted electron, we can find the correlations among the direction of motion of μ^+ and e^+, and the spiralities of them as shown in Fig. 2.

ii) *(STP)-coupling* :

In $\mu \to e + 2\nu(2\bar{\nu})$, tensor interaction vanishes, i. e., $\rho\,(\equiv (12\,|\,g_T\,|^2)/(7\,(|\,g_S\,|^2 + |\,g_P\,|^2)) = 0$, which contradicts with the present experimental results,[14] so only $\mu \to e + \nu + \bar{\nu}$ is permitted.



Case I_μ *Case* II_μ

Fig. 2. Each *Case* I_μ and II_μ are separated into two posibilities *a* and *b* according to the spirality of μ-meson at the instant of its decay.

The argument is in the same manner as before. The interaction Hamiltonian density is

$$\begin{aligned} H_{\text{int}} = \{ & f_S(\bar{\psi}_\mu \psi_e)(\bar{\psi}_\nu \psi_\nu) + f_P(\bar{\psi}_\mu \gamma_5 \psi_e)(\bar{\psi}_\nu \gamma_5 \psi_\nu) \\ & + f_S'(\bar{\psi}_\mu \psi_e)(\bar{\psi}_\nu \gamma_5 \psi_\nu) + f_P'(\bar{\psi}_\mu \gamma_5 \psi_e)(\bar{\psi}_\nu \psi_\nu) \\ & + (f_T/2)(\bar{\psi}_\mu \sigma_{[\rho\lambda]} \psi_e)(\bar{\psi}_\nu \sigma_{[\rho\lambda]} \psi_\nu) + (f_T'/2)(\bar{\psi}_\mu \gamma_5 \sigma_{[\rho\lambda]} \psi_e)(\bar{\psi}_\nu \sigma_{[\rho\lambda]} \psi_\nu) \} \\ & + H.\,C., \end{aligned}$$

where by $I\,\Gamma_5\,T$ invariance, $f_S = \pm f_P'$, $f_S' = \pm f_P$, $f_T = \pm f_T'$. We denote *Case* III_μ (*Case* IV_μ) for the upper (lower) signs.

The calculated spectrum of $e^-(e^+)$:

$$dN = \frac{1}{(2\pi)^4} x^2 \cdot dx \cdot d\Omega \cdot W^5 \left[\frac{|g_S|^2 + |g_P|^2 + 6|g_T|^2}{3} \left\{ \{3(1-x)\right.\right.$$

$$+\frac{2}{3}\rho(4x-3)\Big\}\mp(\pm)r\Big\{(|g_S|^2+|g_P|^2)(1-x)$$

$$-\frac{2}{3}|g_T|^2(1+x)\Big\}\cos\theta\Big],$$

$$(\textit{angular distribution of } e^-(e^+)) = \text{const.}\,(1\mp(\pm)(r/3)\cdot\lambda\cdot\cos\theta),$$

where

$$\lambda\equiv\frac{3(|g_S|^2+|g_P|^2)-14|g_T|^2}{|g_S|^2+|g_P|^2+6|g_T|^2}=3-(16/3)\rho.$$

For e^+ we use the double signs in brackets. In the same way as (VA), $e^+(e^-)$ is almost completely polarized (antipolarized) along the direction of its motion for *Case* III_μ. For *Case* IV_μ, e^+ and e^- are interchanged. The correlation among the direction of motion of μ^+ and e^+, and their spiralities are shown in Fig. 4, where λ is taken as negative corresponding to the recent experimental value of ρ.[14]

We know the experimental fact that e^+ predominates backwards relative to the direction of motion of μ^+.[10] This fact and our derived results lead to the four possible correlations among the directions of motion of μ^+ and e^+, and their spiralities, i. e., *Case* I_μ *a*, *Case* II_μ *b* in Fig. 2 and *Case* III_μ *b*, *Case* IV_μ *a* in Fig. 4. If the spin of e^- is parallel to its motion, then either the two component neutrino theory, originally introduced by Lee and Yang,[1] or the lepton number conservation law must be abandoned.*

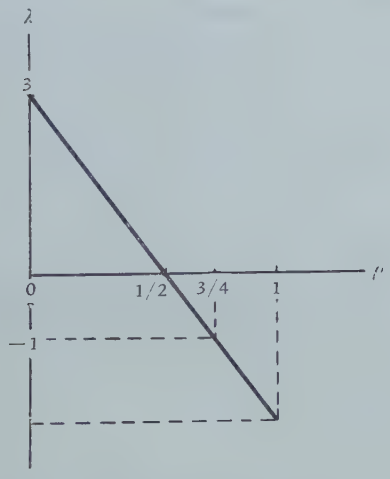
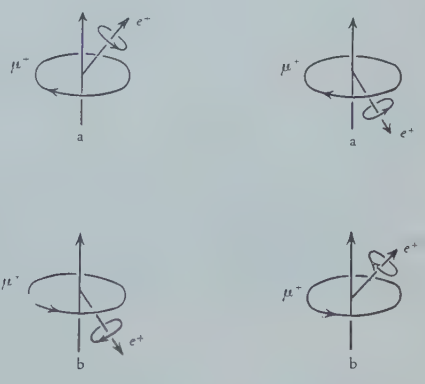


Fig. 3. If we take $\rho=3/4$, then $\lambda=-1$, so the angular distribution for *(STP)-coupling* coincides with that for *(VA)-coupling*.



Case III_μ *Case* IV_μ

Fig. 4. The correlations for *(STP)*-coupling similar to Fig. 2 (*for (VA)-coupling*) are illustrated.

* According to Lee and Yang's two component neutrino theory, where the tensor coupling in β-decay is assumed to be dominant, together with the experimental facts $\rho\neq0$ in $\mu-e$ decay and (*polarization of* e^-) $\sim -v/c$ in β-decay, the lepton numbers are fixed for each lepton. By this assignment the polarization of μ^+ emitted in the π^+ decay is parallel to the direction of μ^+ motion, then backwards emitted e^+ is antiparallel to the direction of its motion.

while in our case we may choose *Case* II_μ *b* or *Case* III_μ *b*, admitting the lepton number conservation law. (See note added in proof.)

§ 4. Discussion

By our tentative assumptions, we have found the violation laws of invariance for weak interactions. Their utility in the lepton processes is rather satisfactory. So far both our assumptions and two component neutrino theory can explain the experimental results of the β-decay of polarized nucleus, such as Co^{60}, and the asymmetries appearing in the $\pi-\mu-e$ process. But, our standpoint has an advantage over Lee-Yang's, because of the recently discovered parity nonconservation in hyperon decays,[5] which cannot be treated by the present two component neutrino theory.

Now, what one eagerly wishes to know is the physical meaning of the $I\Gamma_5 T$. S. Hori and A. Wakasa interpreted $I\Gamma_5 T$ invariance as due to the impossibility to determine the signs of mass of all elementary particles.[7] But we postpone to put the interpretation as it seems to us that the $I\Gamma_5 T$ invariance is one of the motive hypotheses to pull out the future theory.

It seems to us that our assumptions may be fairy effective to push on our project to construct the future theory which can treat the internal structure of particles, such as mass spectrum, classification of the families and so forth.

Of course, $I\Gamma_5 T$ invariance alone cannot realize our project. In the course of research the following problems must also be considered simultaneously: i) From where does the difference between families arise? ii) From where does the apparent universality of the coupling constants appear? iii) Why are the lepton processes always accompanied with neutrino, and why is the scattering type lepton process (e.g. $\nu+\mathfrak{N}\rightarrow\mathfrak{N}+\nu$) not strong?

It seems to us that these strange properties have their origins in the internal structure of elementary particles, too. Our assumptions and unresolved problems i), ii) and iii) may shed light on the profound nature of elementary particles.

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Note added in proof

We did not pay any attention to the mutually correlated choice of signs in $(1\pm\gamma_5)$ of all weak interactions, but as we shall show in the next work there is a charming regularity.

Appendix

If we assume the invariance of β-decay interaction Hamiltonian density under $I\,\Gamma_5\,T$, the following relations among the coupling constants C's and the phase factors η's are obtained.

$$-\eta_p^{p*}\cdot\eta_n^{p}\cdot\eta_e^{p*}\cdot\eta_\nu^{p}\cdot C_i=C_j',\quad -\eta_p^{p*}\cdot\eta_n^{p}\cdot\eta_e^{p*}\cdot\eta_\nu^{p}\cdot C_j'=C_i,$$

(where $(i, j)=(S, P)$, (P, S), (T, T), (V, A) or (A, V)),

$$\pm\eta_p^{n*}\cdot\eta_n^{n}\cdot\eta_e^{n*}\cdot\eta_\nu^{n}\cdot C_i=C_j',\quad \pm\eta_p^{n*}\cdot\eta_n^{n}\cdot\eta_e^{n*}\cdot\eta_\nu^{n}\cdot C_j'=C_i,$$

(where $+$sign is taken for $(i, j)=(S, P)$, (P, S), (T, T), and $-$sign for $(i, j)=(V, A)$, (A, V),

$$\mp\eta_p^{e*}\cdot\eta_n^{e}\cdot\eta_e^{e*}\eta_\nu^{e}\cdot C_i=C_i',\quad \mp\eta_p^{e*}\cdot\eta_n^{e}\cdot\eta_e^{e*}\cdot\eta_\nu^{e}\cdot C_i'=C_i,$$

(where $-$sign is taken for $i=S,\ T,\ P$, and $+$sign for $i=V,\ A$,),

$$\eta_p^{\nu*}\cdot\eta_n^{\nu}\cdot\eta_e^{\nu*}\cdot\eta_\nu^{\nu}\cdot C_i=C_i',\ \text{and}\ \eta_p^{\nu*}\cdot\eta_n^{\nu}\cdot\eta_e^{\nu*}\cdot\eta_\nu^{\nu}\cdot C_i'=C_i.$$

Clearly the coexistence of $(S, T, P)-$ and $(V, A)-$interactions is forbidden. It is easy to see that the following eight cases, which are mutually exclusive, are possible.

(i) $C_S=C_S'=C_P=C_P',\quad C_T=C_T',$

(ii) $C_S=-C_S'=-C_P=-C_P',\quad C_T=-C_T',$

(iii) $C_S=C_S'=-C_P=-C_P',$

(iv) $C_S=-C_S'=-C_P=C_P',$

(v) $C_V=C_V'=C_A=C_A',$

(vi) $C_V=-C_V'=-C_A=C_A',$

(vii) $C_V=C_V'=-C_A=-C_A',$

(viii) $C_V=-C_V'=C_A=-C_A'.$

For the case (i) (or (ii)) the products of phases of four fields are taken as

$$-\eta_p^{p*}\cdot\eta_n^{p}\cdot\eta_e^{p*}\cdot\eta_\nu^{p}=\eta_n^{p*}\cdot\eta_n^{n}\cdot\eta_e^{n*}\cdot\eta_\nu^{n}=-\eta_p^{e*}\cdot\eta_n^{e}\cdot\eta_e^{e*}\cdot\eta_\nu^{e}$$
$$=\eta_p^{\nu*}\cdot\eta_n^{\nu}\cdot\eta_e^{\nu*}\cdot\eta_\nu^{\nu}=+1\ \ (\text{or}\ -1).$$

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Phenomenological Analysis of the Decay Processes of the Σ- and Λ-Hyperons

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On the basis of the so-called $\Delta I=1/2$ law, the decay processes of Σ- and Λ-hyperons are discussed. In particular, the asymmetry coefficients of the pion coming from the hyperon decay are expressed in terms of ratios of partial decay rates of Σ^+-and Σ^--hyperons, and these are useful to test the original assumption of the $\Delta I=1/2$ law. The connection between the Σ- and Λ-decay interactions is also mentioned briefly.

§ 1. Introduction

Recently, it was discovered experimentally[1),2)] that the decay processes of Σ- and Λ-hyperons are parity non-conserving. Table I is a summary of experimental information available up to the present.

Table I

$\Sigma^+\to n+\pi^+$	$a_+P_+=-0.36\pm0.21$	Bologna, Columbia, Michigan and Pisa
$\Sigma^+\to p+\pi^0$	$a_0P_+=-0.37\pm0.19$	
$\Sigma^-\to n+\pi^-$	$a_-P_-=-0.13\pm0.26$	
$\Lambda\to p+\pi^-$	$a_\Lambda P_\Lambda=+0.44\pm0.11$	Berkeley

a is the asymmetry coefficients, while P is the (average) degree of polarization of the original hyperon.

Furthermore, we have a pretty good reason now[2)] to believe in spin $1/2$ for both Σ- and Λ-hyperons, thus making the discussion of decay processes simpler. This fact has already been used in making Table I.

The branching ratio[3)] of Λ-decay is in good agreement with the prediction from the $\Delta I=1/2$ law, while the situation in the K-meson decay is not. Therefore, it will be interesting to see whether the $\Delta I=1/2$ law does hold in the case of the Σ-decay. It is now well known that if the Σ-decay interaction is parity violating, then the branching ratio of $\Sigma^\pm$-decay and the ratio of the Σ^+- to the Σ^--decay rate are still compatible[4),5)] with the $\Delta I=1/2$ law. A test of this law in Σ-decay should be made for the asymmetry coefficients.

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Before going into detail, we would like to point out an important fact. The final state of Λ- or Σ-decay consists of one pion and one nucleon. If one neglects the electromagnetic interaction (and the mass differences between members of charge multiplets), then one can assign a total isotopic spin I, total angular momentum J, and parity to each particular final state. Since one can assume spin $1/2$ for Λ- or Σ-hyperons one finds $J=1/2$ in the final state provided our decay interaction is invariant under the famous combined operation CTP. One can then denote the transition matrix element for the specific final state with I and an s- (or p-) wave pion by $R_{2I,s}e^{i\delta_{2I}}$ (or $R_{2I,p}e^{i\delta_{2I}}$), where δ_{2I} (or $\delta_{2I,1}$) is the pion-nucleon phase shift (see Table II). Moreover, if (a) the decay interaction is invariant under time reversal, then $R_{2I,s}$ and $R_{2I,p}$ are both real, while if (b) the decay interaction is invariant under charge conjugation, then $R_{2I,s}$ is real, $R_{2I,p}$ is purely imaginary.[4),6),7)] If one uses the results given in Table I, one can conclude that case (b) is rejected at least for Λ-decay, because in that case one can see that $|a_\Lambda|<0.2$ without assuming the $\Delta I=1/2$ law,[4),6),8)] a result in contradiction to the empirical fact: $|a_\Lambda|\geq|a_\Lambda P_\Lambda|=0.44\pm0.11$.* Accordingly, we would like throughout this paper to accept the case (a): time reversal invariance for *both* Λ- and Σ- decay interactions.

If we assume the $\Delta I=1/2$ law, we can write down the decay probabilities of Σ- and Λ-hyperons as follows[3),4),7)]:

$$w(\Sigma^+\to n+\pi^+)=1/9\left[|R_{3s}e^{i\delta_3}+2R_{1s}e^{i\delta_1}|^2+|R_{3p}e^{i\delta_{31}}+2R_{1p}e^{i\delta_{11}}|^2\right], \tag{1}$$

$$w(\Sigma^+\to p+\pi^0)=2/9\left[|R_{3s}e^{i\delta_3}-R_{1s}e^{i\delta_1}|^2+|R_{3p}e^{i\delta_{31}}-R_{1p}e^{i\delta_{11}}|^2\right], \tag{2}$$

$$w(\Sigma^-\to n+\pi^-)=|R_{3s}e^{i\delta_3}|^2+|R_{3p}e^{i\delta_{31}}|^2, \tag{3}$$

$$w(\Sigma^+,\ \text{total})=1/3\left[(R_{3s})^2+(R_{3p})^2\right]+2/3\left[(R_{1s})^2+(R_{1p})^2\right], \tag{4}$$

$$w(\Lambda\to p+\pi^-)=2w(\Lambda\to n+\pi^0)=2/3\left[(R_{\Lambda s})^2+(R_{\Lambda p})^2\right], \tag{5}$$

and

$$w(\Lambda,\ \text{total})=(R_{\Lambda s})^2+(R_{\Lambda p})^2. \tag{6}$$

In the case of Λ-decay, we have only final states with total isotopic spin $1/2$. Therefore we have used for the magnitudes of the decay matrix elements the symbols $R_{\Lambda s}$ and $R_{\Lambda p}$ in this case.

Let θ be the angle between the pion and the direction of spin polarization of the initial hyperon in the rest system of the hyperon. Then the angular distribution of the decay pion will be[4),6),7)]

$$1+aP\cos\theta, \tag{7}$$

where P is the degree of polarization of the initial hyperon.

We shall define the asymmetry coefficients a_+, a_0, a_- and a_Λ corresponding to

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* According to ref. 2) $|P_\Lambda|$ is estimated as $\leq0.68\pm0.03$, so that $|a_\Lambda|\leq0.77\pm0.15$; or $0.44\pm0.11\leq|a_\Lambda|\leq0.77\pm0.15$.

$\Sigma^+\to n+\pi^+$, $\Sigma^+\to p+\pi^0$, $\Sigma^-\to n+\pi^-$ and $\Lambda\to p+\pi^-$. [a for $\Lambda\to n+\pi^0$ is equal to (11).]

The explicit forms of the a's are given by[4),7)]

$$a_+=\frac{2Re\left[\left(R_{3s}e^{-i\delta_3}+2R_{1s}e^{-i\delta_1}\right)\left(R_{3p}e^{i\delta_{31}}+2R_{1p}e^{i\delta_{11}}\right)\right]}{|R_{3s}e^{i\delta_3}+2R_{1s}e^{i\delta_1}|^2+|R_{3p}e^{i\delta_{31}}+2R_{1p}e^{i\delta_{11}}|^2}, \tag{8}$$

$$a_0=\frac{2Re\left[\left(R_{3s}e^{-i\delta_3}-R_{1s}e^{-i\delta_1}\right)\left(R_{3p}e^{i\delta_{31}}-R_{1p}e^{i\delta_{11}}\right)\right]}{|R_{3s}e^{i\delta_3}-R_{1s}e^{i\delta_1}|^2+|R_{3p}e^{i\delta_{31}}-R_{1p}e^{i\delta_{11}}|^2}, \tag{9}$$

$$a_-=\frac{2Re\left(R_{3s}e^{-i\delta_3}R_{3p}e^{i\delta_{31}}\right)}{(R_{3s})^2+(R_{3p})^2}, \tag{10}$$

and

$$a_\Lambda=\frac{2Re\left(R_{\Lambda s}e^{-i\delta_3}R_{\Lambda p}e^{i\delta_{31}}\right)}{(R_{\Lambda s})^2+(R_{\Lambda p})^2}. \tag{11}$$

In § 2, we shall write down a_+, a_0 and a_- as functions of ratios of $\Sigma^\pm$-decay rates, and in § 3, we shall discuss some interesting special cases.

One may wonder why this analysis of hyperon-decays should be tried once more even though many papers have already been published on this subject. The answer to this question will become evident in § 3.

§ 2. Explicit forms of asymmetry coefficients in terms of the decay rates

In this section one is solely concerned with Σ-decay. Since experimentally we know the decay rates w's better than the asymmetry coefficients a's, we want in this section to express the a's in terms of the w's. We introduce two parameters ϵ and η as follows:

$$w(\Sigma^+,\ \text{total})/w(\Sigma^-\to n+\pi^-)=2\epsilon, \tag{12}$$

and

$$w(\Sigma^+\to n+\pi^+)/w(\Sigma^+,\ \text{total})=\eta/2. \tag{13}$$

Table II. Pion-nucleon phase shifts[7)]

(a) $\Sigma\to$(nucleon) + (pion)

	$s_{1/2}$	$p_{1/2}$
$I=1/2$	$\delta_1=13.1^0$	$\delta_{11}=-2.5^0$
$I=3/2$	$\delta_3=-8.2^0$	$\delta_{31}=-2.5^0$

(b) $\Lambda\to$(nucleon) + (pion)

	$s_{1/2}$	$p_{1/2}$
$I=1/2$	$\delta_1=6.4^0$	$\delta_{11}\fallingdotseq 0$
$I=3/2$	$\delta_3=-4.4^0$	$\delta_{31}\fallingdotseq 0$

Experimental values[9] are :

$$\left.\begin{aligned}\epsilon &= 1.1 \pm 0.25,\\ \eta &= 0.90 \pm 0.12.\end{aligned}\right\} \tag{14}$$

Since we know the pion-nucleon phase shifts (see Table II), we can write down R_{3s}/R_{1s} and R_{3p}/R_{1p} as functions of ϵ, η and the ratio R_{1p}/R_{1s} using eqs. (1), (2), (3), (4), (12) and (13).

$$R_{1p}/R_{1s} \equiv \gamma, \quad R_{3s}/R_{1s} \equiv \beta, \tag{15}$$

$$\begin{aligned} R_{3p}/R_{1p} \equiv \beta' = {} & \frac{1+\gamma^2}{1+\gamma^2(\cos^2\delta_p/\cos^2\delta_s)} \cdot \frac{\cos\delta_p}{\cos\delta_s} \cdot \frac{12\epsilon-1-9\epsilon\eta}{2(6\epsilon-1)} \\ & \times\left[-1 \pm \frac{1}{\gamma}\left\{\gamma^2 - \frac{1+\gamma^2(\cos^2\delta_p/\cos^2\delta_s)}{1+\gamma^2}\right.\right. \\ & \left.\left.\times\left(\frac{1+\gamma^2}{\cos^2\delta_p} - \frac{8(6\epsilon-1)}{(12\epsilon-1-9\epsilon\eta)^2} \cdot \frac{\cos^2\delta_s}{\cos^2\delta_p}\right)\right\}^{1/2}\right], \end{aligned} \tag{16}$$

$$\beta\cos\delta_s + \beta'\gamma^2\cos\delta_p = -\frac{12\epsilon-1-9\eta\epsilon}{2(6\epsilon-1)}(1+\gamma^2), \tag{17}$$

where

$$\left.\begin{aligned}\cos\delta_s &\equiv \cos(\delta_3 - \delta_1) = 0.95,\\ \cos\delta_p &\equiv \cos(\delta_{31} - \delta_{11}) = 1.00.\end{aligned}\right\} \tag{18}$$

Neglecting $(1-\cos\delta)$ as compared with unity, one may put

$$\cos(\text{difference of any two pion nucleon phase shifts}) = 1\,; \tag{19}$$

then one can obtain quite simple results :

$$\left.\begin{aligned} R_{3s}/R_{1s} &\equiv \beta = -g + f\gamma,\\ R_{3p}/R_{1p} &\equiv \beta' = -(g + f\cdot 1/\gamma),\\ R_{1p}/R_{1s} &\equiv \gamma\,; \end{aligned}\right\} \tag{20}$$

where f and g are determined by ϵ and η :

$$\left.\begin{aligned} f &= \pm\frac{\sqrt{8(6\epsilon-1) - (12\epsilon-1-9\epsilon\eta)^2}}{2(6\epsilon-1)},\\ g &= \frac{12\epsilon-1-9\epsilon\eta}{2(6\epsilon-1)}. \end{aligned}\right\} \tag{21}$$

Throughout in this paper we shall assume (19). If one would like to perform more accurate analyses, one can start from (16) and (17). But considering the accuracy of present experimental results, we shall content ourselves within the approximation (19).

Eqs. (20) suggest that the parameters f and g are more handy than the original ϵ and η. In Table III we list some typical values of ϵ, η, f and g.

Table III

ϵ	η	$\lvert f\rvert$	g
1	1	0.600	0.200
1.1	0.9	0.590	0.294
0.9	0.9	0.611	0.285
1.1	1.1	0.586	0.117
9/8	8/9	0.504	0.304
9/10	8/9	0.606	0.295

Under the assumption (19), and using (20), the a's can now be written as follows:

$$\left.\begin{aligned} a_+ &\fallingdotseq \frac{2(f\gamma+2-g)\{(2-g)\gamma-f\}}{\{f^2+(2-g)^2\}(1+\gamma^2)}, \\ a_0 &\fallingdotseq -\frac{2(f\gamma-1-g)\{(1+g)\gamma+f\}}{\{f^2+(1+g)^2\}(1+\gamma^2)}, \\ a_- &\fallingdotseq -\frac{2(f\gamma-g)(g\gamma+f)}{(f^2+g^2)(1+\gamma^2)}. \end{aligned}\right\} \tag{22}$$

Unless $a\approx 0$, our approximation (19) does not cause any serious errors even for the a's.

§ 3. Discussion of some special cases

As was seen in the previous section, one can express the asymmetry coefficients a's in terms of ϵ, η and $\gamma \equiv R_{1p}/R_{1s}$. Therefore, if one knows ϵ and η, and fixes γ by imposing just one appropriate additional condition, the a's will be completely determined. The prediction will conveniently be used as a check of the original assumption of the $\Delta I=1/2$ law.

First of all, in § 3·1, we discuss the special case $\epsilon=\eta=1$, a choice which is not in disagreement with the present experimental information.

Next, we proceed to discuss two other interesting cases, in which we take:

$$a_-=0 \quad (\S\, 3\cdot 2)$$

$$\text{or} \quad a_+=a_0 \quad (\S\, 3\cdot 3).$$

The motivation for these choices will be evident from Table I. Finally we discuss a possible relation between the Σ- and the Λ-decay mechanisms.

We would like to remark again that the cosines of the differences of any two pion-nucleon phase shifts are taken to be unity (cf. (19)) in this whole section.

3·1 *Special case I,* $\epsilon=\eta=1$

Let us begin with the case

$$\epsilon=\eta=1. \tag{1·1}$$

From eqs. (21) we find

$$f=\pm 3/5, \quad g=1/5, \tag{1·2}$$

$$\left.\begin{aligned}\beta(\gamma, \pm)&=-1/5\pm 3/5\cdot\gamma,\\ \beta'(\gamma, \pm)&=-(1/5\pm 3/5\cdot 1/\gamma).\end{aligned}\right\} \tag{1·3}$$

Evidently we can see that

$$\left.\begin{aligned}\beta(\gamma, +)&=+\beta(-\gamma, -),\\ \beta'(\gamma, +)&=+\beta'(-\gamma, -).\end{aligned}\right\} \tag{1·4}$$

Inserting (1·3) into (22), we obtain

$$\left.\begin{aligned}a_+(\gamma, +)&=-a_-(\gamma, +)=\frac{2(\gamma+3)(3\gamma-1)}{(\gamma+3)^2+(3\gamma-1)^2},\\ a_0(\gamma, +)&=-\frac{2(\gamma-2)(2\gamma+1)}{(\gamma-2)^2+(2\gamma+1)^2},\end{aligned}\right\} \tag{1·5}$$

$$\left.\begin{aligned}a_+(\gamma, -)&=-a_+(-\gamma, +),\\ a_0(\gamma, -)&=-a_0(-\gamma, +),\\ a_-(\gamma, -)&=-a_-(-\gamma, +).\end{aligned}\right\} \tag{1·6}$$

It should be noted that there are identities

$$a_+(\gamma, \pm)=-a_-(\gamma, \pm). \tag{1·7}$$

In Fig. 1 we have plotted $a_+(\gamma, +)$ and $a_0(\gamma, +)$.

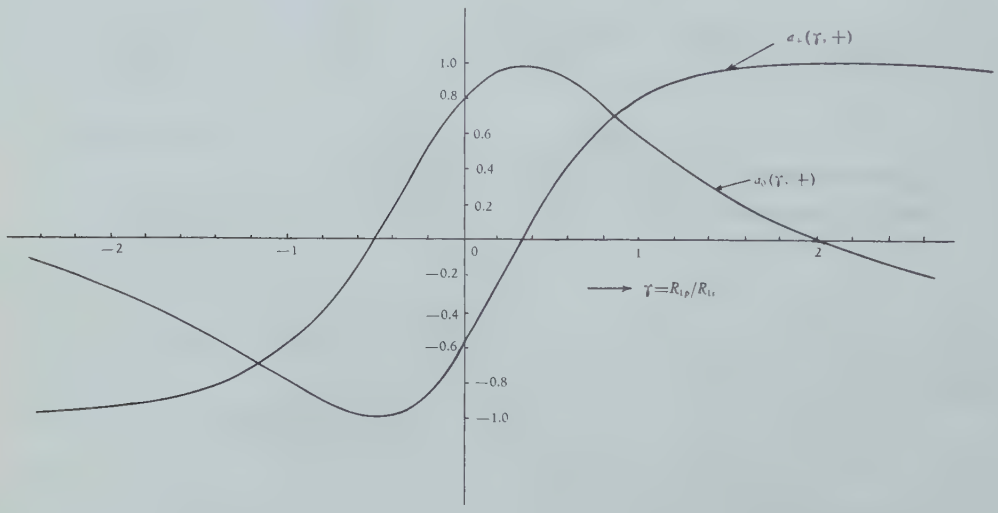

Fig. 1. $a_+(\gamma, +)$ and $a_0(\gamma, +)$ are shown here, see 3·1.

3·2 Special case II, $a_-=0$

Next, we shall assume

$$a_-(\gamma)=0. \tag{2·1}$$

From (2·1) and (22) we can determine γ, and consequently we can calculate $a_+(\gamma)$ and $a_0(\gamma)$:

$$\left.\begin{aligned} \gamma&=\gamma_\pm, \quad \gamma_+=+g/f, \\ &\qquad \gamma_-=-f/g; \end{aligned}\right\} \tag{2·2}$$

$$\left.\begin{aligned} a_+(g/f)&=\frac{4f\{(2-g)g-f^2\}}{(f^2+g^2)\{(2-g)^2+f^2\}}, \\ a_0(g/f)&=\frac{2f\{(1+g)g+f^2\}}{(f^2+g^2)\{(1+g)^2+f^2\}}, \\ a-(g/f)&=0; \end{aligned}\right\} \tag{2·3}$$

$$\left.\begin{aligned} a_+(-f/g)&=(g/f)a_+(g/f), \\ a_0(-f/g)&=-a_0(g/f), \\ a_-(-f/g)&=0. \end{aligned}\right\} \tag{2·4}$$

Some examples are shown in Table IV. Furthermore, if $\epsilon=\eta=1$, (and therefore $g=1/5$, $f=\pm 3/5$), then one can conclude from eq. (1·7) that

$$a_+(\gamma_\pm)=0$$

Table IV

f	g	γ_+ / γ_-	a_+	a_0	a_-
0.6	0.1	1/6	−0.278	0.972	0
		−6	−0.046	−0.972	0
0.6	0.2	+1/3	0	1	0
		−3	0	−1	0
0.6	0.3	1/2	0.246	0.976	0
		−2	0.123	−0.976	0

3·3 Special case III, $a_+=a_0$

We now postulate

$$a_+(\gamma)=a_0(\gamma). \tag{3·1}$$

From eqs. (22) and (3·1) we can solve γ :

$$\left.\begin{aligned} \gamma&=\gamma_\pm=-\alpha\pm\sqrt{\alpha^2+1}, \\ \alpha&=\frac{f(1-2g)}{f^2+2+g-g^2}; \end{aligned}\right\} \tag{3·2}$$

and thus from (22) we obtain :

$$a_+(\gamma_+)=a_0(\gamma_+)=\frac{1}{\sqrt{\alpha^2+1}}\left\{\frac{2(2-g)(2-g-\alpha f)}{f^2+(2-g)^2}-1\right\},$$
$$a_-(\gamma_+)=\frac{1}{\sqrt{\alpha^2+1}}\left\{\frac{2g(\alpha f+g)}{f^2+g^2}-1\right\}; \tag{3·3}$$

$$\left.\begin{aligned} a_+(\gamma_-)&=a_0(\gamma_-)=-a_+(\gamma_+),\\ a_-(\gamma_-)&=-a_-(\gamma_+).\end{aligned}\right\} \tag{3·4}$$

Some examples are shown in Table V.

Table V

f	g	α	γ_+ / γ_-	$a_+=a_0$	a_-
0.6	0.2	1/7	0.867	0.707	−0.707
			−1.153	−0.707	0.707
0.6	0.3	24/257	0.911	0.718	−0.523
			−1.097	−0.718	0.523

3·4 Special case IV, Feld's choice

According to Feld,[7] one may assume that the degree of parity non-conservation is independent of the isotopic spin, i. e.

$$\left.\begin{aligned} &R_{3p}/R_{3s}=-R_{1p}/R_{1s},\\ \text{or}\quad &\beta=-\beta'\end{aligned}\right\} \tag{4·1}$$

Then one can find the values of γ from (4·1) and (20):

$$\gamma=\gamma_\pm=g/f\pm\sqrt{(g/f)^2+1}. \tag{4·2}$$

Resulting expressions for a_+, a_-, and a_0 are as follows:

$$a_+(\gamma_+)=\frac{f(4-f^2-g^2)}{\sqrt{f^2+g^2[f^2+(2-g)^2]}},$$
$$a_0(\gamma_+)=\frac{f(1-f^2-g^2)}{\sqrt{f^2+g^2[f^2+(1+g)^2]}}, \tag{4·3}$$
$$a_-(\gamma_+)=-\frac{f}{\sqrt{f^2+g^2}};$$

and

$$\left.\begin{aligned} a_+(\gamma_-)&=-a_+(\gamma_+),\\ a_0(\gamma_-)&=-a_0(\gamma_+),\\ a_-(\gamma_-)&=-a_-(\gamma_+).\end{aligned}\right\} \tag{4·4}$$

A few examples of this case are shown in Table VI. When $f=3/5$ and $g=1/5$, one knows that $\epsilon=\eta=1$ (Table I). Therefore, as was stated in 3·1, one sees that $a_+ \fallingdotseq -a_-$ in this particular case.

Feld[7] took into account the accurate values of the cosines of the differences of pion-nucleon phase shifts. This explains the small differences in Table VI.

Table VI

f	g	$\frac{\gamma_+}{\gamma_-}$	a_+	a_0	a_-
0.6	0.2	1.387 −0.721	0.95 −0.95	0.33 −0.33	−0.95 +0.95
0.6	0.3	1.618 −0.618	0.98 −0.98	0.24 −0.24	−0.89 +0.89
Feld[7] $f=0.6$ $g=0.2$			±0.95	±0.32	∓0.94

§ 4. Λ-decay and its relation to Σ-decay

Using the $\Delta I=1/2$ law, and eq. (19), one finds

$$\left.\begin{aligned} a_\Lambda &= \frac{2\gamma_\Lambda}{1+\gamma_\Lambda{}^2}, \\ \gamma_\Lambda &= R_{\Lambda p}/R_{\Lambda s}. \end{aligned}\right\} \tag{23}$$

Let ω_Λ and p_Λ (or ω_Σ and p_Σ) be the energy and momentum of the pion coming from the Λ- (or Σ-) hyperon at rest. The density of the final state will be proportional to

$$\rho_\Lambda = \frac{\omega_\Lambda p_\Lambda}{1+\omega_\Lambda/M} \quad \left(\text{or } \rho_\Sigma = \frac{\omega_\Sigma p_\Sigma}{1+\omega_\Sigma/M}\right), \tag{24}$$

where M is nucleon mass. Let us suppose a very special case where the Σ-decay interaction to the final states with the total iso-spin $I=1/2$ is identical with that of Λ's. Then we shall find

$$\left.\begin{aligned} (R_{1s}/R_{\Lambda s})^2 &= (\rho_\Sigma/\omega_\Sigma)/(\rho_\Lambda/\omega_\Lambda), \\ (R_{1p}/R_{\Lambda s})^2 &= (p_\Sigma{}^2\rho_\Sigma/\omega_\Sigma)/(p_\Lambda{}^2\rho_\Lambda/\omega_\Lambda); \end{aligned}\right\} \tag{25}$$

accordingly,

$$\gamma_\Lambda = (p_\Lambda/p_\Sigma)\gamma. \tag{26}$$

If our assumption is correct, we can predict γ_Λ (and therefore a_Λ) from (26) provided we have known γ from information about Σ-decay and vice versa. In fact, we do not yet know γ. Therefore, by using eqs. (25), (26) and the ratio of lifetimes:

$$w(\Sigma^- \to n+\pi^-)/w(\Lambda, \text{ total}) = \zeta, \tag{27}$$

one can determine γ (and γ_Λ). The result is

$$\gamma^2=\frac{\zeta(\rho_\Lambda\omega_\Sigma)/\rho_\Sigma\omega_\Lambda)-(f^2+g^2)}{(f^2+g^2)-\zeta(p_\Lambda{}^2\rho_\Lambda\omega_\Sigma/p_\Sigma{}^2\rho_\Sigma\omega_\Lambda)}, \tag{28}$$

where we have used representation (20).

It is interesting to consider a special case:

$$\epsilon=\eta=1, \quad f=\pm 3/5, \quad g=1/5$$

and

$$\zeta=2.$$

From eq. (28) one can find

$$\gamma=\pm 3,$$

and thus obtains the following results:

	$a_+=-a_-$	a_0	a_Λ
$\gamma=3$	0.96	−0.28	0.89
$\gamma=-3$	0	−1	−0.89

These results are not compatible with Table I and show that our initial assumption, contained in eqs. (25) and (26), seems to be too stringent. This does not, of course, mean that the $\Delta I=1/2$ law should be abandoned.

Unfortunately, we do not have enough knowledge about the degrees of polarization of the initial hyperons (P in Table I) to make it possible to proceed to further theoretical analysis.

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On the Collective Mode of Internal Motion of the Nucleus to be coupled with the Irrotational Surface Motion

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From the analyses of unsatisfactory points in the usual irrotational collective model, it has been considered that there should exist a kind of "internal" rotational collective motion to be coupled with the usual irrotational surface motion. Using Watanabe's method, a generalization of the collective description with redundant variables is developed and it is shown that the introduction of the rotational collective motion is indispensable in describing the collective motion in a closed form consistently. It is confirmed that this rotational mode plays an important role in explaining the discrepancy between the experimental and theoretical values of the effective moments of inertia of nuclei. The fundamental assumption of the various models concerning the moments of inertia is discussed.

§ 1. Introduction

The nuclear collective model proposed by Bohr-Mottelson[1] and Hill-Wheeler[2] has had considerable success in accounting for many features of the nuclear structure. Particularly the confirmation of the existence of low energy rotational states in nuclei with mass numbers between 150 and 190 and above 222 has made the reality of this model decisive. Such a striking success of the model has stimulated the interest of many researchers in analysing some of the underlying problems.

An important problem among those is the one concerning the values of the effective moments of inertia, J_{eff}, which are deduced from the separations of rotational energy levels. Recently it was recognized that the collective model in the original work of Bohr-Mottelson, based on the assumption of incompressible and irrotational collective flow, gives the values of moments of inertia smaller than the empirical values by a factor $1/3 \sim 1/5$, while the latter are found to be smaller than the rigid-body value by a factor $1/3 \sim 1/2$.[3] A number of attempts[4)-15)] has been made to elucidate the above situation by trying to take into account not only the surface rotation caused by the irrotational collective motion, but also another kind of rotations due to the motion of individual particles. For example, Tomonaga[10] assumed the existence of the "internal" rotational collective motion in addition to the usual irrotational collective motion and considered the effective moments of inertia to be increased by the strong coupling between these two kinds of collective motions. In order to formulate this physical consideration, however, we need to introduce some hypothetical quantities for the new internal modes of collective motion, and as long as we use such quantities it is impossible to derive the

relation existing necessarily between the new internal modes and the usual irrotational modes from the formulation itself. Another approach was made by Inglis[4] using the so-called "cranking model." In this model the moments of inertia are discussed in terms of the motion of nucleons in a deformed and averaged potential which is cranked externally. However, the validity of this model is questionable because in this model the collective motion which should be generated by the many particle system itself is imposed by cranking externally.

Another important one of the underlying problems in the nuclear collective model is the quantum mechanical description of the collective modes of motion. In the original work of Bohr-Mottelson, the "collective" variables which describe the shape and orientation of the nucleus were treated as dynamical variables. As the natural consequence, theoretical works[16)-26)] have been developed to relate these variables to the collective motion of a many nucleon system. Two main types of theories have been presented on the collective motion, that is, those theories using "redundant variables" and that of Tomonaga. In the latter method a transformation of variables is carried out so that some of the new coordinates are directly related to the collective motion and the remaining new coordinates are considered to describe the individual particle motion. In the "redundant variables" treatment, certain auxiliary variables to describe the collective motion are introduced together with appropriate subsidiary conditions preserving the correct number of degrees of freedom, and the canonical transformations are employed in such a way that after the transformations the original particle coordinates are directly related to the individual particle motion. Although these methods are successful, they show certain unsatisfactory features when the irrotational collective modes are treated by them. In Tomonaga's method, we must introduce the collective momenta $\pi_\rho(x_1, x_2, \cdots x_N; p_1, p_2, \cdots p_N) \equiv \pi_\rho(x, p)$ which should be canonically conjugate to the new collective coordinates. Unfortunately these collective momenta do not satisfy the commutation relations $[\pi_\rho, \pi_\sigma]=0$, and this fact means that the irrotational collective modes are not separable from each other. In the redundant variable treatment, the same quantities as $\pi_\rho(x, p)$ must be used in the canonical transformations[24)] and this situation prevents us from describing the collective motion in a closed form. Here it should be noted that the non-commutability between π_ρ is related to the existence of a kind of "internal" rotational modes.

Thus we are forced to consider as follows: though the usual collective model, based on the assumption of the irrotational and incompressible collective flow, reflects the actual collective motion correctly in some respects, there also exists the "internal" rotational mode of motion and we have to take it into consideration rightly as one of the collective modes which plays an important role, so that we may obtain a unified formulation and make clear the interrelation between rotational and irrotational modes.

In this paper we shall formulate quantum-mechanically the method of "point transformation" proposed by Watanabe[26)] where each particle coordinate is transformed with common parameters and they are treated as the variables describing the collective modes. And, following the above mentioned consideration, we shall try to generalize the usual methods with redundant variables in such a way that the rotational collective mode is

introduced on an equal footing with the usual irrotational mode. The main purpose of this paper is to show the following: only by taking into account the rotational mode of motion in addition to the irrotational mode of motion, we can develope the method with the redundant variables in a closed form and make clear the interrelation between both modes. Using this formulation, we can clarify the unsatisfactory points in the usual redundant variables treatment and give the right status to the various attempts concerning the effective moments of inertia.

In § 2 the formulation will be developed along the line of Watanabe's. This is the quantum mechanical recapitulation of his method. In § 3 it will be shown that the introduction of the rotational collective mode is indispensable in constructing the theory in a closed consistent form. Also some discussions will be made about the rotational collective mode thus introduced. In § 4 the several models concerning the moments of inertia will be discussed on the basis of our method and the foundation of these models is given. § 5 will be devoted to the concluding remarks.

§ 2. Formulation

(I) *Introduction of collective variables*

Let us consider a nucleus with N nucleons. The coordinates of i-th nucleon are designated by $\bar{x}_i^\mu$ where $^\mu$ denotes the components of the vector. In order to introduce the collective variables to describe the collective motion of this system, we start from a linear point transformation of coordinates of each nucleon:

$$\bar{x}_i^\mu = x_i^\mu + u_\nu^\mu x_i^\nu \quad (i=1, 2, \cdots, N), \tag{2·1}$$

where u_ν^μ are the common parameters to the coordinates of each nucleon and specify the linear transformation. Hereafter indices appearing twice in one term shall be summed over them.

If we regard x_i^μ as variables describing the "individual" particle motion and u_ν^μ as newly introduced variables describing the "collective" motion, the equations (2·1) mean that the motion of nucleons is decomposed into the "individual" particle and the "collective" motions. The coordinates of nucleons agree with the individual particle coordinates when $u_\nu^\mu = 0$, and undergo displacements $\delta\bar{x}_i^\mu = u_\nu^\mu x_i^\nu$ when collective variables u_ν^μ change their values from 0 to u_ν^μ.

If the motion caused by u_ν^μ is irrotational and incompressible, its velocity field $\boldsymbol{v}$ satisfies

$$\mathrm{rot}\,\boldsymbol{v} = 0 \quad \text{and} \quad \mathrm{div}\,\boldsymbol{v} = 0, \tag{2·2}$$

and collective variables u_ν^μ are restricted as

$$u_\nu^\mu = u_\mu^\nu \quad \text{and} \quad \sum_\mu u_\mu^\mu = 0\ ^{27)}. \tag{2·2'}$$

In order to introduce the rotational collective mode on the equal footing with the irrotational collective mode, we leave out these conditions (2·2). Corresponding to the

nine degrees of freedom of the most general form of linear transformations, here is introduced a set of collective variables α^ρ ($\rho=0, 1, \cdots, 8$). Then the transformation matrix is written as follows:

$$\begin{pmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{pmatrix} = \begin{pmatrix} 1+\alpha^0+\alpha^3+\alpha^5 & \alpha^1-\alpha^8 & \alpha^4+\alpha^7 \\ \alpha^1+\alpha^8 & 1+\alpha^0-\alpha^3+\alpha^5 & \alpha^2-\alpha^6 \\ \alpha^4-\alpha^7 & \alpha^2+\alpha^6 & 1+\alpha^0-2\alpha^5 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \equiv \begin{pmatrix} S^x(x, \alpha) \\ S^y(x, \alpha) \\ S^z(x, \alpha) \end{pmatrix}. \quad (2\cdot3)$$

The antisymmetric parts contained in the non-diagonal elements correspond to the degrees of freedom of infinitesimal rotations about each coordinate axis and α^0 represents that of the compressible mode. Accordingly, (2·2) is expressed as

$$\alpha^0=0, \quad \alpha^6=0, \quad \alpha^7=0, \quad \alpha^8=0 \quad (2\cdot4)$$

and the matrix u reduces to the symmetric one with spur zero:

$$(u^\mu_\nu) = \begin{pmatrix} \alpha^3+\alpha^5 & \alpha^1 & \alpha^4 \\ \alpha^1 & -\alpha^3+\alpha^5 & \alpha^2 \\ \alpha^4 & \alpha^2 & -2\alpha^5 \end{pmatrix}. \quad (2\cdot4')$$

α^ρ ($\rho=1, 2, 3, 4, 5$) correspond to the irrotational and incompressible parts of the collective motion and are connected with α_{2m} of Bohr-Mottelson's through the following relations:

$$\begin{aligned} \alpha^1 &= \frac{i}{4}\sqrt{\frac{15}{2\pi}}(\alpha_{2,2}-\alpha_{2,-2}), \quad \alpha^2 = \frac{1}{4i}\sqrt{\frac{15}{2\pi}}(\alpha_{2,1}+\alpha_{2,-1}), \\ \alpha^3 &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}(\alpha_{2,2}+\alpha_{2,-2}), \quad \alpha^4 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(\alpha_{2,-1}-\alpha_{2,1}), \\ \alpha^5 &= -\frac{1}{4}\sqrt{\frac{5}{2\pi}}\alpha_{2,0}. \end{aligned} \quad (2\cdot5)$$

Thus it is apparent that the nine parameters α^ρ of the linear point transformation given by (2·3) can be regarded as the generalized collective variables which contain those of usual irrotational collective mode.

(II) *Subsidiary conditions*

Our next task is to find suitable subsidiary conditions to suppress the increase of the degrees of freedom. We shall employ here Watanabe's method[26] where the subsidiary conditions are set up as the relations between the particle and the collective momenta derived from the Lagrangian formalism.

In the original representation which we call "the particle representation" or the (P)-representation, the Lagrangian $L^{(P)}(\bar{x}, \dot{\bar{x}})$ of our system is given by

$$L^{(P)} \equiv L^{(P)}(\bar{x}, \dot{\bar{x}}) = M/2 \cdot \sum_i \dot{\bar{x}}^\mu_i \dot{\bar{x}}^\mu_i - V(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N) \quad (2\cdot6)$$

where M is the nucleon mass and $V(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$ the interaction potential between

nucleons. By the transformation (2·3), the Lagrangian (2·6) becomes

$$L^{(\mathrm{I})} \equiv L^{(\mathrm{I})}(x, \dot{x}\,;\ \alpha, \dot{\alpha}) = M/2 \cdot \sum_i \left(\frac{\partial S^\mu}{\partial x_i^\nu} \dot{x}_i^\nu + \frac{\partial S^\mu}{\partial \alpha^\rho} \dot{\alpha}^\rho \right)^2 - V(x, \alpha) \tag{2·7}$$

which we call the Lagrangian in the (I)-representation. In this (I)-representation we define the canonical conjugate momenta p_ν^i and β_ρ to the coordinates x_i^ν and α^ρ, respectively, as follows:

$$p_\nu^i \equiv \frac{\partial L^{(\mathrm{I})}}{\partial \dot{x}_i^\nu} = M \left(\frac{\partial S^\mu}{\partial x_i^{\nu\prime}} \dot{x}_i^{\nu\prime} + \frac{\partial S^\mu}{\partial \alpha^\rho} \dot{\alpha}^\rho \right) \frac{\partial S^\mu}{\partial x_i^\nu}, \tag{2·8a}$$

$$\beta_\sigma \equiv \frac{\partial L^{(\mathrm{I})}}{\partial \dot{\alpha}^\sigma} = M \sum_i \left(\frac{\partial S^\mu}{\partial x_i^{\nu\prime}} \dot{x}_i^{\nu\prime} + \frac{\partial S^\mu}{\partial \alpha^\rho} \dot{\alpha}^\rho \right) \frac{\partial S^\mu}{\partial \alpha^\sigma}. \tag{2·8b}$$

Owing to the group property of the transformation* which is expressed by the following set of differential equations:

$$\Lambda_\rho^\nu(x_i) \frac{\partial S^\mu}{\partial x_i^\nu} + G_\rho^\sigma(\alpha) \frac{\partial S^\mu}{\partial \alpha^\sigma} = 0, \tag{2·9}$$

the above defined momenta are not independent of each other. Here $\Lambda_\rho^\nu(x_i)$ are defined by

$$\Lambda_\rho^\nu(x_i) \equiv \left. \frac{\partial f^\nu(x_i, \alpha)}{\partial \alpha^\rho} \right|_{\alpha \to 0} \tag{2·10}$$

and given by

$$\left(\Lambda_\rho^\nu(x_i) \right) = \begin{pmatrix} -x_i & -y_i & 0 & -x_i & -z_i & -x_i & 0 & -z_i & y_i \\ -y_i & -x_i & -z_i & y_i & 0 & -y_i & z_i & 0 & -x_i \\ -z_i & 0 & -y_i & 0 & -x_i & 2z_i & -y_i & x_i & 0 \end{pmatrix}, \tag{2·11}$$

$f^\nu(\bar{x}_i, \alpha)$ being the inverse transformation to (2·3), i.e. $x_i^\nu = f^\nu(\bar{x}_i, \alpha)$. The explicit form of $G_\rho^\sigma(\alpha)$ is given in the Appendix. Using (2·9), the relations between the momenta defined by (2·8) are expressed in a simple form:

$$\phi_\rho^{(\mathrm{I})} \equiv \sum_i \Lambda_\rho^\nu(x_i) p_\nu^i + G_\rho^\sigma(\alpha) \beta_\sigma = 0. \tag{2·12}$$

These relations are adopted as the subsidiary conditions.

Now we shall find the subsidiary conditions in the (P)-representation which are equivalent to (2·12) in the (I)-representation. The transformation to the (P)-representation is performed by the canonical one:

$$W = \sum_i S^\mu(x_i, \alpha) \bar{p}_\mu^i + \alpha^\sigma \bar{\beta}_\sigma, \tag{2·13}$$

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* For details on this point see the Appendix in Watanabe's paper or, for example, G. Racah, *Group Theory and Spectroscopy*, Mimeographed Lecture Note, Princeton, 1951.

generating the inverse transformation to (2·3). Consequently (2·12) are transformed into those of the (P)-representation and they become

$$\phi_{\rho}^{(\mathrm{P})}(\bar{\alpha}, \bar{\beta}) \equiv G_{\rho}^{o}(\bar{\alpha})\bar{\beta}_{o}=0. \tag{2·14}$$

As discussed by Watanabe, these conditions do not restrict the equation of motion of the original system.

(III) *Quantization*

In Watanabe's classical treatment, the transformation of the representation is performed by the canonical transformation corresponding to a coordinate transformation and then the old variables are expressed by the new ones. For quantum mechanical treatment, we need the quantum mechanical transformation corresponding to the classical one. In the following we shall summarize this procedure in the general form.

Let us denote the old coordinates by q^k and the new ones by Q^k, where $k=1, 2, \cdots, S$ and S is the number of the degrees of freedom. When the coordinate transformation is given by

$$Q^k=Q^k(q^1, q^2, \cdots, q^S), \tag{2·15}$$

the classical generating function is

$$W=\sum_k Q^k(q)P_k, \tag{2·16}$$

where P_k are the canonical conjugate momenta to Q^k and those to q^k are denoted by p_k. By (2·16), the equations of the transformations of momenta become

$$p_j=\sum_k \frac{\partial Q^k(q)}{\partial q^j}\cdot P_k \quad (j=1, 2, \cdots S). \tag{2·17}$$

The quantum mechanical canonical transformation corresponding to (2·16) is given for any operator O as follows:[28]

$$O^{(Q)}=T^{(Qq)}O^{(q)}(T^{-1})^{(qQ)}. \tag{2·18}$$

Here $O^{(Q)}$ and $O^{(q)}$ are the forms of the operator O appropriate to the Q- and q-coordinate systems, respectively. $T^{(Qq)}$ is the canonical transformation operator which plays the role of rewriting q-variables in terms of Q variables and of multiplying the resulting quantities by the square root of the Jacobian $D(Q)=\partial(q^1, q^2, \cdots, q^S)/\partial(Q^1, Q^2, \cdots, Q^S)$. Taking the momentum operator p_j as O, the relation (2·18) becomes

$$p_j=\sum_k\left(\frac{\partial Q^k(q)}{\partial q^j}\right)^{(Q)}\left\{P_k+\frac{i\hbar}{2}\frac{\partial}{\partial Q^k}\ln D(Q)\right\}. \tag{2·19}$$

Therefore, we get

$$p_j=1/2\cdot\sum_k\left\{\left(\frac{\partial Q^k(q)}{\partial q^j}\right)^{(Q)}P_k+P_k\left(\frac{\partial Q^k(q)}{\partial q^j}\right)^{(Q)}\right\}, \tag{2·20}$$

which is the symmetrized form of (2·17). Thus by using (2·15) and (2·20), we

can perform the transformation of the representation in the quantum theory.

(IV) *Quantum mechanical description of the collective motion*

According to the previous discussions, we shall develop the quantum mechanical description of the collective motion in parallel with Watanabe's classical one.

In the (P)-representation, the Hamiltonian of our system is

$$H^{(\mathrm{P})} \equiv H^{(\mathrm{P})}(\bar{x}, \bar{p}) = 1/2M \cdot \sum_i \bar{p}_\mu^i \bar{p}_\mu^i + V(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N) \tag{2·21}$$

with the subsidiary conditions for the wave function $\Psi^{(\mathrm{P})}$

$$\phi_\rho^{(\mathrm{P})}(\vec{\alpha}, \bar{\beta})\, \Psi^{(\mathrm{P})}(\bar{x}, \bar{p}\,; \vec{\alpha}; \bar{\beta}) = 0 \tag{2·22}$$

where

$$\phi_\rho^{(\mathrm{P})}(\vec{\alpha}, \bar{\beta}) \equiv 1/2 \cdot \{G_\rho^\sigma(\vec{\alpha})\, \bar{\beta}_\sigma + \bar{\beta}_\sigma G_\rho^\sigma(\vec{\alpha})\}, \tag{2·23}$$

which are the symmetrized forms of classical expressions (2·14).

By the coordinate transformation, (2·3) and $\alpha^\rho = \bar{\alpha}^\rho$, we turn to the (I)-representation. The transformed Hamiltonian and the subsidiary conditions become

$$H^{(\mathrm{I})}(x, p\,; \alpha, \beta) = 1/2M \cdot \sum_i g^{\mu\nu}(\alpha)\, p_\mu^i p_\nu^i + V^{(\mathrm{I})}(x, \alpha) \tag{2·24}$$

and

$$\phi_\rho^{(\mathrm{I})}(x, p\,; \alpha, \beta)\, \Psi^{(\mathrm{I})}(x, p\,; \alpha, \beta) = 0, \tag{2·25}$$

where

$$\phi_\rho^{(\mathrm{I})}(x, p\,; \alpha, \beta) \equiv 1/2 \cdot \{\sum_i A_\rho^\mu(x_i)\, p_\mu^i + G_\rho^\sigma(\alpha)\, \beta_\sigma + H.C.\}. \tag{2·26}$$

Here $g^{\mu\nu}(\alpha) = \partial f^\lambda / \partial \bar{x}_i^\lambda \cdot \partial f^\nu / \partial \bar{x}_i^\lambda$ and they become $\delta^{\mu\nu}$ when α^ρ tend to zero. *H.C.* means the hermitian conjugate to the foregoing quantities.

The (I)-representation is not suitable, because in the subsidiary conditions both the collective and the particle variables are contained. Therefore we turn into the (II)-representation where the collective modes become real modes free from the individual particles. As discussed by Watanabe, requirements for the (II)-representation are summarized as follows:

i) The subsidiary conditions are expressed in terms of the individual particle variables only.

ii) The coupling terms between the collective and particle momenta vanish in the Hamiltonian.

iii) New collective coordinates A^σ are described by certain symmetric functions with respect to x_i.

As for these requirements, no difference exists between the quantum and the classical treatments.

The transformation into the (II)-representation is performed by the canonical transformation corresponding to the classical generating function:

$$W = \sum_i x_i^\mu P_\mu^i + A^\sigma(x, \alpha)\, B_\sigma, \tag{2·27}$$

where $(X_i^\mu, P_\mu^i; A^\sigma, B_\sigma)$ are new variables. The requirement i) is satisfied by choosing $A^\sigma(x, \alpha) = A^\sigma(\bar{x})$.* The subsidiary conditions are transformed into

$$\phi_\rho^{(\mathrm{II})}(X, P)\, \Psi^{(\mathrm{II})}(X, P; A, B) = 0 \tag{2·28}$$

where

$$\phi_\rho^{(\mathrm{II})}(X, P) \equiv 1/2 \cdot \{ \sum_i \Lambda_\rho^\mu(X_i) P_\mu^i + H.C. \}. \tag{2·29}$$

The Hamiltonian in the (II)-representation cannot be obtained by the mere symmetrization of the classical result because of its quadratic dependence on momenta, but it differs only by some additional terms which are functions of the coordinates only. Thus the Hamiltonian in the (II)-representation is given as follows:

$$\begin{aligned} H^{(\mathrm{II})}(X, P; A, B) = {} & 1/2 \cdot [1/2M \cdot \sum_i g^{\mu\nu}(\alpha) P_\mu^i P_\nu^i \\ & + 1/2M \cdot \sum_i g^{\mu\nu}(\alpha) \cdot \partial A^\sigma(\bar{x}) / \partial x_i^\mu \cdot \partial A^\rho(\bar{x}) / \partial x_i^\nu \cdot B_\sigma B_\rho \\ & + \kappa/M \cdot \gamma^{\sigma\rho}(\alpha) B_\sigma \phi_\rho^{(\mathrm{II})}(X, P) + 1/M \cdot \sum_i g^{\mu\nu}(\alpha) A_\nu'^\sigma(x_i, \alpha) P_\mu^i B_\sigma + H.C.] \\ & + \hbar^2/M \cdot U(X, A) + V^{(\mathrm{II})}(X, A), \end{aligned} \tag{2·30}$$

where x_i^μ and α^σ should be written in terms of X_i^μ and A^σ. κ, $\gamma^{\sigma\rho}(\alpha)$ and $A_\nu'^\sigma(x_i, \alpha)$ are defined through the equations,

$$\partial A^\sigma(\bar{x}) / \partial x_i^\nu = \kappa \gamma^{\sigma\rho}(\alpha)\, g_{\nu\mu}(\alpha) \Lambda_\rho^\mu(x_i) + A_\nu'^\sigma(x_i, \alpha), \tag{2·31}$$

κ being a constant factor, $\gamma^{\sigma\rho}(\alpha) \xrightarrow[\alpha \to 0]{} \delta^{\sigma\rho}$ and $g_{\nu\mu}(\alpha) = \partial S^\lambda / \partial x_i^\nu \cdot \partial S^\lambda / \partial x_i^\mu$. $\hbar^2/M \cdot U(X, A)$ is the additional term resulting from commutators such as $[P, f(X, A)]$ and $[B, f(X, A)]$ ($f(X, A)$ being a function of X and A) and tends to zero in the limit $\alpha^\rho \to 0$. This term may be regarded as an additional potential due to a kinematical and quantum mechanical origin.** In (2·30), we can drop the third term owing to the subsidiary conditions (2·29), and if we can choose $A^\sigma(\bar{x})$ in such a way as $A_\nu'^\sigma(x_i, \alpha) = 0$, the coupling terms such as $P_\mu B_\sigma$ vanish exactly. Concerning this point we shall discuss in the next section.

Now let us consider the transformation of the total angular momentum L_T for the sake of later discussions. In the (P)-representation,

$$(L_T^{(\mathrm{P})})_{x,y,z} = (\sum_i \bar{\boldsymbol{x}}_i \times \bar{\boldsymbol{p}}^i)_{x,y,z} = -1/2 \cdot \{ \sum_i \Lambda_\xi^\mu(\bar{x}_i) \bar{p}_\mu^i + H.C. \} \tag{2·32}$$

where ξ takes (6, 7, 8) corresponding to (x, y, z). In the (I)-representation

$$\begin{aligned} (L_T^{(\mathrm{I})})_{x,y,z} = -1/2 \cdot \gamma^{\xi\rho}(\alpha) \{ \sum_i \Lambda_\rho^\nu(x_i) \dot{p}_\nu^i + H.C. \} &= \gamma^{\xi\rho}(\alpha) \pi_\rho(x, p) \\ &= \gamma^{\xi\rho}(\alpha) L_\rho(\alpha, \beta) \end{aligned} \tag{2·33}***$$

* By the use of (2·9) we can verify this fact.

** Hereafter the potential terms are abbreviated in one letter; for example, $\hbar^2/M \cdot U(X, A) + V^{(\mathrm{II})}(X, A)$ in (2·30) is rewritten as $V^{(\mathrm{II})}(X, A)$.

*** In deriving (2·33) we have used the relation $\Lambda_\sigma^\nu(\bar{x}_i) = \gamma^{\sigma\rho}(\alpha) \partial S^\nu / \partial x_i^\mu \cdot \Lambda_\rho^\mu(x_i)$ deduced from (2·31).

where

$$\pi_\rho(x, p) \equiv -1/2 \cdot \{\sum_i \Lambda_\rho^\nu(x_i)\, p_\nu^i + H.C.\}$$
$$L_\rho(\alpha, \beta) \equiv 1/2 \cdot \{G_\rho^\sigma(\alpha)\, \beta_\sigma + H.C.\}. \qquad (2\cdot34)$$

In the (II)-representation,

$$(L_T^{(\mathrm{II})})_{x,y,z} = -1/2 \cdot \{j^{-1\rho}(\alpha)\, \phi_\rho^{(\mathrm{II})}(X, P) + H.C.\}$$
$$-1/2\{j^{-1\rho}(\alpha) \sum_i \Lambda_\rho^\mu(x_i) \cdot \partial A^\sigma / \partial x_i^\mu \cdot B_\sigma + H.C.\}$$
$$= -1/2 \cdot \{\sum_i \Lambda_{\bar{\nu}}^\mu(\bar{x}_i) \cdot \partial A^\sigma(\bar{x}) / \partial \bar{x}_i^\mu \cdot B_\sigma + H.C.\}. \qquad (2\cdot35)$$

§ 3. Necessity of introducing the "internal" rotational mode

In this section we shall discuss the necessity of introducing the rotational mode and clarify the interrelations between this mode and the usual irrotational collective mode by investigating the characters of the commutation relations among subsidiary conditions.

In the description of the collective motion using redundant variables, it must be guaranteed exactly that subsidiary conditions satisfy the following relations:

$$[\phi_\sigma, \phi_\rho] = \text{a linear combination of } \phi_\tau.^{29)} \qquad (3\cdot1)$$

Firstly, we investigate the commutation relations (3·1) with the explicit expression of ϕ_ρ in the (II)-representation, though we can do this in any representation because of the invariance of commutators under the canonical transformation.

The subsidiary conditions in the (II)-representation are given by (2·28), (2·29) and (2·34):

$$\phi_\rho^{(\mathrm{II})}(X, P)\, \Psi^{(\mathrm{II})}(X, P; A, B) = 0,$$
$$\phi_\rho^{(\mathrm{II})}(X, P) = 1/2 \cdot \sum_i [\Lambda_\rho^\mu(X_i) P_\mu^i + H.C.] \equiv -\pi_\rho(X, P).$$

π_ρ have the following explicit forms

$$\left.\begin{aligned}
\pi_0(X, P) &= 1/2 \cdot \sum_i (X_i P_X^i + Y_i P_Y^i + Z_i P_Z^i + H.C.) \\
\pi_1(X, P) &= \sum_i (Y_i P_X^i + X_i P_Y^i) \\
\pi_2(X, P) &= \sum_i (Z_i P_Y^i + Y_i P_Z^i) \\
\pi_3(X, P) &= \sum_i (X_i P_X^i - Y_i P_Y^i) \\
\pi_4(X, P) &= \sum_i (Z_i P_X^i + X_i P_Z^i) \\
\pi_5(X, P) &= \sum_i (X_i P_X^i + Y_i P_Y^i - 2Z_i P_Z^i) \\
\pi_6(X, P) &= \sum_i (Y_i P_Z^i - Z_i P_Y^i) = \sum_i (\boldsymbol{X}_i \times \boldsymbol{P}^i)_X \\
\pi_7(X, P) &= \sum_i (Z_i P_X^i - X_i P_Z^i) = \sum_i (\boldsymbol{X}_i \times \boldsymbol{P}^i)_Y \\
\pi_8(X, P) &= \sum_i (X_i P_Y^i - Y_i P_X^i) = \sum_i (\boldsymbol{X}_i \times \boldsymbol{P}^i)_Z .
\end{aligned}\right\} \qquad (3\cdot2)$$

From (3·2) we obtain

$$[\pi_\sigma(X, P),\ \pi_\rho(X, P)]=i\hbar C_{\sigma\rho}^{\tau}\pi_\tau(X, P), \tag{3·3}$$

hence (3·1) is satisfied, that is,

$$[\phi_\sigma^{(\mathrm{II})}(X, P),\ \phi_\rho^{(\mathrm{II})}(X, P)]=-i\hbar\, C_{\sigma\rho}^{\tau}\phi_\tau^{(\mathrm{II})}(X, P) \tag{3·4}$$

where $C_{\sigma\rho}^{\tau}$ are the pure constant numbers shown in the Appendix. Transforming the relation (3·4) back into that of the (I)-representation, we obtain

$$[\phi_\sigma^{(\mathrm{I})}(x, p\,;\alpha, \beta),\ \phi_\rho^{(\mathrm{I})}(x, p\,;\alpha, \beta)]=-i\hbar\, C_{\sigma\rho}^{\tau}\phi_\tau^{(\mathrm{I})}(x, p\,;\alpha, \beta) \tag{3·5}$$

where

$$\phi_\rho^{(\mathrm{I})}(x, p\,;\alpha, \beta)=-\pi_\rho(x, p)+L_\rho(\alpha, \beta)$$

by the definition (2·34). Because $\pi_\rho(x, p)$ have the same structure as $\pi_\rho(X, P)$, and $L_\rho(\alpha, \beta)$ are the functions of the collective variables only, we have

$$\left.\begin{aligned}&[\pi_\sigma(x, p),\ \pi_\rho(x, p)]=i\hbar\, C_{\sigma\rho}^{\tau}\pi_\tau(x, p)\\&[L_\sigma(\alpha, \beta),\ L_\rho(\alpha, \beta)]=-i\hbar\, C_{\sigma\rho}^{\tau}L_\tau(\alpha, \beta).\end{aligned}\right\} \tag{3·6}$$

The commutators of π_ρ, L_ρ and ϕ_ρ have the following characters, as seen from the table of $C_{\sigma\rho}^{\tau}$ given in the Appendix:

For any ρ, $[\pi_0, \pi_\rho]=0$. (3·7a)

For σ, $\rho=1, \cdots, 5$ and $\tau=6, 7, 8$,

$$[\pi_\sigma, \pi_\rho]=\text{a linear combination of } \pi_\tau. \tag{3·7b}$$

For $\sigma=1, \cdots, 5$, $\rho=6, 7, 8$ and $\tau=1, \cdots, 5$,

$$[\pi_\sigma, \pi_\rho]=\text{a linear combination of } \pi_\tau. \tag{3·7c}$$

For the case that (σ, ρ, τ) is a cyclic permutation of $(6, 7, 8)$,

$$[\pi_\sigma, \pi_\rho]=i\hbar\,\pi_\tau. \tag{3·7d}$$

Thus from the above relations, the main features of the interrelation between collective modes can be concisely stated as follows:

i) The mode of compressible motion ($\rho=0$) can be discussed separately from the remaining modes, because the relation (3·1) is retained even in the case of incompressible motion ($\rho=1, 2, \cdots, 8$).

ii) For the case of incompressible and irrotational motion ($\rho=1, \cdots, 5$) the relation (3·1) is not assured, and the introduction of the "internal" rotational collective modes is indispensable in describing the collective motion in a closed and consistent form.

iii) The "internal" rotational part ($\rho=6, 7, 8$) has the same commutation relations as an angular momentum operator, reflecting the fact that it is the antisymmetric part of the transformation matrix in (2·3).

The essential point lies in the character ii). If we take up only the irrotational modes ($\rho=0, \cdots, 5$) as collective modes in a similar manner to the usual treatment of collective motion, it can be seen from (3·7b) that we encounter the inconsistency that the number of independent subsidiary conditions exceeds the number of the redundant variables introduced previously.* The non-commutability between the irrotational collective momenta, $\pi_\rho(x, p)$ ($\rho=1, 2, \cdots, 5$),** in Tomonaga's method[21] is based just on the character ii) and in the usual method using the redudant variables the same circumstance prevents us from describing the collective motion in a closed form.[24]

Next we consider the choice of $A^\sigma(\bar{x})$, on which the transformation to the (II)-representation essentially depends. As seen from (2·30) and (2·31), in order to make $P_\mu B_\sigma$-terms vanish, $A^\sigma(\bar{x})$ must satisfy

$$\partial A^\sigma(\bar{x})/\partial \bar{x}_i^\mu = \kappa \Lambda_\sigma^\mu(\bar{x}_i).$$

For $\sigma=0, 1, \cdots, 5$, we can easily find the solutions:

$$A^\sigma(\bar{x}) = \begin{vmatrix} -\kappa/2\cdot\sum_i(\bar{x}_i^2+\bar{y}_i^2+\bar{z}_i^2) \\ -\kappa\sum_i \bar{x}_i\bar{y}_i \\ -\kappa\sum_i \bar{y}_i\bar{z}_i \\ -\kappa'/2\cdot\sum_i(\bar{x}_i^2-\bar{y}_i^2) \\ -\kappa\sum_i \bar{x}_i\bar{z}_i \\ -\kappa'/2\cdot\sum_i(\bar{x}_i^2+\bar{y}_i^2-2\bar{z}_i^2) \end{vmatrix}, \qquad (3\cdot8)$$

which represent the velocity potentials of the irrotational flow. On the other hand, such velocity potentials do not exist for $\sigma=6, 7, 8$ because of their rotational character. Therefore we have $\Lambda_\mu'^\sigma \neq 0$ ($\sigma=6, 7, 8$) and consequently there remain the coupling terms between the particle and the collective momenta.

To find the conditions fora reasonable choice cf $A^\rho(\bar{x})$ ($\rho=6, 7, 8$), we consider the total angular momentum in the (II)-representation. If we denote hereafter the irrotational part ($\sigma=0, 1, \cdots, 5$) by σ-suffix and the rotational part ($\rho=6, 7, 8$) by ρ-suffix, from (2·35) we obtain

$$(L_T^{(II)})_{x,y,z} = 1/2\cdot\Big[-\kappa\sum_i \Lambda_\xi^\mu(\bar{x}_i)\Lambda_\sigma^\mu(\bar{x}_i)B_\sigma - (i\hbar)^{-1}[\pi_\xi(\bar{x}, \bar{p}), A^\rho(\bar{x})]B_\rho + H.C.\Big]. \qquad (3\cdot9)$$

If we take $A^\rho(\bar{x})$ as

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* Mathematically, this corresponds to the fact that the symmetric part of the transformation matrix (2·3) alone cannot construct a Lie's transformation group as suggested by Nataf.[12]

** In Tomonaga's method[21] where the redundant variables are not used, $\pi_\rho(x, p)$ ($\rho=1, \cdots, 5$) are adopted as the irrotational collective momenta.

$$[\pi_\rho(\bar{x}, \bar{p}), A^\rho(\bar{x})] = i\hbar,^* \tag{3·10}$$

we can regard B_ρ as the angular momenta of the rotational collective motion.

Since $A^\sigma(\bar{x})$ can be written as (3·8), $A'^\sigma_\mu = 0$ and we finally obtain

$$\begin{aligned} H^{(\mathrm{II})}(X, P; A, B) = 1/2 \cdot [& 1/2M \cdot \sum_i g^{\mu\nu}(\alpha) P^i_\mu P^i_\nu + \kappa^2/2M \cdot \sum_i \Lambda^\mu_\sigma(\bar{x}_i) \Lambda^\mu_{\sigma'}(\bar{x}_i) B_\sigma B_{\sigma'} \\ & + 1/2M \cdot \sum_i \partial A^\rho(\bar{x}) / \partial \bar{x}^\mu_i \cdot \partial A^{\rho'}(\bar{x}) / \partial \bar{x}^\mu_i \cdot B_\rho B_{\rho'} \\ & + \kappa/M \cdot \sum_i \Lambda^\mu_\sigma(\bar{x}_i) \partial A^\rho(\bar{x}) / \partial \bar{x}^\mu_i \cdot B_\sigma B_\rho \\ & + 1/M \cdot \sum_i g^{\mu\nu}(\alpha) A'^\rho_\mu(x_i, \alpha) P^i_\nu B_\rho + H.C.] + V^{(\mathrm{II})}(X, A) \end{aligned} \tag{3·11}$$

with the subsidiary conditions

$$\phi^{(\mathrm{II})}_\rho(X, P) \Psi^{(\mathrm{II})}(X, P; A, B) = 0.$$

In $H^{(\mathrm{II})}$, the first term represents the kinetic energy of the individual particle motion and the second and the third terms are the kinetic energy of irrotational and rotational collective motions respectively. The fourth term represents the coupling term between irrotational and rotational collective motions. It is to be emphasized that only the coupling term between rotational collective and individual particle momenta, i.e. the fifth term, remains, and the coupling term between irrotational collective and individual particle momenta does not appear.

§ 4. Comparison with various methods concerning the moments of inertia

In the previous section, we have shown that the rotational collective motion must be taken into consideration on an equal footing with the irrotational collective motion and that the Hamiltonian (3·11) containing its contribution must be employed. This contribution is coupled with the usual "surface" rotation and the rotational levels caused only by the surface motion may be shifted so that the values of moments of inertia are

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* Here the summation with that dummy suffix is not implied.

$A^\rho(\bar{x})$ determined by (3·10) correspond to the rotation angles with negative sign. Therefore, the following expressions may be obtained, e.g.

$$A^\delta(\bar{x}) = -1/2 \cdot \tan^{-1}\{2\sum_i \bar{x}_i \bar{y}_i / \sum_i (\bar{x}_i^2 - \bar{y}_i^2)\},\ -1/N \cdot \sum_i \tan^{-1}(\bar{y}_i/\bar{x}_i),\ -\sum_i (\bar{x}_i^2 + \bar{y}_i^2) \tan^{-1}(\bar{y}_i/\bar{x}_i) / \sum_i (\bar{x}_i^2 + \bar{y}_i^2)$$

and so forth. The first and second type, however, are not appropriate, because the former is not independent of (A^1, A^3) and the latter leads to a singularity in the velocity field. Therefore, A^ρ such as the third type may be allowable and in this case

$$\begin{cases} \partial A^\delta / \partial \bar{x}_i = (1/\sum_i (\bar{x}_i^2 + \bar{y}_i^2)) \{\bar{y}_i - 2\bar{x}_i (\tan^{-1}(\bar{y}_i/\bar{x}_i) + A^\delta(\bar{x}))\} \\ \partial A^\delta / \partial \bar{y}_i = (1/\sum_i (\bar{x}_i^2 + \bar{y}_i^2)) \{-\bar{x}_i - 2\bar{y}_i (\tan^{-1}(\bar{y}_i/\bar{x}_i) + A^\delta(\bar{x}))\} \\ \partial A^\delta / \partial \bar{z}_i = 0 \end{cases}$$

where the first terms correspond to $\Lambda_8{}^\mu$.

altered effectively. This circumstance serves to explain the difference between the empirical value and the hydrodynamical value. In fact, various methods or models concerning the moments of inertia can be discussed on the basis of our method, i.e., it will be shown that the "internal" rotational collective motion discussed above is just what other authors have considered. In this section it will be made clear in what representations the "internal" rotational collective motion is treated and what kind of approximations are made in other authors' works.

We can classify these methods into two different types in treating the "internal" rotational collective motion: The one is to define the collective variables for the "internal" rotational mode explicitly and the other to treat it implicitly. We shall first discuss the former treatment by taking up Tomonaga's method[10] as a typical one. Then we proceed to discuss the latter which corresponds to the usual method taking up only the irrotational mode as the collective motion. Finally, the cranking model of Inglis will be discussed briefly. To clarify the physical picture, discussions will be confined to the case of the two dimensional nucleus. The extention to the three dimensional case does not involve any fundamental difficulty.

(I) *Comparison with Tomonaga's method* [10]

First we develop our method in a form with which the equivalence to the former treatment can be discussed.

In the case of the two dimensional nucleus, the point transformation takes the following form:

$$\begin{pmatrix} \bar{x}_i \\ \bar{y}_i \end{pmatrix} = \begin{pmatrix} 1+\alpha^0+\alpha^3 & \alpha^1-\alpha^2 \\ \alpha^1+\alpha^2 & 1+\alpha^0-\alpha^3 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \equiv \begin{pmatrix} S^x(x,\alpha) \\ S^y(x,\alpha) \end{pmatrix}, \tag{4·1}$$

and the various quantities defined in the preceding sections are expressed as follows:

$$\left(A^\nu_\mu(x_i)\right) = \begin{pmatrix} -x_i & -y_i & y_i & -x_i \\ -y_i & -x_i & -x_i & y_i \end{pmatrix} \tag{4·2}$$

$$\left(G^\sigma_\rho(\alpha)\right) = \begin{matrix} \rho \downarrow \\ \end{matrix} \begin{pmatrix} \overset{\sigma \rightarrow}{1+\alpha^0} & \alpha^1 & \alpha^2 & \alpha^3 \\ \alpha^1 & 1+\alpha^0 & -\alpha^3 & -\alpha^2 \\ -\alpha^2 & -\alpha^3 & 1+\alpha^0 & \alpha^1 \\ \alpha^3 & \alpha^2 & \alpha^1 & 1+\alpha^0 \end{pmatrix} \tag{4·3}$$

$$\left(A^\nu(\bar{x})\right) = \begin{pmatrix} -\kappa/2\cdot\sum_i(\bar{x}_i^2+\bar{y}_i^2) \\ -\kappa\sum_i \bar{x}_i\bar{y}_i \\ A^2(\bar{x}) \\ -\kappa/2\cdot\sum_i(\bar{x}_i^2-\bar{y}_i^2) \end{pmatrix}. \tag{4·4}$$

We shall treat only the incompressible motion by imposing the condition that the determinant of the transformation matrix in (4·1) is unity:

$$\det\{\alpha\}=1.$$

This is possible because the incompressible motion is closed by itself; that is, the relations (3·1) are satisfied as was shown in § 3.

The Hamiltonian in the (II)-representation is

$$\begin{aligned}H^{(\mathrm{II})}=&1/2\cdot[1/2M\cdot\sum_i g^{\mu\nu}(\alpha)P_\mu^i P_\nu^i+\kappa^2/2M\cdot\sum_i(\bar{x}_i^2+\bar{y}_i^2)\cdot(B_1^2+B_3^2)\\&-\kappa/M\cdot\sum_i(\bar{x}_i\partial A^2(\bar{x})/\partial\bar{y}_i+\bar{y}_i\partial A^2(\bar{x})/\partial\bar{x}_i)B_1B_2\\&-\kappa/M\cdot\sum_i(\bar{x}_i\partial A^2(\bar{x})/\partial\bar{x}_i-\bar{y}_i\partial A^2(\bar{x})/\partial\bar{y}_i)B_3B_2\\&+1/2M\cdot\sum_i\partial A^2(\bar{x})/\partial\bar{x}_i^\mu\cdot\partial A^2(\bar{x})/\partial\bar{x}_i^\mu\cdot B_2^2\\&+1/M\cdot\sum_i g^{\mu\nu}(\alpha)\partial A^2(\bar{x})/\partial x_i^\mu\cdot P_\nu^i B_2+H.C.]\\&+V^{(\mathrm{II})}(X,A)\end{aligned}\tag{4·5a}$$

with the subsidiary conditions

$$\sum_i A_\rho^\mu(X_i)P_\mu^i\Psi^{(\mathrm{II})}=0\quad(\rho=1,2,3),\tag{4·5b}$$

where the old variables are to be expressed in terms of the new ones. The total angular momentum is

$$L_T^{(\mathrm{II})}=2(A^3B_1-A^1B_3)-B_2,\tag{4·5c}$$

when $A^2(\bar{x})$ is chosen in such a way as to satisfy the relation (3·10).

In order to make comparison with Tomonaga's method, we decompose the irrotational collective motion into the "surface" rotational and the vibrational modes. This decomposition is performed by the canonical transformation corresponding to a coordinate transformation

$$\begin{aligned}&X_i^\mu=X_i^\mu\qquad\vartheta(A)=\vartheta\\&\varphi(A)=\varphi\qquad\beta(A)=\beta\end{aligned}$$

generated by

$$W=\sum_i X_i^\mu P_\mu^i+\varphi(A)\mathcal{L}_\varphi+\vartheta(A)\mathcal{L}_\vartheta+\beta(A)\mathcal{P}_{\mathrm{vib}},\tag{4·6}$$

where

$$\varphi(A)\equiv 1/2\tan^{-1}(A^1/A^3),\quad\vartheta(A)\equiv-A^2,\quad\beta(A)\equiv 2\sqrt{(A^1)^2+(A^3)^2}.$$

The Hamiltonian $H^{(\mathrm{III})}$ obtained after the decomposition is

$$\begin{aligned}H^{(\mathrm{III})}=&1/2\cdot[1/2M\cdot\sum_i g^{\mu\nu}(\alpha)P_\mu^i P_\nu^i+\kappa^2/2M\cdot\sum_i(\bar{x}_i^2+\bar{y}_i^2)/\beta^2\cdot\mathcal{L}_\varphi^2\\&-2\kappa/M\beta^2\cdot\{\sum_i(\bar{y}_i\partial A^2(\bar{x})/\partial\bar{x}_i+\bar{x}_i\partial A^2(\bar{x})/\partial\bar{y}_i)A^3\end{aligned}$$

$$-\sum_i (\bar{x}_i \partial A^2(\bar{x})/\partial \bar{x}_i - \bar{y}_i \partial A^2(\bar{x})/\partial \bar{y}_i) A^1\} \mathcal{L}_\varphi \mathcal{L}_\vartheta$$
$$+1/2M \cdot \sum_i \partial A^2(\bar{x})/\partial \bar{x}_i^\mu \cdot \partial A^2(\bar{x})/\partial \bar{x}_i^\mu \cdot \mathcal{L}_\vartheta^2 - 1/M \cdot \sum_i g^{\mu\nu}(\alpha) \partial A'^2/\partial x_i^\mu \cdot P_\nu^i \mathcal{L}_\vartheta$$
$$+ \textit{vibrational terms} + H.C.] + V^{(III)}(X, \varphi, \vartheta, \beta) \qquad (4\cdot 7a)$$

with the subsidiary conditions:

$$\sum_i \Lambda_\nu^\mu(X_i) P_\mu^i \Psi^{(III)} = 0, \qquad (4\cdot 7b)$$

and the total angular momentum is

$$L_T^{(III)} = \mathcal{L}_\varphi + \mathcal{L}_\vartheta. \qquad (4\cdot 7c)$$

In these expressions $\mathcal{L}_\varphi$ is the so-called angular momentum of the "surface" rotation and the second term in (4·7a) is its kinetic energy, and

$$\mathcal{J}_\varphi \equiv \{\kappa^2 \sum_i (\bar{x}_i^2 + \bar{y}_i^2)/M\beta^2\}^{-1} \quad (\kappa = 1/N\langle \bar{x}_i^2 + \bar{y}_i^2 \rangle)^* \qquad (4\cdot 8)$$

is just the moment of inertia for the "surface" rotation, representing the hydrodynamical value given by Bohr-Mottelson. $\mathcal{L}_\vartheta$ is the angular momentum of the "internal" rotational collective motion and the fourth term in (4·7a) is its kinetic energy, and

$$\mathcal{J}_\vartheta \equiv (1/M \cdot \sum_i \partial A^2(\bar{x})/\partial \bar{x}_i^\mu \cdot \partial A^2(\bar{x})/\partial \bar{x}_i^\mu)^{-1} \qquad (4\cdot 9)$$

is the moment of inertia for the "internal" rotation. The third term in (4·7a) represents the coupling between the momenta of the "surface" and the "internal" rotations and the fifth term is that between the "internal" rotation and the individual particle motion. As we are interested in the rotational states, we shall omit the vibrational terms in the following discussions.

Now, if we can take an adequate $A^2(\bar{x})$ so that the above mentioned coupling terms may be negligible, the Hamiltonian (4·7a) is reduced to a simple form:

$$H^{(III')} = 1/2 \cdot [1/2\, M \cdot \sum_i g^{\mu\nu}(\alpha) P_\mu^i P_\nu^i + 1/2 \mathcal{J}_\varphi \cdot \mathcal{L}_\varphi^2 + 1/2 \mathcal{J}_\vartheta \cdot \mathcal{L}_\vartheta^2 + H.C.] + V^{(III')}. \qquad (4\cdot 10)$$

This Hamiltonian is nothing but the one used by Tomonaga. By physical insight he introduced certain canonical variables $(\vartheta, \mathcal{L}_\vartheta)$ which describe the "internal" rotational collective motion. Then as the starting point of his model, he assumed the Hamiltonian in the form of (4·10) and supposed the coupling between both modes of the collective motion through $V^{(III)}$ to be strong. In an analogous way to the separation of the center of mass motion, the Hamiltonian (4·10) is rewritten in the following form:

$$H^{(III'')} = 1/2 \cdot [1/2M \cdot \sum_i g^{\mu\nu}(\alpha) P_\mu^i P_\nu^i + 1/2 (\mathcal{J}_\varphi + \mathcal{J}_\vartheta) \cdot L_T^2$$
$$+ (\mathcal{J}_\varphi + \mathcal{J}_\vartheta)/2 \mathcal{J}_\varphi \mathcal{J}_\vartheta \cdot \{\mathcal{J}_\vartheta/(\mathcal{J}_\varphi + \mathcal{J}_\vartheta) \cdot \mathcal{L}_\varphi - \mathcal{J}_\varphi/(\mathcal{J}_\varphi + \mathcal{J}_\vartheta) \cdot \mathcal{L}_\vartheta\}^2$$
$$+ H.C.] + V^{(III'')}(X, \beta, (\vartheta - \varphi)). \qquad (4\cdot 11)$$

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* κ for the irrotational motion is determined so as to satisfy $[\pi_\rho(\bar{x}, \bar{p}), A'^\rho(\bar{x})] = i\hbar$ in the large number approximation. Here the summation with that dummy suffix is not implied. On this point see reference 21).

Here the third term expresses the relative motion of both of the rotations. By the strong coupling assumption the terms concerning the relative motion are treated as small perturbations. The effective moment of inertia is given by $\mathcal{J}_\varphi + \mathcal{J}_\vartheta$ and it may fit the empirical value.

From our point of view, Tomonag's model is obtained by choosing a suitable $A^2(\bar{x})$ and adopting the (II)-representation and, further, assuming the strong coupling between both rotational modes.

(II) *Comparison with the methods based on the irrotational collective model*

Now we shall proceed to discuss the method where the "internal" rotatinal motion is not treated explicitly as any of the collective modes but is described in terms of the particle variables. In many methods[12),13),15)] concerning moments of inertia the irrotational collective modes are described with the aid of the collective variables while the "internal" rotational ones are taken into consideration as such as the Coriolis effect due to the individual particle motion. In order to facilitate the comparison of our method with these methods, we shall first transform the (I)-representation into that of these methods. This procedure is performed by the canonical transformation corresponding to a coordinate transformation

$$x_i^\mu = X_i^\mu, \qquad A^\sigma(\bar{x}) = A^\sigma, \qquad \alpha^2 = A^2$$

generated by

$$W = \sum_i x_i^\mu P_\mu^i + A^\sigma(\bar{x}) B_\sigma + \alpha^2 B_2 , \tag{4·12}$$

where and hereafter σ stands for only 1 or 3 corresponding to the irrotational modes. We call this represention "the usual representation" or (U). The Hamiltonian H is

$$H^{(\mathrm{U})} = \frac{1}{2}\Bigg[\frac{1}{2M}\cdot\sum_i g^{\mu\nu}(\alpha) P_\mu^i P_\nu^i + \frac{1}{2M}\cdot\sum_i \frac{\partial A^\sigma(\bar{x})}{\partial \bar{x}_i^\mu}\cdot\frac{\partial A^{\sigma'}(\bar{x})}{\partial \bar{x}_i^\mu}\cdot B_\sigma B_{\sigma'}$$

$$+\frac{1}{M}\cdot\sum_i g^{\mu\nu}(\alpha)\frac{\partial A^\sigma(\bar{x})}{\partial x_i^\mu}\cdot P_\nu^i B_\sigma + \mathrm{H.C.}\Bigg] + V^{(\mathrm{U})}(X, A) \tag{4·13a}$$

with the subsidiary conditions

$$1/2\cdot\{\sum_i \Lambda_\nu^\mu(X_i) P_\mu^i + G_\rho^\nu(\alpha) B_2 + \mathrm{H.C.}\}\Psi^{(\mathrm{U})} = 0 \qquad (\rho = 1, 2, 3). \tag{4·13b}$$

In this representation, the terms responsible for the "internal" rotational collective motion do not appear explicitly in the Hamiltonian (4·13a). In order to compare with the usual treatment where the deformation parameters α^σ are taken up to the first order, we shall restrict ourselves up to the first order in α^ρ $(\rho = 1, 2, 3)$ in the inverse transformation to (4·1). The first order approximation in the rotational parameter α^2 would be allowable provided that the coupling between the irrotational and the rotational modes of motion are strong. Then $G_\rho^\sigma(\alpha)$ tend to δ_ρ^σ and the subsidiary conditions (4·13b) are reduced to

$$\sum_i \Lambda_\sigma^\mu(X_i) P_\mu^i \Psi^{(\mathrm{U})} = 0 \qquad (\sigma = 1, 3) \tag{4·14a}$$

$$\{B_2-\sum_i(X_iP_Y^i-Y_iP_X^i)\}\Psi^{(U)}=0. \tag{4·14b}$$

In this case the irrotational collective modes can be treated independently of the particle variables and the rotational collective mode is related to the particle variables through (4·14b). The (U)-representation in this approximation is nothing but what has been used in the usual methods.

However, it should be noted that in this approximation the inconsistency in the usual treatment of the collective motion appears just as we have discussed in § 1 and § 3. The subsidiary conditions (4·14a) and (4·14b) do not satisfy the conditions (3·1). To clarify this circumstance, let us consider the same inconsistency in the (I)-representation. The subsidiary conditions in the above mentioned approximation become as follows:

$$\{\sum_i \Lambda_\rho^\mu(x_i)\,p_\mu^i+{}_i\beta_\rho\}\Psi^{(I)}=\{{}_i\beta_\rho-\pi_\rho(x,\,p)\}\Psi^{(I)}=0 \qquad (\rho=1,\,2,\,3).$$

While $[{}_i\beta_\rho,\,{}_i\beta_{\rho'}]=0$ by the definition of ${}_i\beta_\rho$, $[\pi_\rho(x,\,p),\,\pi_{\rho'}(x,\,p)]\neq 0$. This corresponds to the same circumstance appeared in other authors' works.[12),24)] In their works, however, this inconsistency does not appear explicitly but in other aspects, because the subsidiary conditions are adopted in a form of the functional relation between the particle and the collective coordinates. In spite of this unsatisfactory point we shall treat the Hamiltonian (4·13a) under the approximation discussed above, for we wish to investigate what is taken into consideration in their method.

Separating the kinetic energy of the collective motion into those of the "surface" rotation and the vibration in an analogous way to (4·6), we obtain the following Hamiltonian:

$$\begin{aligned}H^{(U')}=\frac{1}{2}\Big[&\frac{1}{2M}\cdot\sum_i g^{\mu\nu}(\alpha)\,P_\mu^iP_\nu^i+\frac{\kappa^2\sum(\bar{x}_i^{\,2}+\bar{y}_i^{\,2})}{2M\beta^2}\cdot\mathcal{L}_\varphi^{\,2}\\&+\frac{1}{M}\cdot\sum_i g^{\mu\nu}(\alpha)\frac{\partial A^\sigma(\bar{x})}{\partial x_i^\mu}\cdot\frac{\partial\varphi(A)}{\partial A^\sigma}\cdot P_\nu^i\mathcal{L}_\varphi\\&+\textit{vibrational terms}+H.C.\Big]+V^{(U')}(X,\,\varphi,\,\vartheta,\,\beta)\end{aligned} \tag{4·15a}$$

with the subsidiary conditions:

$$\sum_i \Lambda_\sigma^\mu(X_i)\,P_\mu^i\Psi^{(U')}=0, \tag{4·15b}$$

$$\{\mathcal{L}_\vartheta-\sum_i(X_iP_Y^i-Y_iP_X^i)\}\Psi^{(U')}=0 \tag{4·15b'}$$

and the total angular momentum is

$$L_T^{(U')}=\mathcal{L}_\vartheta+\mathcal{L}_\varphi. \tag{4·15c}$$

The third term in the Hamiltonian (4·15a) always appears in the usual methods as coupling term between the "irrotational" collective motion and the individual particle motion, and it has been pointed out frequently[5),12),25)] that this term is by no means small compared with the kinetic energy terms of the irrotational collective motion. To investigate this circumstance, let us rewrite this term by using the large number approxi-

mation[21] and taking (4·15b) into account:

$$\frac{1}{M}\cdot\sum_i g^{\mu\nu}(\alpha)\frac{\partial A^\sigma(\vec{x})}{\partial x_i^\mu}\cdot\frac{\partial\varphi(A)}{\partial A^\sigma}\cdot P_\nu^i\mathcal{L}_\varphi\simeq\frac{\kappa}{M}\cdot\sum_i(X_iP_Y^i-Y_iP_X^i)\cdot\mathcal{L}_\varphi. \tag{4·16}$$

Taking (4·15b′) into account, it can be recognized that the net contribution of the individual particle momenta in the coupling term is the same as that of the "internal" collective rotation, and so the coupling term between the irrotational collective and the individual particle motions in the (U)-representation is just the one between the irrotational and the rotational collective motions.

Let us further make the canonical transformation corresponding to a coordinate transformation

$$X_i^\mu=X_i^\mu \qquad \theta(\varphi,\vartheta)=\theta$$
$$\phi(\varphi)=\phi \qquad {}_i\beta={}_i\beta$$

generated by

$$W=\sum_i X_i^\mu P_\mu^i+\phi(\varphi)L_1+\theta(\varphi,\vartheta)L_2+\beta P_{\text{vib}}, \tag{4·17}$$

where

$$\phi(\varphi)\equiv\varphi, \qquad \theta(\varphi,\vartheta)\equiv\vartheta-\varphi.$$

Then the Hamiltonian (4·15a) becomes

$$\begin{aligned}H^{(\text{U}'')}=\frac{1}{2}\Bigg[&\frac{1}{2M}\cdot\sum_i g^{\mu\nu}(\alpha)P_\mu^iP_\nu^i+\frac{1}{2\mathcal{J}_\varphi}\cdot(L_1-L_2)^2\\&+\frac{1}{M}\cdot\sum_i g^{\mu\nu}(\alpha)\frac{\partial A^\sigma(\vec{x})}{\partial x_i^\mu}\cdot\frac{\partial\varphi(A)}{\partial A^\sigma}\cdot P_\nu^i(L_1-L_2)\\&+\textit{vibrational terms}+H.C.\Bigg]+V^{(\text{U}'')}(X,\phi,\theta,\beta)\end{aligned} \tag{4·18a}$$

with the subsidiary conditions

$$\begin{aligned}&\sum_i A_\sigma^\mu(X_i)P_\mu^i\Psi^{(\text{U}'')}=0\\&\{L_2-\sum_i(X_iP_Y^i-Y_iP_X^i)\}\Psi^{(\text{U}'')}=0,\end{aligned} \tag{4·18b}$$

and the total angular momentum is

$$L_T^{(\text{U}'')}=L_1. \tag{4·18c}$$

In this representation the angle ϕ canonical conjugate to L_1, the total angular momentum, is interpreted as the angle of the principal axis of the deformed nucleus. On the other hand, L_2 represents the relative angular momentum, θ corresponding to the relative angle. It should be noted that L_2 is related to the particle variables through (4·18b). In most of the methods concerning moment of inertia the coordinate system is transformed into the rotating system moving with the principal axis of the deformed nucleus, and relative "internal" rotation is treated in this coordinate system.[12),13),15),25)] In fact, when

L_2 is rewritten in terms of the individual particle variables by the use of the subsidiary conditions (4·18b), we find that our Hamiltonian (4·18a) is just the one used in these methods.

Thus we can conclude that while Tomonaga's method explicitly takes account of the "internal" rotation as a collective mode in our (II)-representation, the usual methods take the same effect into consideration implicitly in terms of the particle variables in our (U)-representation.

(III) *Comparison with the cranking model of Inglis*

Finally we shall briefly refer to the cranking model of Inglis. The third term in the Hamiltonian $H^{(\mathrm{U}'')}$, (4·18a), can be written down in the same form as (4·16):

$$\frac{1}{M}\cdot\sum_i g^{\mu\nu}(\alpha)\frac{\partial A^\sigma(\bar{x})}{\partial x_i^\mu}\cdot\frac{\partial\varphi(A)}{\partial A^\sigma}\cdot P_\nu^i(L_1-L_2)\simeq\frac{\kappa}{M}\cdot(L_1-L_2)\sum_i(X_iP_Y^i-Y_iP_X^i)\,. \tag{4·19}$$

In order to obtain the correspondence to the cranking model we treat the irrotational collective motion classically and define the classical angular velocity $\Omega\equiv\kappa/M\cdot\langle L_1-L_2\rangle$. Then the Hamiltonian can be written as follows,

$$\begin{aligned}\langle H^{(\mathrm{U}'')}\rangle=&\Big\langle\frac{1}{2}\Big[\frac{1}{2M}\cdot\sum_i g^{\mu\nu}(\alpha)P_\mu^iP_\nu^i+\frac{1}{2\mathcal{J}_\sigma}(L_1-L_2)^2+\textit{vib. terms}\\&+H.C.\Big]+V^{(\mathrm{U}'')}\Big\rangle+\Omega\sum_i(X_iP_Y^i-Y_iP_X^i)\\\equiv&H_0(X,P)+\Omega\sum(X_iP_Y^i-Y_iP_X^i)\end{aligned} \tag{4·20}$$

where $\langle\ \ \rangle$ denotes the mean value taken only over collective variables. This Hamiltonian is simply the one used by Inglis.

Taking account of the subsidiary dondition (4·18b), it can be seen that the main perturbation term in the cranking model is nothing but the effect due to the rotational collective mode. On the other hand the effects of the irrotational collective motion are considered as those given externally; the surface rotation is "cranked" and the surface vibration corresponds to the artificially deformed potential.

§ 5. Concluding remarks

Basing upon the analyses of the unsatisfactory points in the usual irrotational collective model, we have considered that there must exist a kind of the rotational collective motion to be coupled with the usual irrotational collective motion. According to this consideration, we have generalized the usual method with redundant variables by using Watanabe's method and have shown that the introduction of the rotational collective mode is indispensable in describing the collective motion in a closed consistent form. Thus we have confirmed the expectation that this rotational mode would play an important role in explaining the discrepancy between the experimental and theoretical values of moments of inertia. In fact, as was shown in § 4, this mode is taken into account in

various forms in the models concerning the moments of inertia which were contrived by many authors recently. Our method which is the generalized one as a collective description with redundant variables has thus given the foundation of those models.

In conclusion, it would be important to discuss the limit of applicability of our theory. Our method as well as the usual ones describing the collective motion is powerless against the question why the specific collective modes of motion can arise in a specific energy region from the actual nuclear motion. Of course this fundamental problem depends essentially on the nature of interactions between nucleons. In this sense our theory is considered as a phenomenological one.

In fact, such limit of our method has appeared already in the following forms. Although we extensively introduced the collective modes by a linear point transformation under the condition that the usual irrotational modes can be included in it, such a transformation merely reflects the "kinematical" nature of the system. Therefore, we can say nothing about the interaction potential between nucleons. In addition, as we have seen in § 3, we cannot find a unique choice of the functional form of $A^{\rho}(\bar{x})$ ($\rho = 6, 7, 8$) in our theory. In the next stage of the study of collective model, it would be necessary to discuss this problem by keeping the close contact with the interaction potential between nucleons.

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Appendix

(I) $G_{\rho}^{\sigma}(\alpha)$ are given in the following matrix:

$$(G_{\rho}^{\sigma}(\alpha)) = \begin{array}{c} \sigma\rightarrow \\ \rho\downarrow \end{array} \begin{pmatrix} 1+\alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 \\ \frac{2}{3}\alpha^1 & 1+\alpha^0+\alpha^5 & (\alpha^4-\alpha^7)/2 & -\alpha^8 \\ \frac{2}{3}\alpha^2 & (\alpha^4+\alpha^7)/2 & 1+\alpha^0-\dfrac{\alpha^3+\alpha^5}{2} & -(\alpha^2-\alpha^6)/2 \\ \frac{2}{3}\alpha^3 & \alpha^8 & -(\alpha^2+\alpha^6)/2 & 1+\alpha^0+\alpha^5 \\ \frac{2}{3}\alpha^4 & (\alpha^2-\alpha^6)/2 & (\alpha^1+\alpha^8)/2 & (\alpha^4+\alpha^7)/2 \\ 2\alpha^5 & \alpha^1 & -(\alpha^2-3\alpha^6)/2 & \alpha^3 \\ -\frac{2}{3}\alpha^6 & (\alpha^4+\alpha^7)/2 & (\alpha^3-3\alpha^5)/2 & -(\alpha^2-\alpha^6)/2 \\ -\frac{2}{3}\alpha^7 & -(\alpha^2-\alpha^6)/2 & (\alpha^1+\alpha^8)/2 & -(\alpha^4+\alpha^7)/2 \\ -\frac{2}{3}\alpha^8 & -\alpha^3 & -(\alpha^4-\alpha^7)/2 & \alpha^1 \end{pmatrix}$$

α^4	α^5	α^6	α^7	α^8
$(\alpha^2+\alpha^6)/2$	$\frac{1}{3}\alpha^1$	$(\alpha^4-\alpha^7)/2$	$-(\alpha^2+\alpha^6)/2$	$-\alpha^3$
$(\alpha^1-\alpha^8)/2$	$-(\alpha^2+3\alpha^6)/6$	$(\alpha^3-3\alpha^5)/2$	$(\alpha^1-\alpha^8)/2$	$-(\alpha^4+\alpha^7)/2$
$(\alpha^4-\alpha^7)/2$	$\frac{1}{3}\alpha^3$	$-(\alpha^2+\alpha^6)/2$	$-(\alpha^4-\alpha^7)/2$	α^1
$1+\alpha^0+\dfrac{\alpha^3-\alpha^5}{2}$	$-(\alpha^4-3\alpha^7)/6$	$-(\alpha^1+\alpha^8)/2$	$(\alpha^3+3\alpha^5)/2$	$(\alpha^2-\alpha^6)/2$
$-(\alpha^4+3\alpha^7)/2$	$1+\alpha^0-\alpha^5$	$(3\alpha^2-\alpha^6)/2$	$-(3\alpha^4+\alpha^7)/2$	α^8
$-(\alpha^1-\alpha^8)/2$	$(3\alpha^2+\alpha^6)/6$	$1+\alpha^0-\dfrac{\alpha^3+\alpha^5}{2}$	$-(\alpha^1-\alpha^8)/2$	$-(\alpha^4+\alpha^7)/2$
$(\alpha^3+3\alpha^5)/2$	$-(3\alpha^4-\alpha^7)/6$	$-(\alpha^1+\alpha^8)/2$	$1+\alpha^0+\dfrac{\alpha^3-\alpha^5}{2}$	$-(\alpha^2-\alpha^6)/2$
$(\alpha^2+\alpha^6)/2$	$-\frac{1}{3}\alpha^8$	$-(\alpha^4-\alpha^7)/2$	$-(\alpha^2+\alpha^6)/2$	$1+\alpha^0+\alpha^5$

(II) $C_{\sigma\rho}^{\tau}$ are given in the following table:

$$[\phi_\sigma, \phi_\rho] = -i\hbar C_{\sigma\rho}^{\tau}\phi_\tau$$

Numbers in ( ) are the values of τ.

$\sigma\downarrow$ / $\rho\rightarrow$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	1(7)	2(8)	−1(6)	0	−1(4)	1(2)	1(3)
2	0	−1(7)	0	−1(6)	1(8)	3(6)	−1/2(3) 1/2(5)	−1(1)	1(4)
3	0	−2(8)	1(6)	0	1(7)	0	1(2)	1(4)	−2(1)
4	0	1(6)	−1(8)	−1(7)	0	−3(7)	1(1)	−1/2(3) −1/2(5)	−1(2)
5	0	0	−3(6)	0	3(7)	0	−3(2)	3(4)	0
6	0	1(4)	1/2(3) −1/2(5)	−1(2)	−1(1)	3(2)	0	1(8)	−1(7)
7	0	−1(2)	1(1)	−1(4)	1/2(3) 1/2(5)	−3(4)	−1(8)	0	1(6)
8	0	−1(3)	−1(4)	2(1)	1(2)	0	1(7)	−1(6)	0

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Dispersion Relations in Nucleon-Nucleon Scattering

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The dispersion relation of Kronig and Kramers is applied to the case of nucleon-nucleon scattering. The real part of its forward scattering amplitude is expressed in terms of an integral of the total cross sections and an integral of the imaginary part over the unphysical region. It is shown that the latter is, in the static limit, proportional to the Fourier transform of the nuclear potential. This furnishes a new method of determining the shape of the nuclear force potential from experimental data.

§ 1. Introduction

The dispersion relation, pointed out by Kronig[1] as early as in 1926 for the case of the classical theory of light, has recently been reinvestigated also in the quantum theory of fields. Goldberger and others[2] have proved this relation for the case of photon-nucleus and pion-nucleon scattering on the basis of micro-causal character of the theory. These relations turned out to be not only theoretically interesting but also very powerful for the analysis of scattering data.

In the present paper we shall apply this technique to the nucleon-nucleon scattering and show that it leads to a certain relation between the scattering amplitudes and the nuclear force potential.

The dispersion relation for nucleon-nucleon scattering has been discussed by many authors. The main obstacle to these efforts has been the treatment of the integral over unphysical region for which no observable quantity corresponds. We shall show that in the static limit, this quantity is proportional to the Fourier transform of the adiabatic nuclear potential. At the present stage of our knowledge, the proof of this statement is possible only in the perturbation expansion, but it seems to us to be meaningful enough because this method of calculation has proved successful at least in a certain range of nuclear forces.

Section 2 will be devoted to the derivation of the S-matrix in the Heisenberg representation. In Section 3 we investigate the location of the singularities which give rise to the so-called absorptive part, and define the "causal" M-matrix. The paper recently published by Goldberger, Nambu and Oehme[3] contains a thorough discussion of these problems and only a brief account is presented here. The reader is referred to this paper for details. In Section 4, we expand the S- and M-matrix into a power series of the pion-nucleon coupling constant, and define the M-matrix corresponding to *respective* Feynman diagrams. In Section 5 we investigate the adiabatic limit of these quantities,

and define a quantity which reduces to the nuclear force potential in this limit. In Section 6, the effect of the deuteron state is discussed and discussions and conclusions are given in Section 7.

§ 2. Formal derivation of the *S*-matrix

Let us derive in this section the *S*-matrix for nucleon-nucleon scattering and investigate some properties of this quantity under the operation of charge conjugation. These can be easily done by making use of the method of Lehmann, Symanzik and Zimmermann[4], which is applicable under the assumption that there is no bound state in our process. We meet, as a matter of fact, the opposite situation in the neutron-proton scattering, *i. e.*, the appearance of the deuteron state. This problem will be discussed in Section 6.

Let $u^{(p\alpha)}(x)$ be a solution of the Dirac equation

$$(\gamma_\mu \partial/\partial x_\mu + m)\, u^{(p\alpha)}(x) = 0$$

with the energy-momentum eigenvalue p and the spin eigenvalue α and normalized in such a way that

$$\int \bar{u}^{(p\alpha)}(x)\gamma_4 u^{(p\alpha)}(x)\, d^3x = 1 \qquad (2\cdot1)$$

With the aid of this function we define the operators $\bar{\psi}^{(p\alpha)}(t)$ and $\psi^{(p\alpha)}(t)$ from the nuclear field operator $\psi(x)$ by

$$\bar{\psi}^{(p\alpha)}(t) = \int_{x_0=t} \bar{\psi}(x)\gamma_4 u^{(p\alpha)}(x)\, d^3x,$$

and (2·2)

$$\psi^{(p\alpha)}(t) = \int_{x_0=t} \bar{u}^{(p\alpha)}(x)\gamma_4 \psi(x)\, d^3x.$$

Corresponding operators are defined for $\psi_{in}(x)$ and $\psi_{out}(x)$, the asymptotic forms of $\psi(x)$ in the far past and future, which we shall assume to satisfy the Dirac equation of a free field.

The matrix element of S between two two-nucleon states is given by

$$S_{p'\alpha',q'\beta';\,p\alpha,q\beta} = (\bar{\psi}_{out}^{(q'\beta')}\bar{\psi}_{out}^{(p'\alpha')}\Omega,\ \bar{\psi}_{in}^{(q\beta)}\bar{\psi}_{in}^{(p\alpha)}\Omega), \qquad (2\cdot3)$$

where $p'\alpha'$, $q'\beta'$ and $p\alpha$, $q\beta$ mean the energy-momentum and spin (including isotopic spin) eigenvalues of the two nucleons in the initial and final states respectively, and Ω represents the state of the true vacuum. From the definition (2·2) of $\psi_{in}^{(q,\beta)}$, we can rewrite (2·3) as follows:

$$S_{p'\alpha',q'\beta';\,p\alpha,q\beta} = \lim_{t\to-\infty}\int (\bar{\psi}_{out}^{(q'\beta')}\bar{\psi}_{out}^{(p'\alpha')}\Omega,\ \bar{\psi}(\eta)\gamma_4 u^{(q\beta)}(\eta)\bar{\psi}_{in}^{(p\alpha)}\Omega)\, d^3\eta$$

$$
\begin{aligned}
&= -\int \frac{\partial}{\partial \eta_0} (\bar{\psi}_{\text{out}}^{(q'\beta')} \bar{\psi}_{\text{out}}^{(p'\alpha')} \Omega,\ \bar{\psi}(\eta) \gamma_4 u^{(q\beta)}(\eta)\, \bar{\psi}_{\text{in}}^{(p\alpha)} \Omega)\, d^4\eta \\
&\quad + ((\bar{\psi}_{\text{out}}^{(q\beta)})^* \bar{\psi}_{\text{out}}^{(q'\beta')} \bar{\psi}_{\text{out}}^{(p'\alpha')} \Omega,\ \bar{\psi}_{\text{in}}^{(p\alpha)} \Omega) \\
&= i \int d^4\eta\, (\bar{\psi}_{\text{out}}^{(q'\beta')} \bar{\psi}_{\text{out}}^{(p'\alpha')} \Omega,\ \bar{\psi}(\eta) \psi_{\text{in}}^{(p\alpha)} \Omega)\, \bar{D}(\eta) u^{(\alpha\beta)}(\eta) \\
&\quad + \delta_{q\beta, q'\beta'} \delta_{p'\alpha', p\alpha} - \delta_{q\beta, p'\alpha'} \delta_{q'\beta', p\alpha}\ ,
\end{aligned}
$$

where $\bar{D}(\eta)$ denotes the operator such that

$$
\bar{\psi}(\eta) \bar{D}(\eta) = -\left(\frac{\partial \bar{\psi}(\eta)}{\partial \eta_\mu} \gamma_\mu - m \bar{\psi}(\eta) \right). \tag{2·4}
$$

Proceeding in a similar manner, we arrive at the result

$$
\begin{aligned}
S_{p'\alpha', q'\beta'; p\alpha, q\beta} &= -\int d^4\eta\, d^4\xi\, \bar{u}^{(q'\beta')}(\xi) D(\xi) \langle p'\alpha' | T(\psi(\xi), \bar{\psi}(\eta)) | p\alpha \rangle \bar{D}(\eta) u^{(\alpha\beta)}(\eta) \\
&\quad + \delta_{q\beta, q'\beta'} \delta_{p'\alpha', p\alpha} - \delta_{q\beta, p'\alpha'} \delta_{q'\beta', p\alpha}\ ,
\end{aligned} \tag{2·5}
$$

where $|p\alpha\rangle$ is the abbreviation of the one nucleon state $\bar{\psi}^{(p\alpha)}\Omega$, $D(\xi)$ is the Dirac operator $\gamma_\mu \partial/\partial x_\mu + m$ and T means the chronological product due to Wick. The terms $\delta_{q'\beta', q\beta} \delta_{p'\alpha', p\alpha}$ and $\delta_{q'\beta', p\alpha} \delta_{p'\alpha', q\beta}$ represent no scattering phenomena, namely, they describe the transit of two particles without mutual interaction, so they are of little interest to us here. In the derivation of the S-matrix described above, we need not mind the anti-symmetrization of the wave functions; this being automatically taken into account in this method.

In the dispersion relations, there appears the value of the scattering amplitude for negative energy of the incident particle. To see the meaning of the S-matrix for negative q_0, we have only to notice the effect of the charge conjugation operation C with the properties[5]

$$
\begin{aligned}
&C C^* = 1, \qquad C^T = -C, \\
&C^{-1} \gamma_\mu C = -\gamma_\mu^T, \qquad \text{and} \quad \psi^{(c)} = \bar{\psi} C
\end{aligned}
$$

on the field operators of the incident particle:

$$
\begin{aligned}
S_{p'\alpha', q'\beta'; p\alpha, q\beta} &= -\iint d^4\xi\, d^4\eta\, \bar{u}^{(q'\beta')}(\xi)\, C \cdot C^{-1} D(\xi)\, C \\
&\qquad \langle p'\alpha' | T(C^{-1}\psi(\xi) \bar{\psi}(\eta) C) | p\alpha \rangle C^{-1} \bar{D}(\eta)\, C \cdot C^{-1} u^{(q\beta)}(\eta) \\
&= \iint d^4\xi\, d^4\eta\, \bar{u}^{c(q\beta)}(\eta) D(\eta) \langle p'\alpha' | T(\psi^c(\eta) \bar{\psi}^c(\xi)) | p\alpha \rangle \bar{D}(\xi) u^{c(q'\beta')}(\xi).
\end{aligned} \tag{2·6}
$$

We see, from this equation, that the scattering amplitude of a negative energy nucleon is the same as the minus of that of an anti-nucleon with the reversed spin direction and initial and final states interchanged.

Performing the operations $D(\xi)$ and $\bar{D}(\eta)$ in (2·4) explicity, we have

$$S_{p'\alpha',q'\beta';p\alpha q\beta}=-\iint d\xi\, d\eta\, \bar{u}^{(q'\beta')}(\xi)\langle p'\alpha'|T(j(\xi)\bar{j}(\eta))$$
$$+i\delta(\xi_0-\eta_0)\{j(\xi),\psi^*(\eta)\}|p\alpha\rangle u^{(q\beta)}(\eta)$$
$$+\delta_{q\beta,q'\beta'}\delta_{p'\alpha',p\alpha}-\delta_{q\beta,p'\alpha'}\delta_{q'\beta',p\alpha}, \qquad (2\cdot7)$$

where

$$j(\xi)=D(\xi)\psi(\xi) \quad \text{and} \quad \bar{j}(\eta)=\bar{\psi}(\eta)\bar{D}(\eta).$$

For the case of forward scattering the second term in the bracket is independent of q since the anticommutator at the equal time gives a factor $\delta^3(\xi-\eta)$. Therefore we shall hereafter concern ourselves mainly with the first term in the bracket.

§ 3. Location of singularities and causal *M*-matrix

Our main concern now is for the first term in the bracket of (2·7), say, S_1. Integration with respect to the centre-of-mass coordinate of ξ and η gives immediately a δ-function that insures the conservation of energy and momentum. S_1 then assumes the forms

$$S_1=(2\pi)^4 i\delta^4(p+q-p'-q')F,$$

with

$$F_{q'\beta',p'\alpha';q\beta,p\alpha}=i\int d^4x\,\bar{u}^{(q'\beta')}(x)\langle p'\alpha'|T(j(x)\bar{j}(0))|p\alpha\rangle u^{(q\beta)}(0). \qquad (3\cdot1)$$

In the present paper we shall concern ourselves with the forward scattering, $q=q'$ and $p=p'$, and refer to q as the energy-momentum of the projectile and p as that of the target which is taken to be at rest in the laboratory system. We put

$$\boldsymbol{q}=\lambda\boldsymbol{e},$$

where $\boldsymbol{e}$ is a fixed unit vector, and

$$\lambda=\sqrt{q_0^2-m^2} \quad \text{for} \quad q_0>m,$$
$$=i\sqrt{m^2-q_0^2} \quad \text{for} \quad |q_0|<m,$$

and

$$=-\sqrt{q_0^2-m^2} \quad \text{for} \quad q_0<-m.$$

With this definition the forward scattering amplitude F is a function of q_0.

In order to see the location of singularities in the complex q_0-plane originating in the conservation of energy and momentum in the intermediate state, we perform the x-integration and obtain

$$F=-(2\pi)^3\bar{u}^{(q\beta')}(0)\sum_n\Big(\frac{\langle p\alpha'|j(0)|n\rangle\langle n|\bar{j}(0)|p\alpha\rangle}{q_0+m-E_n+i\varepsilon}\delta(\boldsymbol{q}-\boldsymbol{p}_n)$$

$$+\frac{\langle p\alpha'|\bar{j}(0)|n\rangle\langle n|j(0)|p\alpha\rangle}{q_0-m+E_n-i\varepsilon}\delta(\boldsymbol{q}+\boldsymbol{p}_n)\Big)u^{(q\beta)}(0). \qquad (3\cdot 2)$$

The problem is to investigate in what region of q_0 each term in the parenthesis in (3·2) may have vanishing denominator (such region is often called spectrum).

The Heisenberg operator $j(\bar{j})$, when applied on any state, decreases (increases) the nucleon number (which is defined as the number of nucleons minus that of antinucleons) by one. Therefore the state n in the first term is the two-nucleon state, so E_n is equal to or larger than $2m$. The values of q_0 that makes the denominator vanish are equal to or greater than m. When n is a state with two nucleons and a pion

$$E_n \geqq \sqrt{(2m+\mu)^2+q^2}=\sqrt{(2m+\mu)^2+q_0^2-\mu^2}.$$

The threshold for this inelastic process, that is, the minimum value of q_0 for which singularities of this type occur is $m+2\mu+\mu^2/2m$.

The intermediate states in the second term are the zero-nucleon states, and the lowest energy different from 0 is the one-pion state. It is easily seen that the position of the pole due to this intermediate state is $q_0=m-\mu^2/2m$, where μ is the mass of the pion. Singularities arising from the two-pion states lie in the region $q_0\leq m-(2\mu)^2/2m$, where the inequality is due to the relative motion of the two pions in the intermediate states. Many pion states and many pion plus nucleon-pair states can be treated in a similar manner.

Let us now introduce, just as in the case of pion-nucleon scattering, the "causal" matrix M defined by

$$M_{\beta'\alpha';\beta\alpha}(q)=i\int d^4x\,\bar{u}^{(q\beta')}(x)\langle p\alpha'|\eta(x)\{j(x),\bar{j}(0)\}|p\alpha\rangle u(0)^{(q\beta)}$$

$$=-(2\pi)^3\bar{u}^{(q\beta')}(0)\sum_n\Big(\frac{\langle p\alpha'|j(0)|n\rangle\langle n|\bar{j}(0)|p\alpha\rangle}{q_0+m-E_n+i\varepsilon}\delta(\boldsymbol{q}-\boldsymbol{p}_n)$$

$$+\frac{\langle p\alpha'|\bar{j}(0)|n\rangle\langle n|j(0)|p\alpha\rangle}{q_0-m+E_n+i\varepsilon}\delta(\boldsymbol{q}+\boldsymbol{p}_n)\Big)u^{(q\beta)}(0) \qquad (3\cdot 3)$$

and divide it into two parts, the dispersive part D and the absorptive part A, *i.e.*,

$$M=D+iA,$$

$$D_{\beta'\alpha';\beta\alpha}(q)=\frac{i}{2}\int d^4x\,\bar{u}^{(q\beta')}(x)\langle p\alpha'|\varepsilon(x)\{j(x),\bar{j}(0)\}|p\alpha\rangle u^{(q\beta)}(0)$$

$$=-\int d\boldsymbol{x}\,u^{(q\beta')}(\boldsymbol{x})\sum_n \mathrm{P}\Big(\frac{\langle p\alpha'|j(\boldsymbol{x})|n\rangle\langle n|\bar{j}(0)|p\alpha\rangle}{q_0+m-E_n}$$

$$+\frac{\langle p\alpha'|j(0)|n\rangle\langle n|j(\boldsymbol{x})|p\alpha\rangle}{q_0-m+E_n}\Big)u^{(q\beta)}(0), \qquad (3\cdot 4\mathrm{D})$$

and

$$A_{\beta'\alpha';\beta\alpha}(q)=\tfrac{1}{2}\int d^4x\,\bar{u}^{(q\beta')}(x)\langle p'\alpha'|\{j(x),\bar{j}(0)\}|p\alpha\rangle u^{(q\beta)}(0)$$

$$=i\pi\int d\boldsymbol{x}\,\bar{u}^{(q\beta')}(\boldsymbol{x})\sum_n(\delta(q_0+m-E_n)\langle p\alpha'|j(\boldsymbol{x})|n\rangle\langle n|\bar{j}(0)|p\alpha\rangle$$

$$+\delta(q_0+m-E_n)\langle p\alpha'|\bar{j}(0)|n\rangle\langle n|j(\boldsymbol{x})|p\alpha\rangle)u^{(q\beta)}(0). \qquad (3\cdot4\mathrm{A})$$

In these equations $\eta(x)$ is the step function such that $\eta(x)=1$ for $x_0>0$ and 0 for $x_0<0$, $\varepsilon(x)=2\eta(x)-1$, and P means that the summation (or integration) should be taken as the principal value integral of Cauchy.

From equations (3·2) and (3·1) we see that

$$\begin{aligned} F(q)&=M(q)=D(q)+iA(q) \quad \text{for} \quad q_0>m,\\ F(q)&=\qquad D(q)-iA(q) \quad \text{for} \quad q^0<m. \end{aligned} \qquad (3\cdot5)$$

We write M, D and A in the form

$$\begin{aligned} M_{\beta'\alpha';\beta\alpha}(q)&=\bar{u}_\varepsilon^{(q\beta')}(0)\,\bar{u}_\delta^{(p\alpha')}(0)\,N_{\varepsilon\delta}^{\sigma\rho}(q)\,u_\sigma^{(q\beta)}(0)\,u_\rho^{(p\alpha)}(0),\\ D_{\beta'\alpha';\beta\alpha}(q)&=\bar{u}_\varepsilon^{(q\beta')}(0)\,\bar{u}_\delta^{(p\alpha')}(0)\,d_{\varepsilon\delta}^{\sigma\rho}(q)\,u_\sigma^{(q\beta)}(0)\,u_\rho^{(p\alpha)}(0),\\ A_{\beta'\alpha';\beta\alpha}(q)&=\bar{u}_\varepsilon^{(q\beta')}(0)\,\bar{u}_\delta^{(p\alpha')}(0)\,a_{\varepsilon\delta}^{\sigma\rho}(q)\,u_\sigma^{(q\beta)}(0)\,u_\rho^{(p\alpha)}(0). \end{aligned} \qquad (3\cdot6)$$

Due to the anticommutativity of $j(x)$ and $\bar{j}(0)$ for space-like x, we can show that each element of the 16-row 16-column matrix $N_{\varepsilon\delta}^{\sigma\rho}(q)$ satisfies dispersion relations. The proof proceeds in a way completely parallel to the case of pion-nucleon scattering and shall never be repeated here*). The dispersion relation reads

$$d(q_0)=\frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}\frac{a(q'_0)}{q_0'-q_0}dq_0'. \qquad (3\cdot7)$$

In this equation and similar equations below we assume the convergence of the integral on the right-hand side. If this is not the case, the subtraction procedure which has been often used in the theory of dispersion relations must be carried out. We shall hereafter write down merely for the sake of simplicity the dispersion relations as the form in (3·7).

The integral in (3·7) can be divided into three parts,

$$\int_{-\infty}^{\infty}=\int_m^{\infty}+\int_{-\infty}^{-m}+\int_{-m}^{m}. \qquad (3\cdot8)$$

The first integral is the region of real scattering and $a(q_0)$ can be expressed in terms of the total cross sections. The second integral can be written, as mentioned at the end of Section 2, by the nucleon-antinucleon scattering cross sections. The third integral extends over the unphysical region. Unlike the case of forward pion-nucleon scattering, this

*) Rigorous proof of this statement would be very difficult. However, it is seen that each term in the perturbation expansion satisfies the dispersion relation. See Section 4 and Appendix.

integral cannot be explicitly carried out. We shall show in the following that in the adiabatic limit $m\to\infty$ this term is proportional to the Fourier transform of the nuclear potential. In this limit the second term in the right-hand side of eq. (3·8) does not contribute.

§ 4. Perturbation expansion

We now turn to the perturbation expansion of F defined in Section 3. This procedure may seem theoreticlly unsatisfactory, but it is the only way at present since an explicit and compact expression for the nuclear force potential is not yet known.

Rewriting F in the form

$$F(q)=\bar{u}_{\xi}^{(\eta\beta')}\bar{u}_{\delta}^{(p\alpha')}G_{\xi\delta}^{\sigma\rho}(q)\,u_{\sigma}^{(\eta\beta)}\,u_{\rho}^{(p\alpha)} \qquad (4\cdot1)$$

corresponding to (3·6), we can easily expand G into a power series of the pion-nucleon coupling constant according to the well-known Feynman-Dyson rules.

Consider, for example, the second order diagram given in Fig. 1 (*a*). The diagram Fig. 1 (*b*) is not contained in F as mentioned in Section 2. The value of G corresponding to Fig. 1 (*a*), say $G^{(1)}$, is readily seen to be

$$G^{(1)}(q)=\Gamma_1\frac{1}{(p-q)^2+\mu^2-i\varepsilon}\Gamma_2$$

where Γ_1 and Γ_2 are the vertex operators such as $if\gamma_5\tau_i$ referring to respective nucleon. The position of the pole of $G^{(1)}(q_0)$ in the complex q_0-plane is that for "one-pion intermediate state" mentioned in Section 3.

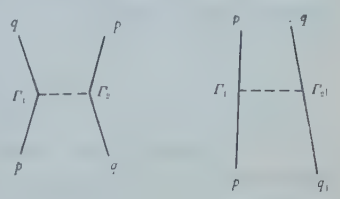

Fig. 1.

The fourth order diagrams are given in Fig. 2. Figs. 2 (*c*) and (*d*) have no absorptive part in the region $-m<q_0<m$. The matrix G corresponding to Fig. 2 (*b*), say $G^{(2b)}(q)$ has branch cut for $q_0>m$, whereas $G^{(2a)}(q)$ has none in this region. Both have cuts for $q_0<m-2\mu^2/m$ corresponding to two- or more-pion intermediate states.

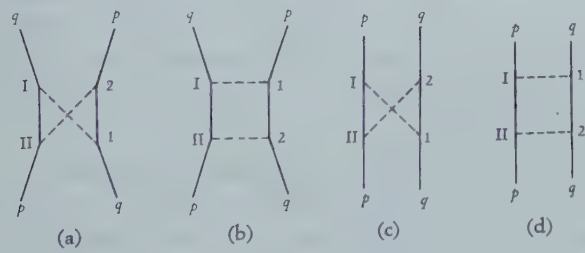

Fig. 2.

In general we see that:

1) If there is no transfer of momentum between two nucleon lines, the graph has no absorptive part in the unphysical region $-m<q_0<m$.
2) If a graph has two parts linked by two nucleon lines, that is, if it has an intermediate state of elastic scattering, it has absorptive parts for $q_0>m$.
3) A graph with momentum transfer has spectrum from $m-(n\mu)^2/2m\,(n\neq1)$ to $-\infty$

where n is the number of exchanged pions.

Our next task is to define the M-matrix or N defined in (3·6) for each diagram and see if each matrix, say $N^{(i)}$ separately satisfies the dispersion relation. In appendix it is shown that $G^{(i)}(q_0)$ calculated according to the Feynman-Dyson rules has poles slightly below the real axis in the region $q_0 > m$ and slightly above the real axis for $q_0 < m$. We define $N^{(i)}$ by displacing the poles of $G^{(i)}$ in the region $q_0 < m$ to be slightly below the real axis. That is, if

$$G^{(i)}(q_0) = d^{(i)}(q_0) - ia^{(i)}(q_0), \qquad (q_0 < m)$$

then

$$N^{(i)}(q_0) = d^{(i)}(q_0) + ia^{(i)}(q_0).$$

Of course, the sum of $N^{(i)}$ over all graphs is N as can be seen from (3·5). The $N^{(i)}$'s thus defined, having no pole in the upper half plane, satisfies the dispersion relation.

On writing down the dispersion relations, we shall hereafter denote as $q_0 = \omega$ and regard $M(q)$, $N(q)$, $F(q)$ and $G(q)$ as functions of ω. These functions, of course, depends on the direction of $\boldsymbol{q}$, in contrast to pion-nucleon case.

§ 5. Adiabatic limit; nuclear forces

After these rather tedious preliminaries, we are now in a position to prove our relation between the nuclear forces and nucleon-nucleon scattering amplitude in the adiabatic limit.

i) Second order

The dispersive part of the second order S-matrix (or F, which is equal to M for $\omega > m$) gives immediately the Fourier transform (or, the Born approximation) of the second order nuclear potential $V^{(1)}(q)$. The dispersion relation, on the other hand, implies that

$$d^{(1)}(\omega) = \frac{\mathrm{P}}{\pi} \int_{-\infty}^{\infty} \frac{a^{(1)}(\omega')}{\omega' - \omega} d\omega. \qquad (5 \cdot 3)$$

We have seen Section 4 that $a(\omega) = 0$ for $\omega > m$, so that (5·3) may be rewritten as

$$d^{(1)}(\omega) = -V^{(1)}(q) = \frac{1}{\pi} \int_{-\infty}^{m} \frac{a^{(1)}(\omega')}{\omega' - \omega} d\omega'. \qquad (5 \cdot 4)$$

ii) Fourth order

In contrast with the second order, the dispersive part of the fourth order S-matrix is not proportional to the Fourier transform of the fourth order nuclear potential. It contains the effect of iteration of the second order potential, and when we calculate the fourth order nuclear potential from the fourth order S-matrix, care must be taken to subtract this effect.[6]

It is the main purpose of this section to show that this subtraction is effected by

restricting the region of the integration of the type of (5·3) to $(-\infty, m)$ instead of $(-\infty, \infty)$.

The causal matrix N corresponding to Fig. 2 (a), say $N^{(2a)}$, satisfies the dispersion relation

$$d^{(2a)}(\omega) = \frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty} \frac{a^{(2a)}(\omega')}{\omega'-\omega}\, d\omega',$$

the right-hand side of which is equal to

$$\frac{\mathrm{P}}{\pi}\int_{-\infty}^{m} \frac{a^{(2a)}(\omega')}{\omega'-\omega}\, d\omega', \tag{5·5}$$

because $a^{(2a)}(\omega)=0$ for $\omega>m$. The $N^{(2b)}$, corresponding to Fig. 2 (b), also satisfies the dispersion relation,

$$d^{(2b)}(\omega) = \frac{\mathrm{P}}{\pi}\int_{m}^{\infty} \frac{a^{(2b)}(\omega')}{\omega'-\omega}\, d\omega' + \frac{\mathrm{P}}{\pi}\int_{-\infty}^{m} \frac{a^{(2b)}(\omega')}{\omega'-\omega}\, d\omega'. \tag{5·6}$$

To consider the problem that what part of the integral in the right-hand side of (5·6) gives the Fourier transform of the nuclear potentials, we must investigate the structure of the matrix elements of the S-matrix corresponding Figs. 2 (a) and 2 (b) in some detail. The G's corresponding to Figs. 2 (a) and 2 (b) are given by the following expressions respectively (apart from some constant factors.);

$$\int dk\, \Delta_F(k)\, \Delta_F(k')\, \Gamma_I S_F(q-k)\, \Gamma_{II} \Gamma_2 S_F(q-k)\, \Gamma_1 \quad \text{for} \quad 2(a) \tag{5·7a}$$

and

$$\int dk\, \Delta_F(k)\, \Delta_F(k')\, \Gamma'_I S_F(p-k)\, \Gamma_{II} \Gamma_1 S_F(q+k)\, \Gamma_2 \quad \text{for} \quad 2(b), \tag{5·7b}$$

where $k'=k+(q-p)$, Γ"s are the vertex operators such as $if\gamma_5\tau_t$, and Δ_F and S_F are the propagation functions of Stückelberg and Feynman.

S_F-functions can be divided into two parts,

$$S_F(l) = S^{(1)}(l) - 2i\bar{S}(l).$$

The first part $S^{(1)}$ contains only components that satisfy the energy momentum relations of a "free" particle, i. e., $l^2+m^2=0$ whereas the second part $\bar{S}$ has no component that satisfies this relation[7]. Now, the region of the integrals (5·5) and the last term of (5·6), as mentioned in Sec. 4, can also be written as $(-\infty, m-2\mu^2/m)$. In calculating the absorptive part in this region, the internal nucleon lines in Figs. 2(a) and 2(b) can never be "free".*

Therefore, we can replace the S_F-functions in (5·7a) and (5·7b) by $-2i\bar{S}$ as far

---

* For the physical region, this can be immediately seen from the energy momentum conservation, and that this is true also in the unphysical region can be proved by an elementary calculation.

as the integrals (5·5) and the last term of (5·6) concern. In the "static" limit, in which the energy-momentum of the mesons are regarded to be far smaller than the rest mass of the nucleon and neglected, $\bar{S}$-functions in (5·7*a*) and (5·7*b*) can be approximated by the following expressions :[8]

$$\bar{S}(q+k) \sim \frac{1}{(2\pi)^4} \mathrm{P} \frac{1}{k_0},$$

$$\bar{S}(q-k) \sim \frac{-1}{(2\pi)^4} \mathrm{P} \frac{1}{k_0} \qquad (5\cdot 8)$$

and

$$\bar{S}(p-k) \sim \frac{-1}{(2\pi)^4} \mathrm{P} \frac{1}{k_0}.$$

Substituting (5·8) in (5·7*a*) and (5·7*b*), we can say that for $\omega < m - 2\mu^2/m$, (5·7*a*) and (5·7*b*) have the same value except for the *sign* and *the order of vertex operators* and therefore, the last integral on the right-hand side of (5·6) has the value equal to (5·5) except for the sign and the order of Γ_1 and Γ_2. Adding together these two integrals we have

$$\frac{\mathrm{P}}{\pi} \int_{-\infty}^{m} \frac{a^{(2a)}(\omega)}{\omega' - \omega} d\omega' + \frac{\mathrm{P}}{\pi} \int_{-\infty}^{m} \frac{a^{(2b)}(\omega')}{\omega' - \omega} d\omega'$$

$$= (2\pi)^4 i \int dk D_F(k) D_F(k+q-p) \frac{-1}{k_0} \frac{-1}{k_0} \Gamma_I \Gamma_{II} [\Gamma_2, \Gamma_1], \qquad (5\cdot 9)$$

which is just the Fourier transform of the fourth order nuclear potential given by Nambu.[6] Hence we have, writing as

$$d_1^{(2)}(\omega) = d^{(2a)}(\omega) + d^{(2b)}(\omega)$$

and

$$a_1^{(2)}(\omega) = a^{(2a)}(\omega) + a^{(2b)}(\omega),$$

$$d_1^{(2)}(\omega) = -U_1^{(2)}(\omega) + \frac{1}{\pi} \int_m^{\infty} \frac{a_1^{(2)}(\omega')}{\omega' - \omega} d\omega',$$

and $U_1^{(2)}(\omega)$ tends to the Fourier transform of the fourth order nuclear potential in the static limit.

Of the remaining fourth order matrix, $N^{(2c)}(\omega)$ has not spectrum for $\omega < m$. $N^{(2d)}(\omega)$ has spectrum for $\omega < -m$, but the contribution from this part vanishes in the limit $m \to \infty$. Thus for the sum of all graphs *a* to *d*,

$$d^{(2)}(\omega) = -U^{(2)}(\omega) + \frac{1}{\pi} \int_m^{\infty} \frac{a^{(2)}(\omega')}{\omega' - \omega} d\omega', \qquad (5\cdot 10)$$

and $U^{(2)}(\omega)$ again tends to the fourth order potential.*

* This is the adiabatic potential and not of the Brueckner-Watson type (Phys. Rev. **90** (1953), 699).

iii) Higher orders

The recipe of construction of the adiabatic potential from S-matrix is, according to Nambu, as follows. If a diagram has no "elastic" intermediate state, that means, if the diagram cannot be separated by opening two internal nucleon lines into two parts each representing the nucleon-nucleon scattering, the diagram is called to be irreducible and the matrix element corresponding to this diagram is to be regarded as a part of the nuclear potential. If a diagram has one elastic intermediate state, the effect of the iteration of the lower order potential must be subtracted to obtain the genuine nuclear force potential.

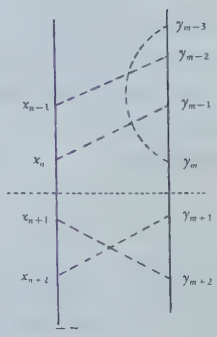

Fig. 3.

Consider, for example, the matrix element corresponding to Fig. 3.

$$S^{(3)}=\int dx_1\,dx_2\cdots dy_1\,dy_2\cdots$$
$$\cdots\Gamma_1(x_{n-1})\,\eta(x_{n-1}-x_n)\,\Gamma_1(x_n)\,\eta(x_n-x_{n+1})\,\Gamma_1(x_{n+1})\cdots$$
$$\times\cdots\Gamma_2(y_{m-1})\,\eta(y_{m-1}-y_m)\,\Gamma_2(y_m)\,\eta(y_m-y_{m+1})\,\Gamma_2(y_{m+1})\cdots$$
$$\times\cdots\Delta_F(x_n-y_{m-1})\,\Delta_F(y_{m-3}-y_m)\,\Delta_F(x_{n+1}-y_{m+2})\cdots, \qquad (5\cdot11)$$

where the step function η is the non-relativistic version of the S_F-function. If we replace $\eta(x_n-x_{n+1})$ in (5·11) by unity, the expression thus obtained represents the iteration of the two lower order potentials which are respectively represented by the upper half and the lower half of Fig. 3. (The separation into two parts is indicated by a dotted line.). This term is to be subtracted. Therefore the net contribution to the potential is obtained by replacing $\eta(x_n-x_{n+1})$ by $\eta(x_n-x_{n+1})-1=-\eta(x_{n+1}-x_n)$. This expression is further transformed by making use of the relation such as

$$\eta(x_{n-1}-x_n)\,\eta(x_{n+1}-x_n)=\eta(x_{n-1}-x_{n+1})\,\eta(x_{n+1}-x_n)+\eta(x_{n+1}-x_{n-1})\,\eta(x_{n-1}-x_n)$$

until the arguments of the η-functions form a continuous chain of x_i, namely such that the product of η functions takes a form $\cdots\eta(a-b)\,\eta(b-c)\,\eta(c-d)\cdots$. Each term is then equal to one of the irreducible graphs except for the sign and the order of vertex operators.

When a graph contains more than one elastic intermediate states, repeated application on the above procedure brings it into a form in which the η-functions are the same with those of an irreducible graph.

Turning back to the dispersion problem, we consider only graphs in which there is a momentum transfer between nucleons, for those without momentum transfer has no absorptive part in the unphysical region. We first note that a graph without any elastic intermediate state has no absorptive part in the elastic region $m<\omega<m+2\mu+\mu^2/2m$. For this kind of graph, say $G^{(I)}$, we have (except for the case of one-pion exchange),

$$d^{(1)}(\omega)=\frac{1}{\pi}\int_{-\infty}^{m-2\mu^2/m}\frac{a^{(I)}(\omega')}{\omega'-\omega}d\omega'+\frac{1}{\pi}\int_{m+2\mu+\mu^2/2m}^{\infty}\frac{a^{(I)}(\omega')}{\omega'-\omega}d\omega'.$$

The second integral is seen to be vanishingly small as compared with the first one in the static limit. For the energy denominator in the first integral is of the order μ^2/m, while that of the second is of the order μ^*. Here ω has been taken to be nearly equal to m since we are dealing with the static limit in this section. This shows that in the static limit,

$$U^{(I)}(\omega)=\frac{1}{\pi}\int_{-\infty}^{m}\frac{a^{(I)}(\omega')}{\omega'-\omega}d\omega'$$

is equal to the part of the potential coming from irreducible graphs.

For those graphs $G^{(II)}$, having one elastic intermediate state, the effect of subtraction must be considered. We compare the contribution of this graph to the potential, V^{II}, and

$$U^{(II)}(\omega)=\frac{1}{\pi}\int_{-\infty}^{m}\frac{a^{(II)}(\omega')}{\omega'-\omega}d\omega'.$$

We have seen that $V^{(II)}$ is obtained from $S^{(II)}$ or $N^{(II)}$ by replacing $\eta(x_n-x_{n+1})$ by $-\eta(x_{n+1}-x_n)$ and the transformed matrix is the sum of irreducible graph except for the sign change and the order of vertex operators. This $V^{(II)}$, having no absoptive part in the elastic region, can be expressed as

$$V^{(II)}(\omega)\doteqdot\frac{1}{\pi}\int_{-\infty}^{m}\frac{a'(\omega')}{\omega'-\omega}d\omega'$$

where the equality holds in the static limit. a' is the absorptive part of $N'^{(II)}$, which is derived from $N^{(II)}$ by replacement of the η-function. It is easily seen that a' is effectively equal to $a^{(II)}$, the absorptive part of $N^{(II)}$. We have previously argued that in discussing the absorptive part in the unphysical region, the nucleon line does not become free and the S_F can be replaced by $-2i\bar{S}$. In the case of static limit, this means that

$$\eta(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{\alpha-i\varepsilon}d\alpha\quad\text{(cf. (5·8))}$$

can be replaced by

$$\tfrac{1}{2}\varepsilon(x)=\frac{\mathrm{P}}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\alpha x}}{\alpha}d\alpha$$

which is an odd function. Thus the replacement of $\eta(x_n-x_{n+1})$ by $-\eta(x_{n+1}-x_n)$ does not change the absorptive part for $\omega<m$. This proves that

$$U^{II}(\omega)=\frac{1}{\pi}\int_{-\infty}^{m}\frac{a^{(II)}(\omega')}{\omega'-\omega}d\omega'\doteqdot V^{(II)}(\omega).$$

* For this argument to hold, the dispersion integral must converge sufficiently rapidly, which we assume to be true.

Those graphs with many elastic intermediate states can be treated in the same way.

Adding all contributions, we have

$$d(\omega) = -U(\omega) + \frac{\mathrm{P}}{\pi}\int_{-m}^{\infty} \frac{a(\omega')}{\omega' - \omega}\, d\omega' \qquad (5\cdot 12)$$

and $U(\omega)$ tends to $V(p)$, $(\omega = m + p^2/m)$, the adiabatic potential in the static limit.*

§ 6. The deuteron

In the derivation of the formula (5·12), the effect of the existence of the bound state of the two nucleons has never been taken into account, for, in the construction of the explicit expression for the S-matrix in section 2, we have utilized the concept of "*in*" and "*out*" field, which are legitimate only under the assumption of non-existence of the bound state. In spite of these assumptions we meet, in fact, the bound state, when we treat the proton-neutron scattering in spin triplet state. We must, therefore, add a term representing the effect of the deuteron state to (5·12). This can be done if we regard the deuteron as a kind of elementary particle.

Returning to eq. (3·3), we see that the absorptive part $a(\omega)$ contains a term of the form *const.* $\times \delta(\omega - \omega_D)$, where ω_D is the energy that is just necessary for the formation of the deuteron in the laboratory system, *i.e.*, $\omega_D = m - 2B + B^2/2m$, B being the binding energy of the deuteron. We must therefore add to the right-hand side of (5·12) a term $C/(\omega_D - \omega)$, where C is a constant matrix that we shall determine in this section.

To determine the constant C in the non-relativistic case,** let us now write down F in terms of phase shifts δ_l in the centre of mass system and investigate its behaviour in the vicinity of $\omega_d = m - B/2$. If we neglect the effect of the tensor force, all partial waves other than the S-wave contribute to F very little for $\omega \approx \omega_d$. In this case the F-matrix for the forward scattering is given by[9]

$$F_c(\omega_c) = \frac{2\pi}{q_c \omega_c} \sin\delta_0 \exp i\delta_0 \qquad (6\cdot 2)$$

where ω_c and q_c are the energy and momentum of the nucleon in the c.m. system and δ_0 is the triplet S phase shift. When we use the effective range approximation[10] $q_c \cot\delta_0 = -a + \frac{1}{2} r q_c^2$, we see that (6·2) behaves in the neighbourhood of ω_d as

$$F_c(\omega_c) \approx -\frac{2\pi}{m^2} \frac{R^{-1}}{(1 - R^{-1} r^1)} \cdot \frac{1}{\omega_c - \omega_d} \qquad (6\cdot 3)$$

where R is the radius of the deuteron $R = \sqrt{mB}$, and r is the effective range for spin

* That the U is somewhat connected with the nuclear potential is stated by Goldberger et al., ref. 3, from a different argument.

** This problem is treated in detail by Goldberger et al., ref. 3. The following arguments are inserted here so that this paper is self-contained.

triplet state. Transforming (6·3) into the laboratory system, we have for $\omega \approx \omega_D$

$$F(\omega) \approx -\frac{16\pi}{m} \frac{R^{-1}}{1-R^{-1}r^1} \frac{1}{\omega-\omega_D}. \tag{6·4}$$

We see from the above arguments that (6·4) is the term that we must add to the right-hand side of (5·12) for the neutron proton scattering in triplet state.

§ 7. Discussions and conclusion

Equations (5·12) and (6·4) are the main results of this paper. These formulae are not the dispersion relations for the S-matrix, or, in our notation, F which is directly connected with the transition probabilities, but those for G or N defined by (3·6). As the spinors u and $\bar{u}$ depend on energies in a rather complicated way, the dispersion relations for S(or F) are not as simple as (5·12). In the non-relativistic approximation (Pauli approximation), however, these spinors reduce to the energy-indepndent two component ones and (5·12) can be regarded as a dispersion relation for F.

We have thus in this approximation the dispersion relation for the S-matrix

$$D_0^{(s)}(\omega) = -U^{(s)}(\omega) + \int_m^\infty \frac{A_0^{(s)}(\omega')}{\omega-\omega'} d\omega' - \delta_{sD}\left(\frac{16\pi}{m} \frac{R^{-1}}{1-R^{-1}r^1} \frac{1}{\omega-\omega_D}\right) + \text{const.} \tag{7·1}$$

with

$$S^{(s)}(\omega) = (2\pi)^4 i F_0^{(s)}(\omega),$$

and

$$F_0^{(s)}(\omega) = D_0^{(s)}(\omega) + i A_0^{(s)}(\omega) \quad \text{for} \quad \omega > m.$$

In this formula (s) denotes the spin and charge state of the two nucleon system (singlet, triplet, direction of resultant spin and the charge state pp, np or nn) and δ_{sD} equals unity for spin triplet neutron-proton state, and zero otherwise. The constant added in (7·1) originates from the term that has been omitted from our considerations at the bottom of section 2.

If we confine ourselves to the non-relativistic region (and, of course, the static approximation for S_F, (5·8)), our formula (7·1) leads to a new relation between the scattering amplitudes and the nuclear force potential and proves to be very useful for the analysis of the experimental data. $A_0^{(s)}(\omega)$ on the right-hand side of (7·1) can be related by the so-called optical theorem to the total scattering cross section $\sigma_{tot}(\omega)$ as

$$\sigma_{tot}^{(s)}(\omega) = \frac{2\omega}{q} A_0^{(s)}(\omega), \tag{7·2}$$

whereas $D_0^{(s)}(\omega)$ is connected to the differential cross section in the forward direction as follows,

$$\sigma_{\text{diff}}^{(s)}=\frac{\omega^2}{(2\pi)^2}[|D_0^{(s)}(\omega)|^2+|A_0^{(s)}(\omega)|^2]. \tag{7·3}$$

Therefore, if we know the total scattering cross section and the differential cross section in the forward direction at a certain energy, say ω, from experiment, then we can obtain the values of $A_0^{(s)}(\omega)$ and $|D_0^{(s)}(\omega)|$. Although we cannot know the sign of $D_0^{(s)}(\omega)$ from these formulae, we can infer it from other theory or experiment at least for low energies. When we know the values of $D_0^{(s)}(\omega)$ and $\sigma_{\text{tot}}^{(s)}(\omega)$ for all energies greater than m, we can immediately calculate the Fourier transform of the nuclear potential making use of (7·1).

Let us investigate this problem in more detail.

The most general form of the nuclear potential in the adiabatic approximation is

$$\begin{aligned}V(\boldsymbol{x},\sigma^{(1)},\sigma^{(2)})&=V_0(x)+(\sigma^{(1)}\cdot\sigma^{(2)})V_1(x)+(\sigma^{(1)}\cdot\nabla)(\sigma^{(2)}\cdot\nabla)V_2(x)\\&=V_0(x)+(\sigma^{(1)}\cdot\sigma^{(2)})\Big(V_1(x)+\frac{2}{3}\frac{1}{x}V_2'(x)+\frac{1}{3}V_2''(x)\Big)\\&\quad+\frac{1}{3}S_{12}\Big(-\frac{1}{x}V_2'(x)+V_2''(x)\Big),\end{aligned} \tag{7·3}$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the spin operators of respective nucleons, $\boldsymbol{x}$ is the relative coordinate of the two nucleons and $x=|\boldsymbol{x}|$. ∇ means the differentiation with respect to the relative coordinate, and V_0, V_1 and V_2 are the functions of x and perhaps other variables, *e.g.* isotopic spin, but do not depend on the direction of $\boldsymbol{x}$ and σ's. S_{12} is the well-known tensor operator $S_{12}=3(\sigma^{(1)}\boldsymbol{n})(\sigma^{(2)}\cdot\boldsymbol{n})-(\sigma^{(1)}\cdot\sigma^{(2)})$ with $\boldsymbol{n}=\boldsymbol{x}/x$ and primes denote the differentiation with respect to x.

The Fourier transform of (7·3) is given by

$$W(q)=-W_0(q)-(\sigma^{(1)}\cdot\sigma^{(2)})W_1(q)+(\sigma^{(1)}\cdot\boldsymbol{e})(\sigma^{(2)}\cdot\boldsymbol{e})q^2W_2(q), \tag{7·4}$$

with

$$W_i(q)=\int V_i(\boldsymbol{x})e^{i\boldsymbol{q}\boldsymbol{x}}d^3\boldsymbol{x}, \quad \text{and}$$

$\boldsymbol{e}$ is the unit vector in the direction of $\boldsymbol{q}$. To obtain the spin-independent part, spin-dependent part and tensor part of the nuclear force separately, we must know the values of $W(q)$ in various spin states. The values of W in respective spin states are

$W=-W_0+3W_1-q^2W_2$ for singlet state,

$W=-W_0-W_1+q^2W_2$ for triplet state, resultant spin parallel or anti-parallel to the direction of injection,

and

$W=-W_0-W_1-q^2W_2$ for triplet state, resultant spin perpendicular to the direction of the injection.

For a completely unpolarized beam

$$W_{mean}=-W_0. \tag{7·5}$$

Although it is very difficult to realize the experiment for various spin states separately, we see from (7·5) that we can obtain at least the central spin-independent part of the potential.

Because of the approximation procedures that has been made in the deduction of our formulae, our theory cannot be applied to very high energies at which the adiabatic potential loses its meaning. Nevertheless our formulae are still useful for the determination of the shape of nuclear potential in the outer region, and furnish us, at least, a powerful means to determine the renormalized meson-nucleon coupling constant in connection with the nuclear forces.[11] The analysis of the experimental data with our formulae will be presented in a separate paper.

Appendix. Location of poles in a matrix element for each Feynman graph

The scattering matrix element for graphs (i) has a form

$$G^{(i)}=\int dk_1 dk_2 \cdots \frac{\psi(k, p, q)}{(\varphi_1(k, p, q)-i\varepsilon)(\varphi_2(k, p, q)-i\varepsilon)\cdots}$$

where φ_i is a quadratic function in p, q and $k(=k_1, k_2, \cdots)$ and ψ is a polynomial including γ-matrices. Making use of Feynman's technique of parameter integration, we have, after the integrations with respect to k_i,

$$G^{(i)}=\int dx_1 dx_2 \cdots \frac{h(x, p, q)}{(f'(x)pq+g'(x)-i\varepsilon)^n} \quad n: \text{integer}$$

where f', g' are polynomials in x_i and h is a polynomial in x, p, q and γ. In terms of the incident energy ω in c. m. system,

$$G^{(i)}=\int dx_1 dx_2 \cdots \frac{h(x, \omega, \sqrt{\omega^2-m^2})}{[f(x)(\omega-m)+g(x)-i\varepsilon]^n}.$$

First of all we notice that $g(x)$ never vanishes since G has no absorptive part at $\omega=m$. It is easily seen that $g(x)$ is positive definite since $g(x)$ is positive for some value of x (for instance, if we put all x to be zero, $g(x)=m^2$ or μ^2). Then G has a singularity at

$$\omega=m-\frac{g(x)}{f(x)}+\frac{i\varepsilon}{f(x)}.$$

This shows that the position of poles in G is either at $\omega=\omega_0-i\varepsilon$ if $\omega_0>m(f(x)<0)$ or $\omega=\omega_0+i\varepsilon$ if $\omega_0<m(f>0)$. Instead of G, we define N by

$$N^{(i)}=\int\cdots\int dx \frac{h}{[f(x)(\omega-m)+g(x)+i\varepsilon f(x)]^n}.$$

which is equal to G for $\omega > m$, but the sign of the absorptive parts is opposite to G for $\omega < m$. The $N(\omega)$, having no pole in the upper half of the complex ω-plane, satisfies the dispersion relation

$$d^{(\pm)}(\omega) = \frac{\mathrm{P}}{\pi} \int_{-\infty}^{\infty} \frac{a^{(\pm)}(\omega')}{\omega' - \omega} \, d\omega'.$$

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Theory of Cohesive Energies and Energy-Band Structures of Diamond-Type Valence Crystals

—The Method of SLCO, II—

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Use is made of the method of SLCO to develop the theory of the cohesive energies of diamond-type valence crystals. A formula determining ionicity of diatomic crystals such as InSb is presented. Furthermore, a new method to calculate the energy-band structures of these crystals is presented.

§ 1. Introduction

The first quantum mechanical computation of the cohesive energy of diamond was attempted by Schmid.[1] He constructed a two-electron wave function, in which two valence electrons together form an electron-pair bond between two neighbouring atoms and he, furthermore, introduced another two-electron function for two core electrons in a lst level together forming a lone pair. Then the total wave function of crystal electrons as a whole was approximated by the product of these two-electron functions antisymmetrized with respect to all electrons. His treatment is a generalization of the Slater-Pauling theory of directional valence. He found, however, that satisfactory results were achieved only when two-electron functions were formulated to allow ionic states to be included into the electron-pair bonds. Asano and Tomishima[2] generalized Schmid's treatment for diatomic crystals and applied it to ZnS crystal.

In his treatment, as is mentioned above, Schmid made use, not of one-electron wave functions, but of two-electron wave functions. However, in order to study the relations of the theory of cohesive energy with the other physical problems such as energy-band structures and diamagnetic properties, it is convenient to formulate it on the basis of one-electron wave functions. The energy-band structures of diamond-type valence crystals have been calculated by many authors[3] in detail, but frequently without further going on to work their connections with the theory of cohesive energy.

In the preceding paper[4] (hereafter referred to as I), we formulated the new method of SLCO to calculate the cohesive energy of ionic crystals and applied it to the calculation of the cohesive energy of LiH crystal. Furthermore, we showed that SLCO was useful to compute the band structures of the conduction and valence bands of these crystals. The method of SLCO also is applicable to valence crystals having an energy gap between valence and conduction bands like ionic crystals.

In the present paper, we shall develop the theory of cohesive energy of diamond-type

valence crystals on the basis of one-electron wave functions by using SLCO, discussing its relation with the energy-band structures of these crystals. As a result, a formula determining ionicity of diatomic crystals such as InSb and a new method to calculate the energy-band structure of these crystals will be presented. Application of our results to the diamagnetic properties of these crystals will be discussed on a separate paper in the near future.

§ 2. Theory of cohesive energy

(a) *General Theory*

The crystal structure of diamond-type crystals is shown in Fig. 1. In the figure, atoms denoted by solid and empty circles form fcc sub-lattices, respectively. Each solid (or empty) atom is surrounded by four empty (or solid) atoms having tetrahedral arrangement and is bonded with them by electron-pair bonds. In zinc blende structure such as InSb crystal, one of the two fcc sub-lattices is populated by In atoms and the other by Sb atoms.

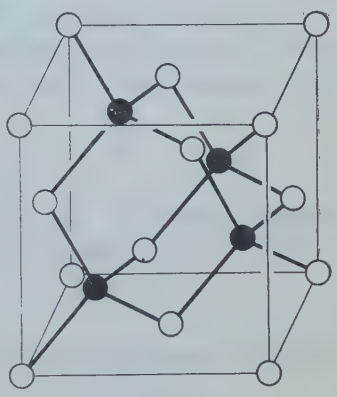

Fig. 1

Among hybridized tetrahedral spin-orbitals, these subject to solid atoms, i.e., one of the two fcc lattices, are denoted by φ_a^0, φ_b^0,... and the ones forming electron-pair bonds with them together, which are subject to the other sublattice-atoms, are denoted by $\varphi_{\bar{a}}^0$, $\varphi_{\bar{b}}^0$,..., respectively. Here the indices a and b are the abbreviations of $\{\alpha, i, p\}$, $\{\beta, j, q\}$, respectively, where we use indices α and β to denote the lattice sites of the solid atoms, i and j ($=1$, 2, 3 and 4) to specify the states of the spin-orbitals, and p and q to specify their spin states. When we need not distinguish solid atoms from empty atoms, we specify spin-orbitals by indices l, m and n.

Now let us denote the row matrix of degree $16N$*, of which the first $8N$ elements are φ_a^0, φ_b^0,... and the last $8N$ elements are $\varphi_{\bar{a}}^0$, $\varphi_{\bar{b}}^0$,$\cdots$, by $\boldsymbol{\varphi}^0$. We then orthogonalize $\boldsymbol{\varphi}^0$ by making use of Löwdin's method[5] and denote it by $\boldsymbol{\varphi}$; that is

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^0 \boldsymbol{\Delta}^{-1/2}, \tag{2·1}$$

where

$$\Delta_{lm} = (\mathbf{1}+\mathbf{S})_{lm} = \int \varphi_l^0 \varphi_m^0 d\tau. \tag{2·2}$$

We define bonding and anti-bonding spin-orbitals by

$$\left.\begin{aligned} \phi_a &= \frac{1}{\sqrt{2}}(\varphi_a + \varphi_{\bar{a}}), \\ \phi_{\bar{a}} &= \frac{1}{\sqrt{2}}(\varphi_a - \varphi_{\bar{a}}) \end{aligned}\right\} \tag{2·3}$$

* $2N$ is the total number of atoms in the crystal.

and then introduce the row matrix $\boldsymbol{\phi}$ of degree $16N$, of which the first $8N$ elements are ϕ_a, ϕ_b,... and the last $8N$ elements are $\phi_{\bar{a}}$, $\phi_{\bar{b}}$,.... Then, making use of the row matrix $\boldsymbol{\phi}$, SLCO $\boldsymbol{\psi}$ is defined by

$$\boldsymbol{\psi}=\boldsymbol{\phi}\boldsymbol{X}, \tag{2·4}$$

where the $16N\times 16N$ unitary matrix $\boldsymbol{X}$ has the block form

$$\boldsymbol{X}=\begin{bmatrix} (\mathbf{1}-\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})^{1/2} & \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^{+} & -(\mathbf{1}-\boldsymbol{\varepsilon}^{+}\boldsymbol{\varepsilon})^{1/2} \end{bmatrix}. \tag{2·5}$$

The $8N\times 8N$ matrix ε having matrix elements $\varepsilon_{a\bar{b}}$, should be determined by the variation principle which will be mentioned below, where it is noted that indices a refer to one fcc sub-lattice and $\bar{b}$ to the other one.

We consider that only the first $8N$ spin-orbitals ψ_a's of $\boldsymbol{\psi}$ are occupied in the ground state and the remaining $8N$ spin orbitals $\psi_{\bar{a}}$'s of $\boldsymbol{\psi}$ may be used to describe certain excited states. Thus we express the wave function* for the ground state by

$$\Psi(x_1, \cdots, x_{8N})=\frac{1}{\sqrt{(8N)!}}\det|\psi_1(x_1)\cdots\psi_{8N}(x_{8N})|, \tag{2·6}$$

where each coordinate x is a combination of a space coordinate $\boldsymbol{r}$ and a spin coordinate σ.

We set up the variation principle:

$$\delta E_g=0. \tag{2·7}$$

E_g is the expectation value of the crystal energy for the ground state; that is,

$$E_g=\int \Psi^*(x_1, \cdots, x_{8N})\mathcal{H}_{(8N)}\Psi(x_1, \cdots, x_{8N})d\tau_{(8N)}, \tag{2·8}$$

where $\mathcal{H}_{(8N)}$ is the Hamiltonian of the crystal. When $\boldsymbol{\varepsilon}=0$, the wave function is reduced to Schmid's wave function with $A=1$.**

As an important result of the variation principle (2·7), as was shown in I, we get the relation

$$\int \psi_{\bar{b}}^* H_F \psi_a d\tau=0, \tag{2·9}$$

where

$$H_F f(x)=\Big[-\frac{1}{2}\Delta-\sum_{\xi}^{(2N)}v(\boldsymbol{r}-\boldsymbol{R}_{\xi})+\sum_a\int\frac{|\psi_a(x_1)|^2}{|\boldsymbol{r}-\boldsymbol{r}_1|}d\tau_1 - \sum_a\frac{\psi_a(x)}{f(x)}\int\frac{\psi_a^*(x_1)f(x_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|}d\tau_1\Big]f(x). \tag{2·10}$$

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* In what follows, for simplicity we treat core electrons as classical charge clouds.

** A is a parameter appearing in Schmid's two-electron wave functions and showing the proportion of ionic states to be included into the electron-pair bonds.

Here $v(\boldsymbol{r}-\boldsymbol{R}_\xi)$ is the potential at $\boldsymbol{r}$ that arises from the ξth ion core locating at $\boldsymbol{R}_\xi$ and $\sum_{\xi}^{(2N)}$ extends over all atoms in the crystal.

(b) *$\boldsymbol{\varepsilon}$-matrix*

We may assume that $\boldsymbol{X}$ and $\boldsymbol{\varepsilon}$ satisfying the variation principle (2·7) have the same rotational and translational symmetry as that of the crystal.* Then we may propose the trial form for $\boldsymbol{\varepsilon}$**

$$\varepsilon_{a\bar{b}}=\begin{cases}\lambda & \text{when } a=b,\\ \mu & \text{when } b \text{ is subject to the same atom and the same spin state as } a, \text{ but } a\neq b,\\ \nu & \text{when } \bar{b} \text{ is subject to the same atom and the same spin state as } \bar{a}, \text{ but } \bar{a}\neq\bar{b},\\ 0 & \text{otherwise.}\end{cases} \tag{2·11}$$

Therefore,

$$(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})_{a\bar{b}}=\begin{cases}\lambda^2+3\mu^2+3\nu^2 & \text{when } a=b,\\ 2\mu^2 & \text{when } b \text{ is subject to the same atom and the same spin state as } a, \text{ but } a\neq b,\\ 2\nu^2 & \text{when } \bar{b} \text{ is subject to the same atom and the same spin state as } \bar{a}, \text{ but } \bar{a}\neq\bar{b},\\ \mu\nu & \text{when } a \text{ and } b \text{ (or } \bar{a} \text{ and } \bar{b}) \text{ are subject to the nearest neighbour atoms and to the same spin state,}\\ 0 & \text{otherwise.}\end{cases} \tag{2·12}$$

This trial form of $\boldsymbol{\varepsilon}$ is for diatomic crystals having zinc blende structure such as InSb. For monatomic crystals such as diamond, the existence of the inversion symmetry demands that

$$\lambda=0 \qquad \text{and} \qquad \mu=-\nu.*** \tag{2·13}$$

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* These $\boldsymbol{X}$ and $\boldsymbol{\varepsilon}$ need not always have the same rotational and translational symmetry as that of the crystal. However such $\boldsymbol{X}$ and $\boldsymbol{\varepsilon}$ may be transformed to ones having these symmetries by the unitary transformation due to a suitable unitary matrix with the block form

$$\boldsymbol{Y}=\begin{bmatrix} y & 0 \\ 0 & y' \end{bmatrix}.$$

For such a unitary transformation, the lattice energy (2·8) is invariant.

** In his calculation of the cohesive energy of diamond, Schmid showed that the quantity A (see the foregoing footnote) was nearly equal to one. That $A=1$ corresponds to $\varepsilon=0$ for our case. This fact suggests that λ, μ and ν may be considered as small quantities. (Also see the second footnote of the next section.) In what follows, therefore, we treat these parameters as first order small quantities.

*** We are intending to compute numerically the cohesive energy of diamond by using this trial form of ε.

§ 3. Ionicity (Charge- and bond-order matrix)

The one-electron density matrix due to Ψ given by (2·6) is

$$\gamma(x'|x)=\boldsymbol{\phi}(x)\boldsymbol{n}\boldsymbol{\phi}^{+}(x), \tag{3·1}$$

where $\boldsymbol{n}$ is the $16N\times 16N$ matrix having the block form

$$\boldsymbol{n}=\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3·2}$$

$\boldsymbol{n}$ is the charge- and bond-order matrix with respect to SLCO ϕ. We see that $\boldsymbol{\phi}$ are the so-called Löwdin's natural spin orbitals[6] for the crystal in ground state and $\boldsymbol{n}$ is the corresponding charge- and bond-order matrix, in which the bond-orders are equal to zero. The charge- and bond-order matrix $\boldsymbol{Q}$ with respect to the orthonormal hybridized spin-orbitals $\boldsymbol{\varphi}$ is given by

$$\boldsymbol{Q}=\boldsymbol{U}^{+}\boldsymbol{X}\boldsymbol{n}\boldsymbol{X}^{+}\boldsymbol{U}, \tag{3·3}$$

where

$$\boldsymbol{U}=\frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{1} & \boldsymbol{\omega}^{+} \\ \boldsymbol{\omega} & -\mathbf{1} \end{bmatrix} \tag{3·4}$$

and

$$\omega_{a\bar{b}}=\delta_{ab}. \tag{3·4'}$$

From (2·5), (3·2) and (3·4), the explicit form of $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q}=\frac{1}{2}\begin{bmatrix} \mathbf{1}+2(\mathbf{1}-\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})^{1/2}\boldsymbol{\varepsilon}\boldsymbol{\omega}^{+} & (\mathbf{1}-2\boldsymbol{\varepsilon}^{+}\boldsymbol{\varepsilon})\boldsymbol{\omega}^{+} \\ (\mathbf{1}-2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})\boldsymbol{\omega} & \mathbf{1}-2(1-\boldsymbol{\varepsilon}^{+}\boldsymbol{\varepsilon})^{1/2}\boldsymbol{\varepsilon}^{+}\boldsymbol{\omega} \end{bmatrix}. \tag{3·5}$$

For the cases of the monatomic crystals, where it holds that

$$\{(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})^{n}\boldsymbol{\varepsilon}\boldsymbol{\omega}^{+}\}_{a\alpha}=0 \qquad n=1,\ 2,\ 3,\cdots \tag{3·6}$$

from (2·11), (2·12) and (2·13), we get as the charge-orders

$$Q_{a\alpha}=Q_{\bar{a}\bar{\alpha}}=\tfrac{1}{2} \tag{3·7}$$

and, as the bond-order between φ_{a} and $\varphi_{\bar{a}}$,

$$Q_{a\bar{a}}=\tfrac{1}{2}(\mathbf{1}-2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{+})_{a\alpha}=\tfrac{1}{2}(1-12\mu^{2}). \tag{3·8}$$

On the other hand, for the cases of the diatomic crystals, we get

$$Q_{a\alpha}=\tfrac{1}{2}(1+2\lambda) \quad \text{and} \quad Q_{\bar{a}\bar{\alpha}}=\tfrac{1}{2}(1-2\lambda) \tag{3·9}$$

as the charge-orders, up to second order.* We see, therefore, that the ionicity of the zinc blende-type crystals is determined by λ.

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* In the InSb crystal, the valence-bonding structure ($In^{-}+Sb^{+}$) corresponds to $\lambda=0$, and the neutral structure (In+Sb) to $\lambda=0.25$. The real value of λ for InSb crystal is between 0.25 and zero.

§ 4. Energy-band structure

Recently the energy-band structures of diamond-type valence crystals have been computed in detail by many authors.[3] The theory of energy-band structure may be formulated correctly only through the theory of cohesive energy. However, some of these computations have been performed independently of the theory of cohesive energy. Therefore, it is not necessarily clear whether the computed band-structures yield reasonably good approximation for the cohesive energy. On the other hand, our method of SLCO yields the theory of the energy-band structure intimately connected with the theory of the crystal energy mentioned in the preceding sections.

We have presented in I the method to compute the energy-band structure of LiH crystal by the use of SLCO, the conduction and valence bands of which are non-degenerate respectively. On the other hand, the conduction and valence bands of the diamond-type valence crystals are four-fold degenerate respectively. However, the generalization of the computation of band structures by SLCO for such degenerate cases is straight-forward.

The valence bands of these crystals are determined by the eigenvalues of the secular equation

$$||H_{ij}(\boldsymbol{k}) - \epsilon_v(\boldsymbol{k})\,\delta_{ij}||=0, \tag{4·1}$$

where $\boldsymbol{k}$ is the wave-number vector,

$$H_{ij}(\boldsymbol{k}) = \int a^*_{i,k}(x)\,H_F a_{j,k}(x)\,d\tau, \tag{4·2}$$

and

$$a_{i,k}(x) = \frac{1}{\sqrt{N}} \sum_{\alpha}^{(N)} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\alpha}\psi_\alpha(x), \quad a=\{\alpha,\ i,\ \sigma\}^*. \tag{4·3}$$

The conduction band structures are determined by the secular equation

$$||H_{\bar{i}\,\bar{j}}(\boldsymbol{k}) - \epsilon_c(\boldsymbol{k})\,\delta_{\bar{i}\,\bar{j}}||=0, \tag{4·4}$$

where

$$H_{\bar{i}\,\bar{j}}(\boldsymbol{k}) = \int a^*_{\bar{i},k}(x)\,H_F a_{\bar{j},k}(x)\,d\tau \tag{4·5}$$

and

$$a_{\bar{i},k}(x) = \frac{1}{\sqrt{N}} \sum_{\alpha}^{(N)} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_\alpha}\psi_{\bar{\alpha}}(x). \tag{4·6}$$

The relative situation of the conduction bands to the valence bands is made clear by the following discussion.

At $\boldsymbol{k}=0$, the diagonal element $H_{\bar{i}\,\bar{i}}(0)$ is independent of i and all off-diagonal elements $H_{\bar{i}\,\bar{j}}(0)$ are equal to each other, because of the symmetry of the crystal. Therefore the conduction bands at $\boldsymbol{k}=0$ consist of a single level and a three-fold degenerated level. These two levels are separated from the diagonal energy $H_{\bar{i}\,\bar{i}}(0)$ by $3|H_{\bar{i}\,\bar{j}}(0)|$ and $|H_{\bar{i}\,\bar{j}}(0)|$ respectively, while the energy gap between them is $4|H_{\bar{i}\,\bar{j}}(0)|$. This situation holds also for the valence bands, if we replace $\bar{i}$ and $\bar{j}$ with i and j.

* We neglect spin-orbit coupling throughout. Therefore we suppress the spin coordinate in the following, because it is irrelevant to the following discussions.

For the valence bands, it has been shown that the three-fold degenerated level has higher energy than the single level. For the conduction bands, it has been believed[7] from the various theoretical and experimental studies that the energy of the three-fold degenerated level is higher for diamond and Si, and lower for Ge compared with that of the single level. Therefore the energy gap at $\boldsymbol{k}=0$ between the conduction and valence bands is given by

$$\Delta\epsilon = H_{\bar{i}\,\bar{i}}(0) - H_{ii}(0) - \tfrac{3}{4}\{|H_{ij}(0)| + |H_{\bar{i}\,\bar{j}}(0)|\} \qquad (4\cdot 7)$$

for Ge, and

$$\Delta\epsilon = H_{\bar{i}\,\bar{i}}(0) - H_{ii}(0) - \{\tfrac{3}{4}|H_{ij}(0)| + \tfrac{1}{4}|H_{\bar{i}\,\bar{j}}(0)|\} \qquad (4\cdot 8)$$

for diamond and Si.

Our method of SLCO to compute the energy-band structure is very similar to the so-called tight-binding method, but has various advantages compared with other methods (i.e., tight-binding method, OPW method and cellar-method) for the following reasons. a) The SLCO method is intimately related to the theory of cohesive energy. The one-electron effective Hamiltonian and SLCO used in this method are given through the theory of cohesive energy. b) In the other methods, the potential terms contained in their effective Hamiltonians either include some amount of ambiguity or, at least, have to be determined self-consistently. It is difficult for almost all cases to perform such self-consistent computations thoroughly. In the SLCO method, the effective Hamiltonian is determined, independent of the computation of energy-band structure, through the theory of cohesive energy. Therefore, one need not be troubled with self-consistent computations. c) In the SLCO method, the conduction and valence bands are determined respectively by separate secular equations. For diamond-type crystals, therefore, one has two 4-4 secular equations. On the other hand, one has an 8-8 secular equation in the tight-binding method.

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A General Theory of Magnetic Resonance Saturation

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The purpose of this paper is to present a general theory for describing a spin assembly uuder a nearly resonant rotating magnetic field of arbitrary strength. According to the relation between the static local field strength σ_0 and the characteristic frequency ϕ_0 of fluctuation of the environment two important cases are classified. A) If the fluctuation of the environment dominates ($\phi_0 \gg \sigma_0$), the equation becomes essentially stochastic in type and we can safely define relaxation times in Bloch's sense with a modification that the restoring force is proportional to the deviation not from the canonical distribution under the constant magnetic field but from the canonical distribution under the total magnetic field in an instantaneously static system of coordinates (i. e. the modified Bloch equation holds). B) If, on the other hand, the static local field dominates ($\sigma_0 \gg \phi_0$), then the transverse lifetime depends essentially on the strength of the rotating field, in fact it is prolonged upon saturation. This leads to a general and reasonable explanation of "saturational narrowing" and the anomalous saturational behaviour of dispersion mode such as have been observed by Redfield and others in the case of nuclear resonance in metals. The same kind of classification applies to the case in which we have two interacting different species of spin, and a general treatment will be given in a forthcoming paper.

Introduction

There are two apparently different types of theory for describing the behaviour of the magnetic resonance, i. e. the Bloch phenomenological theory[1] proposed for the case of liquids and the moment method of Van Vleck[2] proposed for solids. Bloembergen, Purcell and Pound[3] gave a link between these two types of theories by introducing the concept of "motional narrowing", and later it was put into a form more soundly based on the quantum theory of the irreversible process.[4)-8)] This was done by Bloch[4] by using a differential representation and by Kubo and Tomita[5] by using an integral representation. In both of these treatments, however, the radiofrequency field was treated as a small perturbation, which is not really the case when we use the method of saturation in measuring the lattice relaxation time. In fact several cases are found where we observe definite deviations from the results expected by using the Bloch equation. In the extremely low value region of the constant magnetic field the original Bloch equation predicts a vanishing absorption which is evidently inconsistent with the observed result, and a heuristic way of modifying the original equation was proposed by several authors.[9)-13)] A second anomaly was found in the saturation behaviour of a Gaussian type line in which the saturational decrease of the dispersion mode was found in the region of considerably higher power level than that for the absorption mode.[14] In addition the width of the

absorption narrows upon saturation before it is broadened.[14)-17)] Redfield[14)] tried to explain the anomalous behaviour of the dispersion by introducing the concept of the canonical ensemble in the rotating system of coordinates, however he did not give any explanation of the saturational narrowing. In this paper we propose to demonstrate that these anomalous results can be derived from a more general theory in a natural way. The results are classified into two typical cases. A) If the fluctuation of the lattice system is rapid enough so that we have an unambiguous definition of relaxation times, the modified Bloch eqation can be derived from the microscopic standpoint by choosing an appropriate canonical distribution as the reference. B) If the lattice system is nearly rigid and no single relaxation time can be defined, it is shown that the spin-spin relaxation time becomes considerably longer upon saturation. The saturational narrowing and the anomalous behaviour of dispersion are found to be closely related to this fact. In a practical application of the general theory we must take into account the fact that a part of the local field arises from the mutual interaction of the spins under consideration and the remaining part comes from the contact of the spin system with the system outside. As an example we treat the case of nuclear resonance in metallic copper and aluminium.

For the case in which two different species of spin system are interacting with each other again we can classify two corresponding situations. As a result of the so-called "double resonance" a general type of the Overhauser effect[18)] is derived when the lattice is fluctuating strongly, whereas the saturational narrowing[19)] is derived for the case of nearly rigid lattice.

§ 1. Fundamental equation

Here we choose a differential representation[4)]* and try to find a differential equation which is obeyed by the density matrix describing the spin system only by starting with the density matrix describing the whole system. The spin system has a Larmor frequency ω_0 under a constant magnetic field H_0. Suppose in addition there exists a transverse, rotating magnetic field which is nearly resonant in its frequency ω and have amplitude H_1. The density matrix ρ describing the whole system changes with time according to the equation

$$\frac{d\rho}{dt}=\frac{i}{\hbar}[\rho, \mathcal{H}] \tag{1·1}$$

where $\mathcal{H}$ is the Hamiltonian for the total system and is given by

$$\mathcal{H}=\hbar(E+D+F+G). \tag{1·2}$$

Here E is the Zeeman energy

$$E=-\gamma H_0 I_z=-\omega_0 I_z \qquad (\boldsymbol{I}=\sum_i \boldsymbol{I}_i), \tag{1·3}$$

* An approach using an integral representation[5)] is also conceivable, however we need the density matrix specifying the initial condition, which is not obvious from the outset and has to be calculated by using a differential representation anyway.

D is the energy of the system with respect to the rotating field ($\gamma H_1 \equiv \omega_1$)

$$D = -\omega_1 (I_x \cos \omega t - I_y \sin \omega t),$$

and G is the interaction between the spin system and the system outside, which is in general written in a form

$$G = \sum_{\mu} I^{\mu} F^{\bar{\mu}} \tag{1·4}$$

where I^{μ} denotes a spin operator associated with μ-quantum jump and $\bar{\mu}$ stands for $-\mu$. F is the Hamiltonian describing the pure lattice. These notations are similar to those of Bloch, however we treat the rotating field exactly and not as a small perturbation. First we change the frame of reference to the one which is rotating with the rotating field H_1 by way of the transformation

$$A^T = e^{iE't} A e^{-iE't} \tag{1·5}$$

where

$$E' = -\omega I_z. \tag{1·6}$$

As a result of this transformation to the rotating frame the equation of motion in the new frame becomes

$$\frac{d\rho^T}{dt} = i[\rho^T,\ E^{Te} + F + G^T] \tag{1·7}$$

where E^{Te} is the Zeeman energy of system with respect to the effective field in the rotating frame, (cf. Fig. 1) i. e.

$$\begin{aligned} E^{Te} &= -\{(\omega_0 - \omega) I_z + \omega_1 I_x\} \\ &= -\{\Delta I_z + \omega_1 I_x\} \qquad (\Delta \equiv \omega_0 - \omega). \end{aligned} \tag{1·8}$$

G^T is the form of interaction in the rotating frame and is given by

$$G^T = \sum_{\mu} I^{\mu T} F^{\bar{\mu}} = \sum_{\mu} I^{\mu} F^{\bar{\mu}} e^{i\omega_{\mu} t}, \tag{1·9}$$

where $\omega_{\mu} \equiv -\mu\omega$ according to the definition of the operator I^{μ}.

Under the assumption that our system is essentially paramagnetic, the part G is considered to be a small perturbation as compared with the other parts of the Hamiltonian, and again we transform to the interaction representation (or the nutating frame of reference) by way of the operation

$$A^* = e^{i(E^{Te} + F)t} A e^{-i(E^{Te} + F)t}. \tag{1·10}$$

The density ρ^{T*} in the new frame obeys the equation

$$\frac{d\rho^{T*}}{dt} = i[\rho^{T*}, G^{T*}] \tag{1·11}$$

where

$$G^{T*}=\sum_{\mu} I^{\mu*}F^{\bar{\mu}*}e^{i\omega_\mu t}=\sum_{\lambda}\sum_{\mu} I^{\mu*}F^{\bar{\mu}}(\lambda)\,e^{i(\omega_\lambda+\omega_\mu)t}$$

$$=\sum_{\lambda}\sum_{\mu}\sum_{\nu} a_{\mu\nu}I(\nu)\,F^{\bar{\mu}}(\lambda)\,e^{i(\omega_\lambda+\omega_\nu+\omega_\nu)t} \qquad (1\cdot12)$$

and

$$F^{\mu*}=e^{iFt}F^{\bar{\mu}}e^{-iFt}\equiv\sum_{\lambda}F^{\bar{\mu}}(\lambda)\,e^{i\omega_\lambda t},$$

$$I(\nu)^*=e^{iE^{T*}t}I(\nu)e^{-iE^{T*}t}=\sum_{\nu}I(\nu)\,e^{i\omega_\nu t}. \qquad (1\cdot13)$$

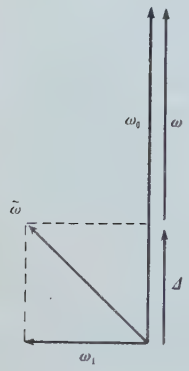

Fig. 1. Various quantities in the rotating system of coordinates.

In order to have a simple result as the Fourier transform (1·13) we have again changed the system of coordinate to the one in which the total effective field lies along the z'-axis. Here $a_{\mu\nu}$ is a function of the angle θ between z- and z'- axis, and

$$\tilde{I}^{\mu}\equiv e^{i\theta I_y}I^{\mu}e^{-i\theta I_y}=\sum_{\nu}a_{\mu\nu}\,I(\nu). \qquad (1\cdot14)$$

If we write down the short time development of the density ρ^{T*} according to the equation (11), we have

$$\rho^{T*}(t)=\rho^{T*}(0)+\Delta_1\rho^{T*}(0)+\Delta_2\rho^{T*}(0)+\cdots$$

$$=\rho^{T*}(0)-i\int_0^t dt_1\,[G^{T*}(t_1),\,\rho^{T*}(0)]$$

$$-\int_0^t dt_1\int_0^{t_1}dt_2[G^{T*}(t_1),[G^{T*}(t_2),\rho^{T*}(0)]]. \qquad (1\cdot15)$$

Let us suppose that the lattice degree of freedom is large enough so that the density matrix describing it is practically independent of the spin state, then the total density ρ can be factorized into those of the spin system σ and the lattice system P_F,

$$\rho=\sigma\times P_F, \qquad (1\cdot16)$$

where P_F may be assumed as the canonical distribution with respect to the lattice

$$P_F=\exp\,\beta(\Omega-\hbar F); \qquad (1\cdot17)$$

$$\beta\equiv 1/kT \qquad \text{and} \qquad \exp-\beta\Omega\equiv\mathrm{Tr}\,\exp-\beta\hbar F.$$

If we take the average of the initial condition over the lattice states, then we have the density matrix describing the spin system as follows.

$$\sigma^{T*}(t)=\sigma^{T*}(0)+\Delta_1\sigma^{T*}(0)+\Delta_2\sigma^{T*}(0)+\cdots$$

$$=\sigma^{T*}(0)-i\int_0^t dt_1\,[\langle G^{T*}(t_1)P_F\rangle_F,\;\sigma^{T*}(0)]$$

$$-\int_0^t dt_1\int_0^{t_1}dt_2\langle[G^{T*}(t_1),[G^{T*}(t_2),\;\sigma^{T*}(0)P_F]]\rangle_F \qquad (1\cdot18)$$

where $\langle\ \rangle_F$ denotes the trace operation after multiplying the lattice density P_F. According to the definition of the trace

$$\langle G^{T*}(t_1)\rangle_F=\sum_{\nu}e^{i\omega_\nu t}\,a_{o\nu}I(\nu)\langle F^0(0)\rangle_F \qquad (1\cdot19)$$

in which only the terms corresponding to $\omega_\mu=0$ are surviving as we have neglected the satellites. The second order term becomes

$$\begin{aligned}\Delta_2\sigma^{T*}(0)=-\sum_\lambda\sum_{\lambda'}\sum_\mu\sum_{\mu'}\sum_\nu\sum_{\nu'}a_{\mu\nu}\,a_{\mu'\nu'}\int_0^t dt_1\int_0^{t_1}dt_2 e^{i(\omega_\lambda+\omega_\mu+\omega_\nu)t_1+i(\omega_{\lambda'}+\omega_{\mu'}+\omega_{\nu'})t_2}\\ \times\{\langle F^{\bar\mu}(\lambda)F^{\bar\mu'}(\lambda')\rangle I(\nu)I(\nu')\sigma^{T*}(0)\\ +\langle F^{\bar\mu'}(\lambda')F^{\bar\mu}(\lambda)\rangle\sigma^{T*}(0)I(\nu')I(\nu)\\ -\langle F^{\bar\mu'}(\lambda')F^{\bar\mu}(\lambda)\rangle I(\nu)\sigma^{T*}(0)I(\nu')\\ -\langle F^{\bar\mu}(\lambda)F^{\bar\mu'}(\lambda')\rangle I(\nu')\sigma^{T*}(0)I(\nu)\}.\end{aligned}\tag{1·20}$$

According to the definition of the trace $\omega_\lambda+\omega_{\lambda'}=0$, and if we neglect the satellites $\omega_\mu+\omega_{\mu'}=0$ is required. Suppose we concentrate our attention on a variation which appears over a period including many fluctuations of the lattice, then the above equation can be rendered into an approximate differential equation for the spin density

$$\frac{d}{dt}\sigma^{T*}=\frac{d}{dt}\Delta_1\sigma^{T*}+\frac{d}{dt}\Delta_2\sigma^{T*},\tag{1·21}$$

where

$$\frac{d}{dt}\Delta_1\sigma^{T*}=-i\sum_\nu e^{i\omega_\nu t}\,a_{o\nu}\langle F^0(0)\rangle_F[I(\nu),\sigma^{T*}]\tag{1·22}$$

and

$$\begin{aligned}\frac{d}{dt}\Delta_2\sigma^{T*}=-\sum_\mu\sum_\nu\sum_{\nu'}a_{\mu\nu}\,a_{\bar\mu\nu'}e^{i(\omega_\nu+\omega_{\nu'})t}\sum_\lambda\int_0^t d\tau\,e^{i(\omega_\lambda+\omega_\mu-\omega_{\nu'})\tau}\\ \times[\langle F^{\bar\mu}(\lambda)F^{\mu}(\bar\lambda)\rangle_F\{I(\nu)I(\nu')\sigma^{T*}-I(\nu')\sigma^{T*}I(\nu)\}\\ +\langle F^{\mu}(\bar\lambda)F^{\bar\mu}(\lambda)\rangle_F\{\sigma^{T*}I(\nu')I(\nu)-I(\nu)\sigma^{T*}I(\nu')\}].\end{aligned}\tag{1·23}$$

Coming back to the precessing frame, we now write down the differential equation for the expectation value of an operator Q^T which is associated with the spin system as follows.

$$\begin{aligned}\frac{d}{dt}\langle Q^T\rangle&=-i\langle[Q^T,\ E^{Te}]\rangle-i\sum_\nu a_{o\nu}\langle F^0(0)\rangle_F\langle[Q_T,\ I(\nu)]\rangle\\ &\quad-\sum_\mu\sum_\nu\sum_{\nu'}a_{\mu\nu}\,a_{\bar\mu\nu'}\sum_\lambda\int_0^t d\tau\,e^{i(\omega_\lambda+\omega_\mu-\omega_{\nu'})\tau}\\ &\quad\times\{\langle F^{\bar\mu}(\lambda)F^{\mu}(\bar\lambda)\rangle_F\langle[Q^T,\ I(\nu)]I(\nu')\rangle-\langle F^{\mu}(\bar\lambda)F^{\bar\mu}(\lambda)\rangle_F\langle I(\nu')[Q^T,I(\nu)]\rangle\}\\ &=-i\langle[Q^T,\ E^{Te}]\rangle-i\sum_\nu a_{0\nu}\langle F^0(0)\rangle_F\langle[Q^T,\ I(\nu)]\rangle\\ &\quad-\sum_\mu\sum_\nu\sum_{\nu'}a_{\mu\nu}\,a_{\bar\mu\nu'}\int_0^t d\tau\,e^{i(\omega_\mu-\omega_{\nu'})\tau}\\ &\quad\times\{\langle F^{\bar\mu}(\tau)F^{\mu}(0)\rangle_F\langle[Q^T,\ I(\nu)]I(\nu')\rangle-\langle F^{\mu}(0)F^{\bar\mu}(\tau)\rangle_F\langle I(\nu')[Q^T,\ I(\nu)]\rangle\}\end{aligned}\tag{1·24}$$

where

$$\langle F^{\bar{\mu}}(\tau)F^{\mu}(0)\rangle_F \equiv \sum_{\lambda} e^{i\omega_\lambda \tau}\ \langle F^{\bar{\mu}}(\lambda)F^{\mu}(\bar{\lambda})\rangle_F$$
$$\langle F^{\mu}(0)F^{\bar{\mu}}(\tau)\rangle_F \equiv \sum_{\lambda} e^{i\omega_\lambda \tau}\ \langle F^{\mu}(\bar{\lambda})F^{\bar{\mu}}(\lambda)\rangle_F. \qquad (1\cdot 25)$$

At this point we introduce an abbreviation for the integral which is involved in (1·23)

$$\Phi^{\mu\nu} \equiv \int_0^t d\tau e^{i(\omega_\mu+\omega_\nu)\tau}\langle F^{\mu}(0)F^{\bar{\mu}}(\tau)\rangle_F \equiv \Phi'^{\mu\nu} + i\Phi''^{\mu\nu}. \qquad (1\cdot 26)$$

It should be remembered that $\Phi^{\mu\nu}$ depends explicitly on the intensity of the radiofrequency field, and so the situation is essentially different from that found with the original Bloch equation where the rate of relaxation does not depend on the saturation level. The behaviour of $\Phi^{\mu\nu}$ is determined by the correlation function $\langle F^{\mu}(0)\ F^{\bar{\mu}}(\tau)\rangle_F$ of the lattice part of the interaction, which is in general a rapidly decaying function of time. Suppose this correlation function is simple exponential in type, i.e.

$$\langle F^{\mu}(0)F^{\bar{\mu}}(\tau)\rangle_F = \sigma_\mu{}^2 e^{-\phi_\mu \tau}\ , \qquad (1\cdot 27)$$

where

$$\sigma_\mu{}^2 = \langle |F^{\mu}(0)|^2\rangle_F\ , \qquad (1\cdot 28)$$

then we have

$$\Phi^{\mu\nu} = \sigma_\mu{}^2 \int_0^t d\tau\ e^{i(\omega_\mu+\omega_\nu)\tau - \phi_\mu \tau}$$
$$= \sigma_\mu{}^2 \left(\frac{e^{\{i(\omega_\mu+\omega_\nu)-\phi_\mu\}t}-1}{i(\omega_\mu+\omega_\nu)-\phi_\mu} \right). \qquad (1\cdot 29)$$

We remember that $\Phi^{\mu\nu}$ stands for the rate of the phase variation of $\langle Q^T\rangle$, so that $\langle Q^T\rangle$ becomes practically vanishing as soon as the product $\Phi^{\mu\nu}t$ becomes greater than unity. According to the relative magnitude of the four parameters, i.e. σ_μ, ω_μ, ω_ν and ϕ_μ, two typical cases are discriminated as the asymptotic behaviour of $\Phi^{\mu\nu}$.

(A) The case in which Max $(\omega_\mu, \omega_\nu, \phi_\mu) \gg \sigma_\mu$.

$$\Phi^{\mu\nu} \sim \sigma_\mu{}^2 \left(\frac{\phi_\mu + i(\omega_\mu+\omega_\nu)}{\phi_\mu{}^2 + (\omega_\mu+\omega_\nu)^2} \right). \qquad (1\cdot 30)$$

(B) The case in which Max $(\omega_\mu, \omega_\nu, \phi_\mu) \ll \sigma_\mu$.

$$\Phi^{\mu\nu} \sim \sigma_\mu{}^2 t. \qquad (1\cdot 31)$$

Owing to the condition $\omega_0 \gg \sigma_\mu$, which is valid except in the case of very weak constant field, the case $\mu \neq 0$ belongs to the situation (A) and $\Phi^{\mu\nu}$ is approximately independent of the saturation level, i.e.

$$\Phi^{\mu\nu} \doteqdot \Phi^{\mu} \equiv \sigma_\mu{}^2 \left(\frac{\phi_\mu + i\omega_\mu}{\omega_\mu{}^2 + \phi_\mu{}^2} \right) = \Phi'^{\mu} + i\Phi''^{\mu}. \qquad (1\cdot 32)$$

Either situation (A) or (B) can exist for the case $\mu = 0$.

As a special case of the above formulae, i.e. when the radiofrequency field is very small, we have the familiar result hitherto obtained by putting $a_{\mu\nu} = \delta_{\mu\nu}$, and $\omega_1 \to 0$. In this particular limit there appears no dependence on the saturation level, and the transition from situation (B) to (A) due to the change in the relative magnitude of

ϕ_0 and σ_0 is what is called "motional narrowing." In this paper we include the magnitude of the radiofrequency field explicitly and discuss the situation under saturation. The first order shift will be included into the resonance frequency itself, because it is not substantially dependent on the saturation level.

§ 2. Relaxation due to the local field

When I^μ stands for a component of a single spin vector in the expression of the interaction G, then the remaining part is considered as a local field acting on it. The hyperfine interaction between a nucleus and an electron is an example. In this case $I(\nu)$ is again a single spin operator and the coordinate transformation formula (1.14) is explicitly given by

$$\begin{bmatrix} \tilde{I}^0 \\ \tilde{I}^+ \\ \tilde{I}^- \end{bmatrix} = \begin{bmatrix} \cos\theta & -(\sin\theta)/2 & -(\sin\theta)/2 \\ \sin\theta & (\cos\theta+1)/2 & (\cos\theta-1)/2 \\ \sin\theta & (\cos\theta-1)/2 & (\cos\theta+1)/2 \end{bmatrix} \begin{bmatrix} I(0) \\ I(+) \\ I(-) \end{bmatrix}. \tag{2·1}$$

Here we choose the resultant spin vector as Q^τ and derive a differential equation for the expectation value. In this case $[Q^\tau, I(\nu)]\, I(\nu')$ involves the product of three spin operators in general, so that the equation (1·24) is not linear in the component of Q^τ. However, it does become linear if we have the situation

$$\hbar\omega_0 \ll kT. \tag{2·2}$$

Retaining the terms up to the first order in $\beta\hbar\omega$, we have the expression

$$-\sum_\mu \sum_\nu \sum_{\nu'} a_{\mu\nu} a_{\bar{\mu}\nu'} \Phi^{\bar{\mu}\nu'} \left\{ \langle [[Q^\tau, I(\nu)]I(\nu')] \rangle - \beta\hbar(\mu\omega - \nu'\tilde{\omega}) \langle I(\nu')[Q^\tau, I(\nu)] \rangle_0 \right\} \tag{2·3}$$

for the third term in (1·24), where $\langle\ \rangle_0$ denotes the operation of taking a simple trace by replacing the density by unity. Thus we obtain the following linear equation in the rotating system of coordinate as a generalization of the Bloch equation.

$$\frac{d}{dt}\begin{bmatrix} I_x \\ I_y \\ I_z \end{bmatrix} + \begin{bmatrix} \Phi_x & -\Delta+\Phi_{xy} & \Phi_{xz} \\ \Delta+\Phi_{yx} & \Phi_y & -\omega_1+\Phi_{yz} \\ \Phi_{zx} & \omega_1+\Phi_{zy} & \Phi_z \end{bmatrix} \begin{bmatrix} I_x \\ I_y \\ I_z \end{bmatrix} = \frac{\beta\hbar I(I+1)}{3} \begin{bmatrix} \Phi_x^* \tilde{\Phi}_x \\ \Phi_y^* \tilde{\Phi}_y \\ \Phi_z^* \tilde{\Phi}_z \end{bmatrix} \begin{bmatrix} \omega \\ \tilde{\omega} \end{bmatrix}. \tag{2·4}$$

Here the relaxation tensors Φ, Φ^* and $\tilde{\Phi}$ are expressed as linear combinations of the quantities $\Phi^{\mu\nu}$ defined by (1·26) as is given in Table 1. In particular the expression becomes simpler when the resonance absorption is well-defind, i. e. (1·32) holds, and the result is given in Table 2. In this case we have the situation in which $\Phi_x = \Phi_y = 1/T_2$ which is dependent on the deviation in frequency from the resonance. In contrast $\Phi_1 = 1/T_1$ does not depend on the deviation. Remembering $s\tilde{\omega} = \omega_1$ and $\omega + c\tilde{\omega} = \omega_0$, we realize that there acts a restoring force which is proportional to the deviation from the equilibrium polarization corresponding to the total instantaneous field, i. e. including the rotating component. The shift in the resonance frequency consists of the first and the second order contribution and is given by

Table 1. Relaxation constants (the quantities in the first row) expressed as linear combinations of $\Phi^{\mu\nu}$. The coefficients are tabulated using the abbreviations $c \equiv \cos\theta$ and $s \equiv \sin\theta$, where θ is the angle between z- and z'-axis. The bar on the top indicates a minus sign.

	Φ^{00}	Φ'^{01}	Φ'^{10}	Φ'^{11}	$\Phi'^{1\bar{1}}$
Φ_x	c^2	s^2		$1+c$	$1-c$
Φ_y	c^2	s^2	$2s^2$	$c(c+1)$	$c(c-1)$
Φ_z			$2s^2$	$(c+1)^2$	$(c-1)^2$
$\Phi_x{}^*$			$2cs$	$s(c+1)$	$s(c-1)$
$\tilde{\Phi}_x$		s		$s(c+1)$	$\overline{s(c-1)}$
$\Phi_z{}^*$			$2s^2$	$(c+1)^2$	$(c-1)^2$
$\tilde{\Phi}_z$				$(c+1)^2$	$\overline{(c-1)^2}$
Φ_{xz}	cs	cs			
Φ_{zx}			$2cs$	$s(c+1)$	$s(c-1)$

	Φ''^{01}	Φ''^{10}	Φ''^{11}	$\Phi''^{1\bar{1}}$
$\Phi_y{}^*$		$2cs$	$s(c+1)$	$s(c-1)$
$\tilde{\Phi}_y$	$\overline{cs}$		$s(c+1)$	$\overline{s(c-1)}$
Φ_{xy}		$\overline{2s^2}$	$c(\overline{c+1})$	$c(\overline{c-1})$
Φ_{yx}			$(c+1)$	$c-1$
Φ_{yz}	s			
Φ_{zy}		$\overline{2cs}$	$s(c+1)$	$s(c-1)$

Table 2. Relaxation constants (the quantities in the first row) expressed as linear combinations of $\Phi^{\mu\nu}$, using the same notations as in Table 1. (The case in which $\Phi^{10}=\Phi^{11}=\Phi^{1\bar{1}}=\Phi^{1}$).

	Φ^{00}	Φ'^{01}	Φ'^{1}
Φ_x	c^2	s^2	2
Φ_y	c^2	s^2	2
Φ_z			4
$\Phi_x{}^*$			
$\tilde{\Phi}_x$		s	$2s$
$\Phi_z{}^*$			4
$\tilde{\Phi}_z$			$4c$
Φ_{xz}	$\overline{cs}$	cs	
Φ_{zx}			

	Φ''^{01}	Φ''^{1}
$\Phi_y{}^*$		
$\tilde{\Phi}_y$	$\overline{cs}$	$2cs$
Φ_{xy}		-2
Φ_{yx}		-2
Φ_{yz}	$\overline{s}$	
Φ_{zy}		

$$\Delta\omega_0=\langle F^0(0)\rangle_F+2\Phi'^1. \tag{2·5}$$

$\Phi_{xy}=\Phi_{yx}=-2\Phi'^1$ denotes the shift in the precessional frequency due to the second order perturbation, i. e. the self-energy effect, and has a positive sign in general (always accelerating the precession). However, the imaginary part of $\Phi^{\mu\nu}$ is much smaller than the real part, because we have the situation $\omega_0\gg(\phi_\mu,\ \omega_1)$. Thus we can safely neglect the non-adiabatic shift and have the simplified equation

$$\frac{d}{dt}\begin{bmatrix}I_x\\I_y\\I_z\end{bmatrix}+\begin{bmatrix}\Phi_x & -\Delta & \Phi_{xz}\\ \Delta & \Phi_y & -\omega_1\\ 0 & \omega_1 & \Phi_z\end{bmatrix}\begin{bmatrix}I_x\\I_y\\I_z\end{bmatrix}=\frac{\beta\hbar I(I+1)}{3}\begin{bmatrix}\Phi_x\omega_1\\0\\\Phi_z\omega_0\end{bmatrix}. \tag{2·6}$$

At this stage the quantities Φ^{00} and Φ'^{01} depend on time, although Φ'^1 does not, therefore it is not easy to solve (2·6) in general. However, the case of vanishing radio-frequency field has already been solved by our relaxation function method,[5] so that along the same line we consider several limiting cases in which $\Phi^{\mu\nu}$ can be simplified. The following two extreme cases can be discriminated according to the effect of the adiabatic perturbation corresponding to $\mu=0$.

§ 2. A. The case of rapidly fluctuating lattice. $(\sigma_0\ll\phi_0)$

When the external system fluctuates more rapidly than the relative precession due to the distribution of the local field, it belongs to the situation (A) and $\Phi^{0\nu}$ is given by

$$\Phi^{0\nu}=\sigma_0^2\left(\frac{\phi_0+i\omega_\nu}{\omega_\nu^2+\phi_0^2}\right)\qquad(\nu=0,1). \tag{2·7}$$

Therefore we have a perfect definition of relaxation constants including the effect of the non-adiabatic perturbation as it is given in Tables 1 and 2. In particular, if we have the condition $\phi_0\gg\tilde{\omega}$, the relaxation constants do not depend on the intensity of radio-frequency field and are given by

$$\begin{cases}\Phi_x=\Phi_y=\Phi'^0+2\Phi'^1, & (2·8)\\ \Phi_z=\qquad 4\Phi'^1. & (2·9)\end{cases}$$

In the special case of extreme narrowing, i. e. $\phi_\mu\gg\omega_0$, we have $\Phi'^0:\Phi'^1=2:1$ and

$$\Phi_x=\Phi_y=\Phi_z, \tag{2·10}$$

which is an indication of complete isotropy. It is evident that the equation (2·6) reduces to the original Bloch equation[1] when the intensity of the radiofrequency field is negligible, however the equation is different from the Bloch equation in the fact that it contains Φ_{xz} and there appears ω_1 on the right-hand side of the equation. Thus the modification of the Bloch equation, which was proposed heuristically by several authors,[9)–13] has been derived in a natural manner. The meaning of this modification becomes clear when we look into the steady state solution of (2·6) which is given by

1. Dispersion (x-component) :

$$I_x=\frac{\hbar\omega}{3kT}I(I+1)\left\{\frac{\omega_1\Phi_z\Delta}{D}+\frac{\omega_1}{\omega}+\frac{\Phi_{xz}(\omega_1^2\Delta-\Phi_z\Phi_y\omega_0)}{D\omega}\right\}, \qquad (2\cdot11)$$

2. Absorption (y-component) :

$$I_y=\frac{\hbar\omega}{3kT}I(I+1)\left\{\frac{\omega_1\Phi_z\Phi_x}{D}+\frac{\Phi_{xz}}{D\omega}(\Phi_z\Delta\omega_0)\right\}, \qquad (2\cdot12)$$

3. z-polarization (z-component) :

$$I_z=\frac{\hbar\omega}{3kT}I(I+1)\left\{\frac{\Phi_z(\Phi_x\Phi_y+\Delta^2)}{D}+\frac{\Delta}{\omega}+\frac{\Phi_{xz}}{D\omega}(\Delta^2\omega_1)\right\}, \qquad (2\cdot13)$$

where

$$D\equiv\Phi_z\Delta^2+\Phi_x\Phi_y\Phi_z+\omega_1^2\Phi_x-\Phi_{xz}\Delta\omega. \qquad (2\cdot14)$$

Φ_{xz} vanishes in the case of exact resonance or vanishing radiofrequency field. It has a negative sign in general, but in most cases it can be neglected, for $\Phi^{00}\doteq\Phi^{01}$ owing to the condition $\tilde{\omega}\ll\phi_0$. Thus the stationary solution becomes nearly Lorentzian in the shape of absorption, but it differs from the Bloch solution in the fact that (i) we have ω instead of ω_0 outside the bracket and (ii) we have additional second terms inside the brackets of (2·11) and (2·13). These are small but important differences. The reasons are : (i) In the limit of vanishing constant field our solution reduces naturally to the Debye-type formula, whereas the original Bloch solution does not. In this sense our modification corresponds exactly to the modification of Lorentz theory by Van Vleck and Weisskopf,[20] and Karplus and Schwinger[21] in the theory of pressure broadening. In general shift and intensity change are expected as the result of this modification which has actually been observed in the low field experiments. However, at a high value of the resonance field the modification becomes trivial and we can safely replace ω by ω_0 outside the bracket, thus having a simple Lorentzian shape of absorption. (ii) The steady polarization to be observed at exact resonance can be described simply by saying that "there exists a canonical distribution in an appropriately chosen system of coordinates". This was anticipated by Redfield,[14]* but again was not the case for the Bloch solution. The detailed results are shown in Table 3, where the column (b) indicates the effective relaxation mechanisms which act against the nutational coherence under the given condition. (cf. Figure 2) It should be noted that the relaxation parameters do not depend on the power level of saturation so far as $\omega_1\ll\phi_0$. The effect of saturation is the simple saturation broadening which can be described by the modified Bloch equation. However, when the intensity of radiofrequency field becomes so large that $\omega_1\gg\phi_0$ holds, the spin-spin relaxation decreases with increasing power level in the following manner,

* Originally Redfield introduced this type of description not for our case (A), but for our case (B) which will be discussed in § 2. B and § 3. B.

$$\Phi_x = \Phi_y = \sigma_0^2 \tau_0 \left\{ \cos^2\theta + \frac{\sin^2\theta}{1+\tilde{\omega}^2\tau_0^2} \right\} + \frac{\sigma_1^2 \tau_1}{1+\omega^2\tau_1^2}, \quad (\tau_\mu = \phi_\mu^{-1}) \qquad (2\cdot 15)$$

or at exact resonance

$$\Phi_x = \Phi_y = \frac{\sigma_0^2 \tau_0}{1+\omega_1^2\tau_0^2} + \frac{\sigma_1^2 \tau_1}{1+\omega_0^2\tau_1^2}. \qquad (2\cdot 16)$$

It means that the line shape of the absorption deviates from the simple Lorentzian type, however it should be borne in mind that not every kind of deviation from the Bloch equation can be attributed to this type of effect. In fact it is rather rare that the situation appears in which $\omega_1 \gg \phi_0 \gg \sigma_0$, and no example of this type of effect seems to have been observed yet* (cf. Figure 3).

Usually the frequency dependent susceptibility $\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$ is defined under an excitations due to linearly polarized radiation, and it satisfies the symmetry condition

$$\chi'(-\omega) = \chi'(\omega), \quad \chi''(-\omega) = -\chi''(\omega). \qquad (2\cdot 17)$$

To satisfy these conditions we have only to augment the above treatment by introducing a counter-rotating magnetic field of the equal amplitude, and the results are given by

$$\chi'(\omega) = \frac{\chi_0}{2} \left\{ 2 + \frac{\omega(\omega_0-\omega)}{(\omega_0-\omega)^2 + \Phi_x\Phi_y(1+s)} - \frac{\omega(\omega_0+\omega)}{(\omega_0+\omega)^2 + \Phi_x\Phi_y(1+s)} \right\}, \qquad (2\cdot 18)$$

$$\chi''(\omega) = \frac{\chi_0}{2}\omega\Phi_x \left\{ \frac{1}{(\omega_0-\omega)^2 + \Phi_x\Phi_y(1+s)} + \frac{1}{(\omega_0+\omega)^2 + \Phi_x\Phi_y(1+s)} \right\}, \qquad (2\cdot 19)$$

Table 3. Steady state polarization described in terms of an appropriate canonical distribution. (The description in (C)-(c) is not always reliable, because we neglected the terms corresponding to $\omega_\mu + \omega_{\mu'} \neq 0$).

(a) The condition for the intensity of the static field.		(b) The condition under which the steady state polarization is described by a canonical distribution.	(c) The system of coordinates to which the canonical distribution is referred.
(A) $\omega_0 > (\Phi_y^2/\Phi_z)$	(A1)	$\omega_1 > (\omega_0\Phi_z)^{1/2}$	Rotating system.
	(A2)	$(\omega_0\Phi_z)^{1/2} > \omega_1 > (\Phi_y\Phi_z)^{1/2}$	None.
	(A3)	$(\Phi_y\Phi_z)^{1/2} > \omega_1$	Static system.
(B) $(\Phi_y^2/\Phi_z) > \omega_0 > (\Phi_y\Phi_z)^{1/2}$	(B1)	$\omega_1 > (\omega_0\Phi_y\Phi_z)^{1/3}$	Rotating system.
	(B2)	$(\omega_0\Phi_y\Phi_z)^{1/3} > \omega_1 > (\Phi_y\Phi_z)^{1/2}$	None.
	(B3)	$(\Phi_y\Phi_z)^{1/2} > \omega_1$	Static system.
(C) $(\Phi_y\Phi_z)^{1/2} > \omega_0$	(C1)	$\omega_1 > (\Phi_y\Phi_z)^{1/2}$	Rotating system.
	(C2)	$(\Phi_y\Phi_z)^{1/2} > \omega_1 > \omega_0$	Instantaneous system.
	(C3)	$\omega_0 > \omega_1$	Static system.

* Bloch[5] ascribed Redfield's results to this case, however his interpretation seems unjustified to the present author.

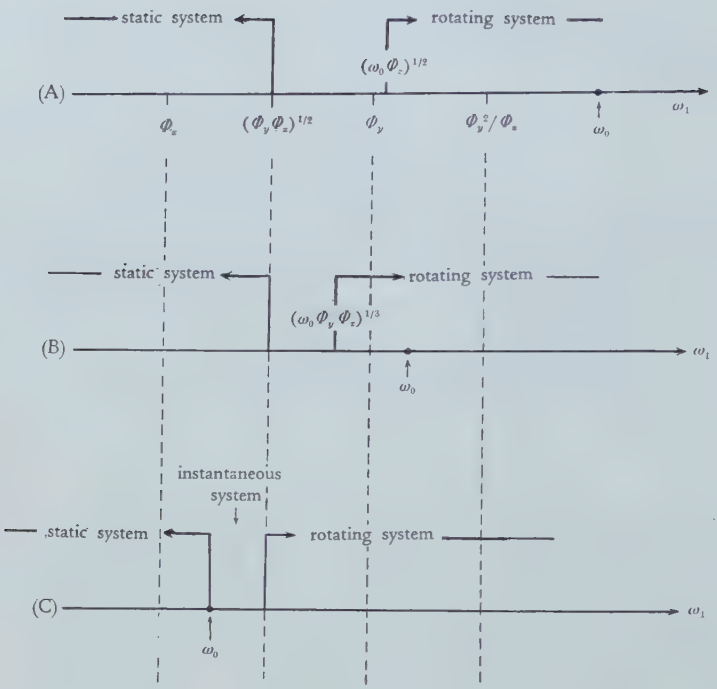

Fig. 2. The system of coordinates in which a canonical distribution can be used to describe the steady state solution.

where the quantity

$$s=\omega_1 T_z T_y \qquad (2\cdot 20)$$

indicates the degree of saturation. Obviously it reduces to the Debye type formula when the constant field is vanishing, whereas at a high value of the constant field the term involving $\omega_0+\omega$ can be neglected as compared with that involving $\omega_0-\omega$, and the main feature is described by (2·11) to (2·13). The formulae (2·18) and (2·19) satisfy the Kramers-Kronig relation when the saturation factor s is neglected, but with s they do not satisfy the relation owing to the loss of linearity upon saturation.

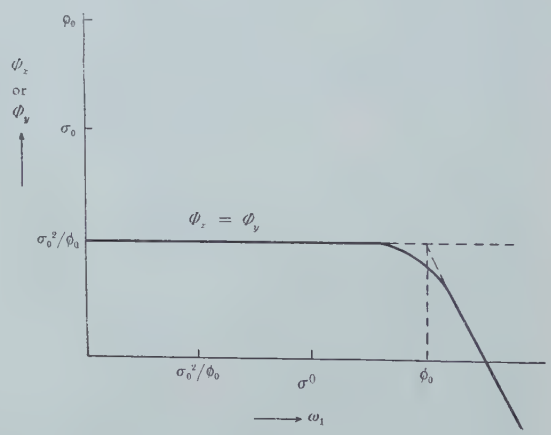

Fig. 3. Dependence of the transverse relaxation upon the degree of saturation. Case (A) in the local field model. (logarithmic scale)

§ 2. B. The case of rigid lattice. ($\sigma_0 \gg \phi_0$)

When the external system does not actually fluctuate over many periods of the relative precession due to the distribution of local field, there is no assurance in general that we can draw any substantial information about the time variation of density matrix out of its short time behaviour. However, it does yield valuable information in a special case in which the origin of the width is due to many perturbers which have regular static arrangement.* A Gaussian shape of distribution is expected in this case as a result of the central limit theorem, therefore one can reproduce the whole line shape once he has the value of the adiabatic second moment. Being aware of this restriction, we here limit ourselves to this special case, then the case $\mu=0$ belongs to the situation (B), and we have

$$\Phi^{0\nu} \fallingdotseq \sigma_0^2 t \tag{2·21}$$

which determines the main feature of the time variation, because $\Phi^{\mu\nu} \ll \sigma_0$ for $\mu \neq 0$ according to (1·32). The shape of the absorption is given by

$$f(\omega) = \left(\frac{1}{2\pi\langle\omega^2\rangle}\right)^{1/2} e^{-\frac{(\omega-\omega_0)^2}{2\langle\omega^2\rangle}} \tag{2·22}$$

in this limit, and we cannot define a single relaxation time. On the other hand we know that the situation must be changed into (A) in the limit of extremely strong radiofrequency field, where we have

$$\Phi'^{0\nu} \fallingdotseq \sigma_0^2 \left(\frac{\phi_0}{\omega_\nu^2 + \phi_0^2}\right). \tag{2·23}$$

At this point we try to interpolate between the two limits. In order to observe the qualitative behaviour we may replace the Gaussian limit by a corresponding Lorentzian line which has an equivalent half-value width. An example of the desired interpolation is given by

$$\Phi^{0\nu} \fallingdotseq \sigma_0' \{1 - s_\nu\}, \tag{2·24}$$

where

$$s_\nu \equiv \exp\left(-\frac{\sigma_0'' \phi_0}{\phi_0^2 + \omega_\nu^2}\right) \tag{2·25}$$

represents the saturation factor. σ_0' is the half-value width in the Gaussian limit, and σ_0'' is defined by the relation

$$\sigma_0^2 = \sigma_0' \sigma_0'', \tag{2·26}$$

where

$$\sigma_0' = 1.175\, \sigma_0. \tag{2·27}$$

The point of introducing the above sort of transcription lies in the fact that in this way we may use the Bloch type equation and its solution formally even for the case of rigid

* When the perturbers are located irregularly, or at random, we cannot expect a Gaussian shape of distribution, even if the arrangement is static.[22]

lattice. However, the transverse relaxations are now dependent on the power level of the radiofrequency, i. e.

$$\Phi_x=\Phi_y=\frac{\sigma_0'}{\Delta^2+\omega_1^2}\left[\Delta^2+\omega_1^2\cdot\left\{1-\exp\left(-\frac{\sigma_0''\phi_0}{\phi_0^2+\omega_1^2}\right)\right\}\right]+\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}, \tag{2·28}$$

although the longitudinal relaxation does not depend on the saturation level. Consequently, the behaviour of the line shape is now different from the simple saturational broadening. First we look into the case of the exact resonance for which the transverse relaxation is given by

$$\begin{aligned}\Phi_x=\Phi_y&=\sigma_0'\left\{1-\exp\left(-\frac{\sigma_0''\phi_0}{\phi_0^2+\omega_1^2}\right)\right\}+\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}\\&=\sigma_0'(1-s_1)+\sigma_1^*=\sigma'(1-\rho s_1),\end{aligned} \tag{2·29}$$

where

$$\left\{\begin{aligned}\sigma_1^*&\equiv\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}, &(2·30)\\ \sigma'&\equiv\sigma_0'+\sigma_1^*, &(2·31)\\ \rho&\equiv\sigma_0'/\sigma'. &(2·32)\end{aligned}\right.$$

In contrast to the case treated in the foregoing paragraph now the effect of saturation manifests itself as soon as ω_1 becomes

$$\omega_1>(\sigma_0''\phi_0)^{1/2}. \tag{2·33}$$

The main difference lies in the fact that the forced-nutational modulation tends to prolong the lifetime of the spin state and induce the narrowing of resonance at least in the early stage of the saturation. In the case of the exact resonance $\Phi_{xy}=0$ and the steady state solution is given by

1. Dispersion derivative (x-component)

$$\begin{aligned}\frac{dI_x}{d\Delta}&=\frac{\hbar\omega}{3kT}I(I+1)\omega_1\left\{\frac{1}{\Phi_x(\Phi_y+\omega_1^2/\Phi_z)}\right\}\\&=\frac{\hbar\omega I(I+1)\omega_1}{3kT\sigma'^2}\left\{\frac{1}{(1-\rho s_1)(1-\rho s_1+\omega_1^2/\sigma'\Phi_z)}\right\}\\&=\left(\frac{dI_x}{d\Delta}\right)^0\left\{\frac{1}{(1-\rho s_1)(1-\rho s_1+\omega_1^2/\sigma'\Phi_z)}\right\},\end{aligned} \tag{2·34}$$

2. Absorption (y-component)

$$\begin{aligned}I_y&=\frac{\hbar\omega}{3kT}I(I+1)\omega_1\left\{\frac{1}{\Phi_y+\omega_1^2/\Phi_z}\right\}\\&=\frac{\hbar\omega I(I+1)\omega_1}{3kT\sigma'}\left\{\frac{1}{1-\rho s_1+\omega_1^2/\sigma'\Phi_z}\right\}\end{aligned}$$

$$\simeq (I_y)^0 \left\{ \frac{1}{1-\rho s_1+\omega_1^2/\sigma' \Phi_z} \right\}, \tag{2·35}$$

3. z-polarization (z-component)

$$I_z = \frac{\hbar\omega}{3kT} I(I+1) \left\{ \frac{1}{1+\omega_1^2/\Phi_y \Phi_z} \right\}$$

$$= \frac{\hbar\omega I(I+1)}{3kT} \left\{ \frac{1}{1+\omega_1^2/\Phi_z \sigma'(1-\rho s_1)} \right\}$$

$$= (I_z)^0 \left\{ \frac{1}{1+\omega_1^2/\Phi_z \sigma'(1-\rho s_1)} \right\}, \tag{2·36}$$

where we denoted the solution below saturation by $(\ )^0$. The behaviour of this steady state solution is somewhat different according to the relation between ϕ_0 and Φ_z. The detailed characteristics are shown schematically in Figure 4, where we adopted the log-log scale to make the situation clearer. The results of the foregoing section are also given in broken lines for comparison. The following facts are to be noted. (1) The dispersion derivative survives the absorption upon increasing the intensity of the radio-frequency field. (2) For the case in which $\Phi_z > \phi_0$ the saturation of the z-polarization appears earlier than expected from the Bloch equation.

To obtain the exact half-value width we must treat (2·35) accurately, however we may take $\Phi_{xz} \fallingdotseq 0$ if we are concerned mainly with the asymptotic behaviour. In this approximation the half-value width $\Delta_{1/2}$ is given by

$$\Delta_{1/2} = \sigma' \left\{ (1-\rho s_1)\left(1-\rho s_1+\frac{\omega_1^2}{\sigma' \Phi_z}\right) \right\}^{1/2}, \tag{2·37}$$

the behaviour of which is shown by the chained line in Figure 4(a). In addition, we have a simple relation

$$\left(\frac{d\chi'}{dH_0}\right)_{\Delta=0} \simeq (\Delta_{1/2})^{-2} \tag{2·38}$$

by which the dispersion derivative at exact resonance is directly related to the half-value width. In fact the survival of the dispersion derivative corresponds to the saturational narrowing. When the radiofrequency field becomes large enough to satisfy the condition

$$\omega_1 > \sigma' [\Phi_z/\sigma_1^*]^{1/2}, \tag{2·39}$$

then there occurs the ordinary saturational broadening due to the life-limiting effect of the nutational modulation. Correspondingly the dispersion derivative at exact resonance begins to decrease in this region.

In the above treatment we assumed that ϕ_0 and Φ_z are entirely independent of each other. However, when a single mechanism is responsible for both the transverse and the longitudinal relaxation, we have

$$\Phi_z \simeq \Phi^{\mu\nu} \simeq \frac{\sigma_\mu^2 \phi_\mu}{\omega_\mu^2} \tag{2·40}$$

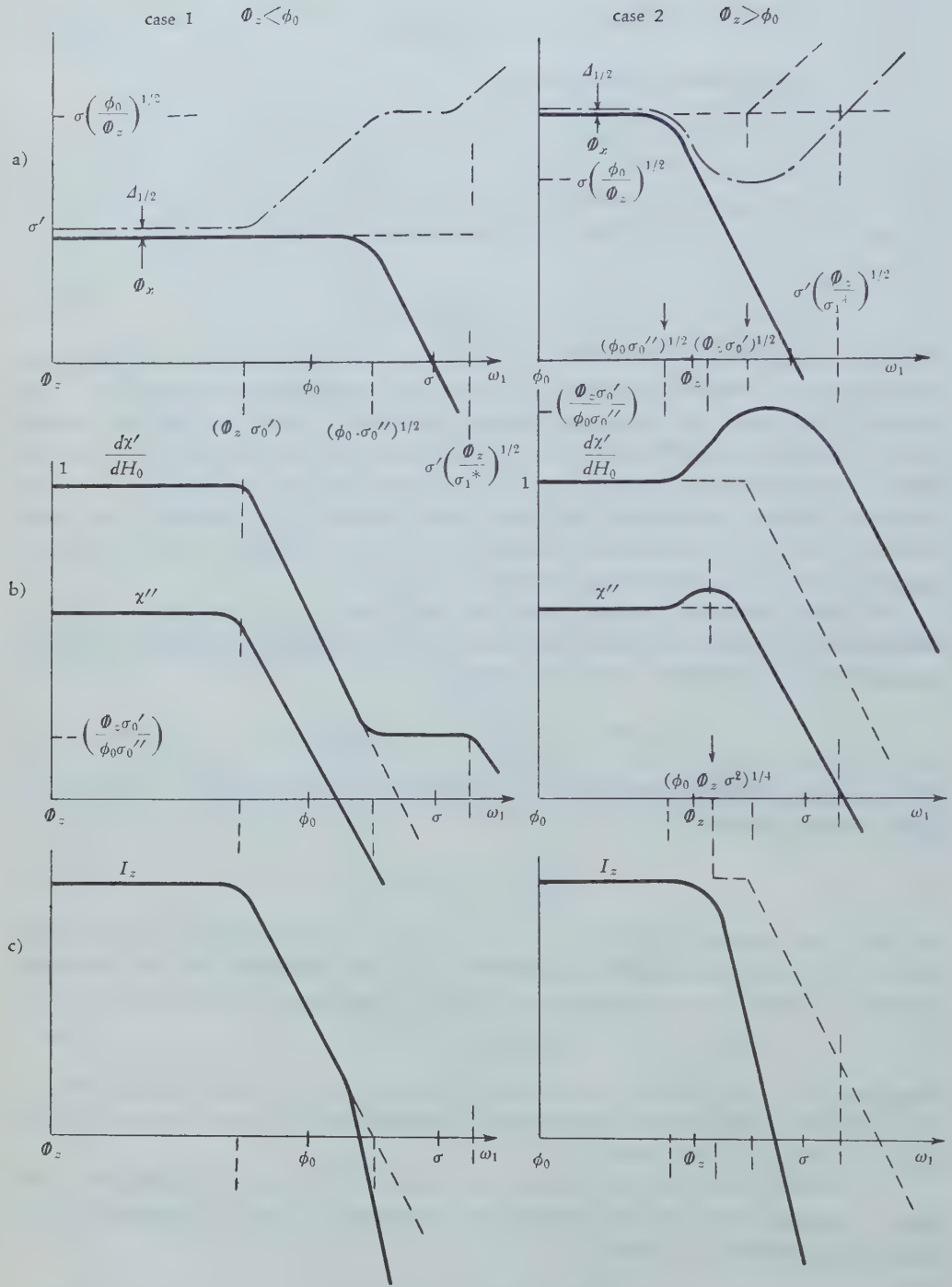

Fig. 4. Steady state solution as a function of the degree of saturation. Case (B) in the local field model. (logarithmic scale)

owing to the condition $\omega_0 \gg \sigma_\mu \gg \phi_\mu$, and

$$\frac{\Phi_z}{\phi_0} \simeq \frac{{\sigma_\mu}^2}{{\omega_0}^2} \ll 1, \tag{2·41}$$

because ϕ_μ is of the same order of magnitude as ϕ_0. (2·41) implies that the situation belongs to Case 1 in Figure 4.

It is possible to characterize the steady state solution by a canonical distribution in an appropriately chosen system of coordinates, however it should be borne in mind that Φ_y is now dependent on the power level of radiofrequency in contrast to the situation (A). In the limit of the extreme saturation the situation can be described by a canonical distribution in the rotating system as was anticipated by Redfield.[14]

§ 2. C. The case in which two types of relaxation mechanism are existent.

There are many examples of nuclear resonance in solids in which the line shape is nearly Gaussian and the width can be explained by calculating the adiabatic second moment due to the dipole-dipole interaction between nuclei. Suppose there is only one mechanism of relaxation in this case, then the corresponding lattice relaxation should be very small and the resonance should be saturated already at a very low power level. In actual fact, however, we have a rather strong steady absorption without the effect of saturation. This indicates the fact that there exists a second mechanism for the lattice relaxation. For example, in the case of nuclear resonance in metals the width is usually determined by the dipole-dipole interaction between nuclei, but the spin-lattice relaxation arises from the contact with the conduction electron. Let us accordingly look into the saturation behaviour when there are two types of mechanism for relaxation.

If we continue to use the Bloch type equation for the whole range as we did in section (B), then the relaxation constant Φ can be simply added in superposing two mechanisms. Of the three possible combinations of situation a pure Bloch equation is obtained in the case (A) + (A). The result is not essentially different from the simple Bloch equation, except for the case in which $\omega_1 \gtrsim \phi_a$ or $\omega_1 \gtrsim \phi_b$ which is scarcely realized. In the case (B) + (B) the lattice relaxation should be so small that it must be difficult to obtain a steady absorption. In view of this situation we treat here only the case (A) + (B) in some detail. So far as the spin-lattice relaxation is concerned, the effect of the process (B) can be neglected, therefore it is determined by the process (A). The spin-spin relaxation is obtained by adding up the contribution of both types, and is given by

$$\begin{aligned} \Phi_x = \Phi_y = \Phi_z &= \Phi_{2a}(1-s_a) + \Phi_{2b}(1-s_b), \\ &= \sigma'(1-\rho_a s_a - \rho_b s_b) = \sigma'(1-\rho s), \end{aligned} \tag{2·42}$$

where

$$\begin{cases} \sigma' \equiv \Phi_{2a} + \Phi_{2b}, & (2·43) \\ \rho_a \equiv \Phi_{2a}/\sigma', & (2·44) \\ \rho_b \equiv \Phi_{2b}/\sigma', & (2·45) \end{cases}$$

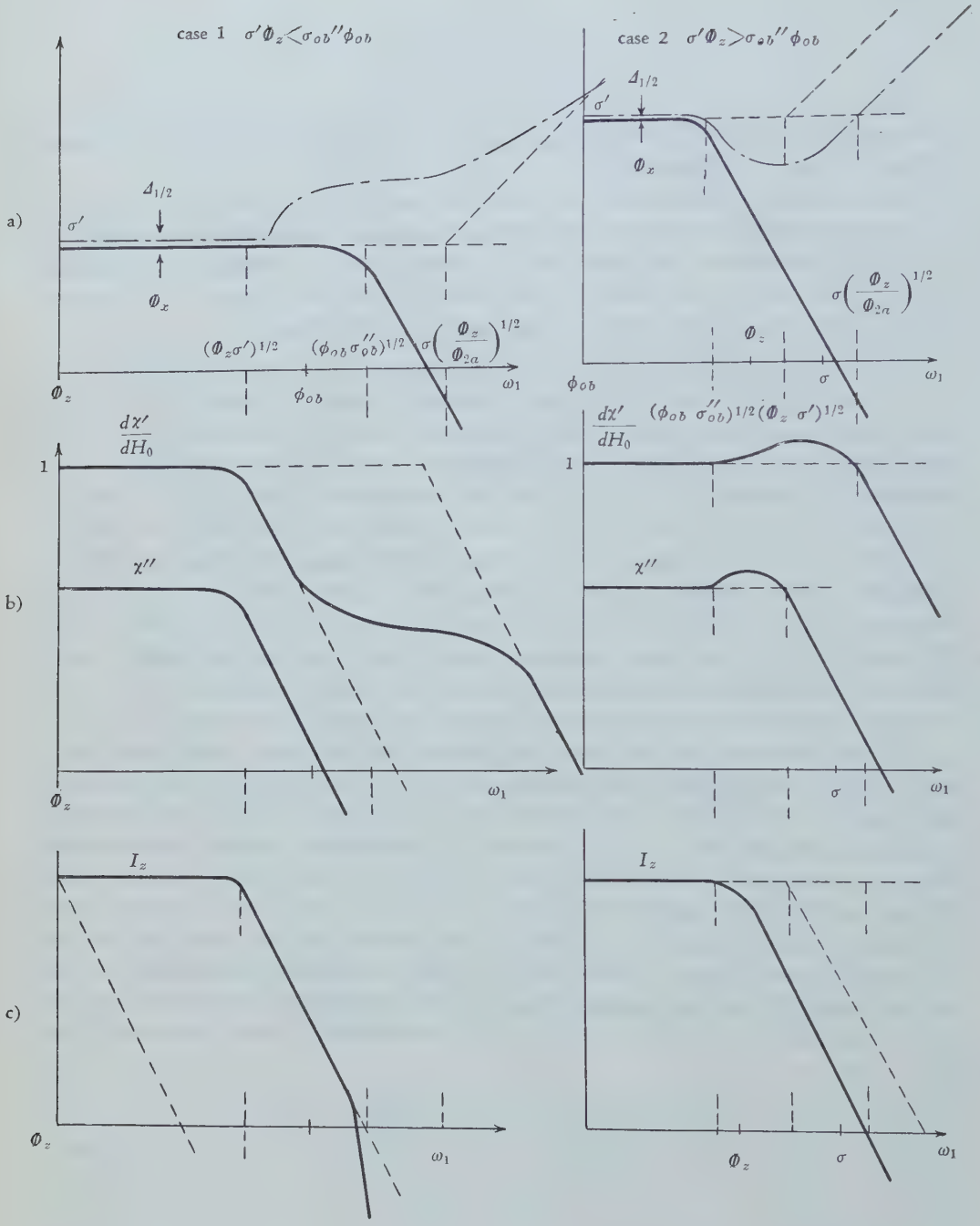

Fig. 5. Steady state solution as a function of the degree of saturation: Case (A) + (B) in the local field model. (logarithmic scale)

and

$$\Phi_{2\alpha} \equiv \sigma_{o\alpha}^2 \tau_{o\alpha}, \quad s_\alpha \equiv \frac{\tilde{\omega}_\alpha^{\,2}}{\phi_{o\alpha}^2 + \tilde{\omega}_\alpha^{\,2}}, \tag{2·46}$$

$$\Phi_{2b} \equiv \sigma'_{ob}, \quad s_b \equiv \exp\left(-\frac{\sigma''_{ob}\phi_{ob}}{\phi_{ob}^2 + \tilde{\omega}_b^{\,2}}\right). \tag{2·47}$$

Then the steady state solution is given again formally by the equations (2·34) to (2·36) in the foregoing section. The asymptotic behaviour of the half-value width $\Delta_{1/2}$ is given by

$$\Delta_{1/2} = \sigma' \left\{ (1-\rho s)\left(1-\rho s+\frac{\omega_1^{\,2}}{\sigma' \Phi_z}\right)\right\}^{1/2} \tag{2·48}$$

where we assumed $\Phi_{xz}=0$. As it is rather rare to have the situation $\omega_1 > \phi_{0\alpha}$, we may safely assume $s_\alpha \simeq 0$, i. e. $\rho s \simeq \rho_b s_b$. The behaviour of various quantities in this case is shown in Figure 5 schematically. The range of saturation of χ'' is given by

$$\omega_1 \gtrsim (\sigma' \Phi_z)^{1/2}. \tag{2·49}$$

Whereas the range of saturation of χ', which corresponds to saturational broadening, is given by

$$\omega_1 \gtrsim \sigma' (\Phi_z / \Phi_{2\alpha})^{1/2}. \tag{2·50}$$

In the case of nuclear resonance in metals the relaxation due to conduction electron is nearly isotropic, i. e.

$$\Phi_z \equiv \Phi_{1\alpha} = \Phi_{2\alpha}, \tag{2·51}$$

then (2·50) is written as

$$\omega_1 \gtrsim \sigma'. \tag{2·52}$$

These criterions are exactly the same as that proposed by Redfield[14] in a heuristic argument. The situation in the present case is characterized by the fact that $\sigma' (\Phi_z/\Phi_{2\alpha})^{1/2} > (\sigma' \Phi_z)^{1/2}$. If in addition there exists a condition

$$\sigma' \Phi_z \gtrsim \sigma''_{ob} \phi_{ob} \quad \text{(case 2 in Fig. 5)}, \tag{2·53}$$

the saturational narrowing occurs for the range

$$\sigma' (\Phi_z/\Phi_{2\alpha})^{1/2} > \omega_1 > (\sigma' \Phi_z)^{1/2}. \tag{2·54}$$

By using (2·49) and (2·52) we can estimate the two kinds of relaxation time by looking at the asymptotes in the logarithmic plot. For example we obtained the results shown in Table 4 (a) by analizing the experimental data on Al^{27} and Cu^{63} due to Redfield.[14] Here we should be careful about the correspondence between two kinds of phase unit

$$\omega = 2\pi\nu \quad \text{and} \quad \tilde{T} = 2\pi T. \tag{2·55}$$

In comparing our result with Redfield's which is shown in Table 4 (b) we conclude that there is a factor 2π difference between the units of $g(\nu_0)$ and T_1. It should be noted that our results are consistent with the value $\tilde{T}_2=2.43\times10^{-4}$ sec/cycle which corresponds to the maximum slope width $\Delta H_{m.s.}=6.2$ gauss observed by Sugawara.[16]

The appearance of saturational narrowing before that of saturational broadening is qualitatively consistent with the situation found by Abell and Knight,[15] Redfield,[14] Holcomb,[17] and Sugawara.[16] After fixing the values of Ψ_z and σ' we may determine the value of ϕ'_{0b} in such a way as to reproduce the behaviour of $(d\chi', dH_0)_{\Delta=0}$ in the intermediate region. In the above example this reproduction is obtained by adopting

$$\phi'_{0b}=\Psi_z. \tag{2·56}$$

The results are shown in Figure 5. In principle it is possible to do a more detailed analysis by comparing the formula for the line width (2·48) with the observed data on saturational narrowing. The relation (2·56) is rather remarkable, however there is no way of explaining it here, as we have treated ϕ'_{0b} and Ψ_z as entirely independent of each other. In order to obtain this kind of explanation we have to take into account the explicit form of interaction responsible for the relaxation, which we will do in the next section.

Table 4 (a). Nuclear magnetic relaxation times of Al^{27} and Cu^{63} as determined from saturation data. (cf. Reference 14)

	Al^{27}	Cu^{63}
1 gauss	1.109_4 k. c.	1.128_5 k. c.
1 k. c. (kilocycle)	0.901_4 gauss	0.886_1 gauss
$\nu_1=\tilde{\sigma}'$ at $H_1=$	3.8 gauss	3.2 gauss
$\nu_1=$	4.215_7 k. c.	3.611_2 k. c.
$\tilde{T}_2\equiv(\tilde{\sigma}')^{-1}=\nu_1^{-1}=$	$2.372_1\times10^{-4}$ sec/cycle	$2.769_1\times10^{-4}$ sec/cycle
$T_2\equiv(2\pi)^{-1}\tilde{T}_2$	$3.775_1\times10^{-5}$ sec/radian	4.407×10^{-5} sec/radian
$\nu_1^2=\tilde{\sigma}'\tilde{\Phi}_z$ at $H_1=$	0.186 gauss	0.22 gauss
$\nu_1=$	0.206_3 k. c.	0.248_3 k. c.
$\nu_1^2=$	$4.258_0\times10^4$ (c)^2	$6.163_8\times10^4$ (c)^2
$\tilde{\Phi}_z=\nu_1^2/\tilde{\sigma}'=$	10.10 cycle	17.07 cycle
$\tilde{T}_1\equiv\tilde{\Phi}_z^{-1}=$	9.901×10^{-2} sec/cycle	5.858×10^{-2} sec/cycle
$T_1=(2\pi)^{-1}\tilde{T}_1$	5.612×10^{-3} sec/radian	3.605×10^{-3} sec/radian

Table 4 (b). Redfield's Table I. (cf. Reference 14)

	Al^{27}	Cu^{63}
$g(\nu_0)$ sec	$2.12\pm0.2\times10^{-4}$	$2.28\pm0.2\times10^{-4}$
T_1 sec (Redfield)	$4.1\pm0.8\times10^{-3}$	$3.0\pm0.6\times10^{-3}$
T_1 sec (Theory)	5.06×10^{-3}	2.26×10^{-3}
$\Delta H/H_0$	0.237×10^{-2}	0.161×10^{-2}
$(m^*/m)v_0P_F$	250	260

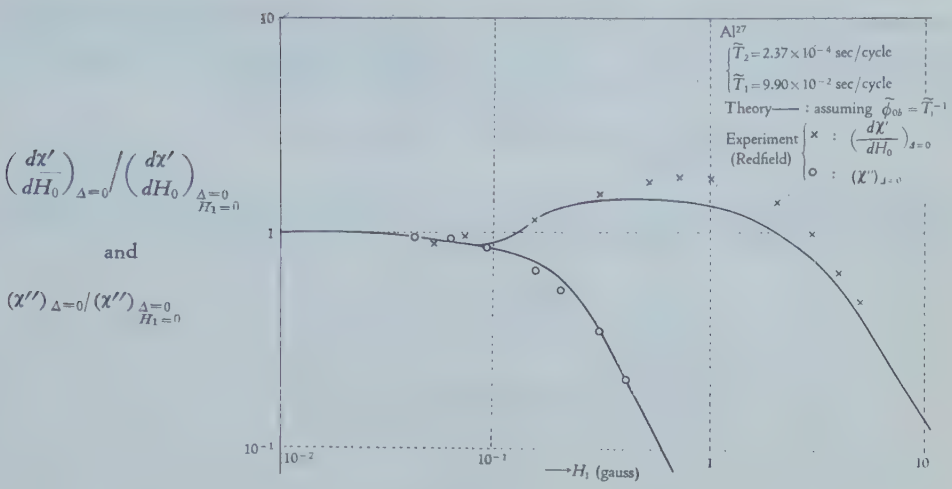

Fig. 6. (a). Nuclear magnetic dispersion and absorption as functions of the degree of saturation in metallic aluminium. Theoretical curves are based on the local field model and an assumption $\phi_{0b} = \phi_z$. Experimental data were taken from Redfield (Reference 14).

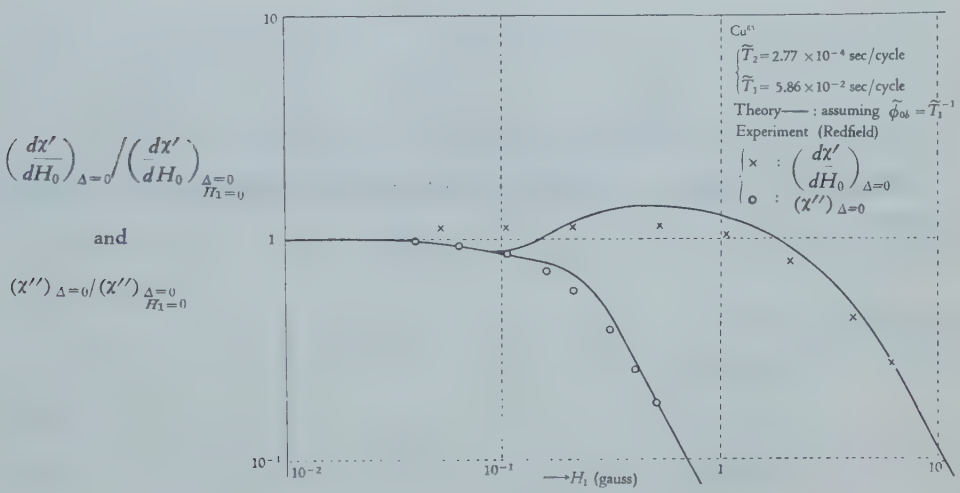

Fig. 6 (b). Nuclear magnetic dispersion and absorption as functions of the degree of saturation in metallic copper. Theoretical curves are based on the local field model and an assumption $\phi_{0b} = \phi_z$. Experimental data were taken from Redfield (Reference 14).

§ 3. Relaxation in an interacting spin assembly

The treatment described in § 2 cannot directly be applied to the case in which the relaxation is due to the interaction between spins of the same kind, for the effect of the quantum mechanical resonance cannot be taken into account in the local field picture.

In this paragraph, therefore, we develop the theory which includes the explicit form of the interaction from the outset. The general scheme, developed in § 1, may be used also in this case, however the form of the interaction now becomes

$$G=\sum_s\sum_t\sum_{\mu_s}\sum_{\mu_t} I_s^{\mu_s} I_t^{\mu_t} F_{st}^{\bar{\mu}_s\bar{\mu}_t} \quad (|\mu_s+\mu_t|=0,\ 1,\ 2) \tag{3·1}$$

in terms of the single spin operators $I_s^{\mu_s}$, etc., and after a substitution which is specified in Table 5 the treatment goes quite similar to that which was given in § 2.

Table 5. Correspondence between the local field case and the case of interacting spin assembly.

Local field model	Interacting spin assembly
$\boldsymbol{I}$	$\sum_s \boldsymbol{I}_s$
μ	$(\mu_s,\ \mu_t)$
ν	$(\nu_s,\ \nu_t)$
$\omega_\mu=-\omega\mu$	$\omega_\mu=-\omega(\mu_s+\mu_t)$
$\omega_\nu=-\tilde{\omega}\nu$	$\omega_\nu=-(\tilde{\omega}_s\nu_s+\tilde{\omega}_t\nu_t)$
Δ	$(\Delta_s,\ \Delta_t)$
ω_1	$(\omega_{1s},\ \omega_{1t})$
θ	$(\theta_s,\ \theta_t)$
$a_{\mu\nu}$	$a_{\mu_s\nu_s}\cdot a_{\mu_t\nu_t}$
$\Phi^{\mu\nu}$	$\Phi_{st}^{\mu_s\nu_s,\ \mu_t\nu_t}$

If we take the magnetic dipole-dipole interaction as an example, then the interaction G is written in a form

$$G=\sum_s\sum_t \{st\}_\mu F_{st}^{\bar{\mu}} \tag{3·2}$$

where $\mu=\mu_s+\mu_t$, and

$$\{st\}_0=I_s^0 I_t^0-\tfrac{1}{4}(I_s^+ I_t^-+I_s^- I_t^+), \tag{3·3}$$

$$\{st\}_{\pm1}=I_s^0 I_t^\pm+I_s^\pm I_t^0, \tag{3·4}$$

$$\{st\}_{\pm2}=I_s^\pm I_t^\pm, \tag{3·5}$$

$$F_{st}^{\ 0}=-\frac{\mu_s\mu_t}{r_{st}^3}(3\cos^2\theta_{st}-1)\equiv\mu_s\mu_t F_0, \tag{3·6}$$

$$F_{st}^{\pm1}=-\frac{\mu_s\mu_t}{r_{st}^3}\left(\frac{3}{2}\right)\cos\theta_{st}\sin\theta_{st}\,e^{\pm i\varphi_{st}}\equiv\mu_s\mu_t\left(-\frac{3}{2}\right)F_1, \tag{3·7}$$

$$F_{st}^{\pm2}=-\frac{\mu_s\mu_t}{r_{st}^3}\left(\frac{3}{4}\right)\sin^2\theta_{st}\,e^{\pm2i\varphi_{st}}\equiv\mu_s\mu_t\left(-\frac{3}{4}\right)F_2. \tag{3·8}$$

In (3·6), (3·7) and (3·8) μ_s and μ_t stand for the magnetic moments and not for quantum numbers. Suppose, in addition, we have only one species of spins which

participates in the process of relaxation, then $t=s$, $\theta_s=\theta_t=\theta$, and the transformation formula corresponding to (14) becomes

$$\{\tilde{st}\}_\mu=\sum_\nu A_{\mu\nu}\{st\}_\nu, \tag{3·9}$$

i. e. explicitly

$$\begin{pmatrix} \{\tilde{st}\}_0 \\ \{\tilde{st}\}_1 \\ \{\tilde{st}\}_{\bar{1}} \\ \{\tilde{st}\}_2 \\ \{\tilde{st}\}_{\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{3c^2-1}{2} & -\frac{3cs}{4} & -\frac{3cs}{4} & \frac{3s^2}{8} & \frac{3s^2}{8} \\ 2cs & \frac{c(c+1)-s^2}{2} & \frac{c(c-1)-s^2}{2} & -\frac{s(c+1)}{2} & -\frac{s(c-1)}{2} \\ 2cs & \frac{c(c-1)-s^2}{2} & \frac{c(c+1)-s^2}{2} & -\frac{s(c-1)}{2} & -\frac{s(c+1)}{2} \\ s^2 & \frac{s(c+1)}{2} & \frac{s(c-1)}{2} & \left(\frac{c+1}{2}\right)^2 & \left(\frac{c-1}{2}\right)^2 \\ s^2 & \frac{s(c-1)}{2} & \frac{s(c+1)}{2} & \left(\frac{c-1}{2}\right)^2 & \left(\frac{c+1}{2}\right)^2 \end{pmatrix} \begin{pmatrix} \{st\}_0 \\ \{st\}_1 \\ \{st\}_{\bar{1}} \\ \{st\}_2 \\ \{st\}_{\bar{2}} \end{pmatrix} \tag{3·10}$$

where $c\equiv\cos\theta$ and $s\equiv\sin\theta$, and ν is the number of the quantum jump in the tilted system of coordinates in the rotating frame. The direction of the resulting field is chosen as the z'-axis in this frame, and θ is the angle between this axis and the original z-axis in the static frame. As we did in the last section, now we try to find an approximate differential equation which is obeyed by the expectation value of an operator Q^r, which in our case will be chosen as the resultant spin vector $\boldsymbol{I}$.

The following commutation relations are useful in deriving the differential equation

Table 6 (a). The values of coefficients C_ν and K_ν.

ν	-2	-1	0	1	2
C_ν	-2	-4	3/2	2	0
K_ν	4/9	4/9	1/6	4/9	4/9

Table 6 (b). The value of coefficient $d_{\nu\nu'}$.

ν \ ν'	2	1	0	-1	-2
2	0	0	0	$-1/2$	1
1	0	0	$-3/8$	1/2	$-1/2$
0	0	3/8	0	$-3/8$	0

$$[I_s(0)+I_t(0),\ \{st\}_\nu]=\nu\{st\}_\nu, \tag{3·11}$$

$$[I_s(+)+I_t(+),\ \{st\}_\nu]=C_\nu\{st\}_{\nu+1}, \tag{3·12}$$

Table 7. Relaxation constants (the quantities in the first row) expressed as linear
tions $c=\cos\theta$ and $s=\sin\theta$, where θ is the angle between z- and z'-axis. (The case

	$\mathcal{O}^{00}$	$\mathcal{O}'^{01}$	$\mathcal{O}'^{02}$	$\mathcal{O}'^{10}$	$\mathcal{O}'^{11}$	$\mathcal{O}'^{1\bar{1}}$	$\mathcal{O}'^{12}$
$\mathcal{O}_x$	$\frac{9}{32}\cdot c^2\cdot(3c^2-1)$	$\frac{9}{8}\cdot c^2s^2$	$\frac{9}{32}\cdot s^2\cdot(s^2+1)$	$\frac{3}{2}\cdot c^2s^2$	$\frac{1}{4}\cdot(c+1)(2c-1)\cdot(2c^2+4c-1)$	$\frac{1}{4}\cdot(c-1)(2c+1)\cdot(2c^2-4c-1)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(c+4)$
$\mathcal{O}_y$	$\frac{9}{32}\cdot(3c^2-1)\cdot(2c^2-1)$	$\frac{9}{4}\cdot c^2s^2$	$\frac{9}{32}\cdot s^2\cdot(2c^2-1)$	$6s^2c^2$	$\frac{1}{4}\cdot(c+1)(2c-1)\cdot(8c^2+c-4)$	$\frac{1}{4}\cdot(c-1)(2c+1)\cdot(8c^2-c-4)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(4c+1)$
$\mathcal{O}_z$				$\frac{3}{2}\cdot c^2s^2$	$\frac{1}{4}\cdot(c+1)^2\cdot(2c-1)^2$	$\frac{1}{4}\cdot(c-1)^2\cdot(2c+1)^2$	$\frac{1}{4}\cdot s^2(c+1)^2$
$\mathcal{O}_x{}^*$				$3\overline{c^3s}$	$\frac{1}{2}\cdot s(c+1)\cdot(2c-1)^2$	$\frac{1}{2}\cdot s(c-1)\cdot(2c+1)^2$	$\frac{1}{2}\cdot s(c+1)\cdot(c^2-c-2)$
$\tilde{\mathcal{O}}_x$					$\frac{1}{2}\cdot s(c+1)\cdot(2c-1)^2$	$\frac{1}{2}\cdot s(c-1)\cdot(2c+1)^2$	$s(c+1)\cdot(c^2-c-2)$
$\mathcal{O}_z{}^*$				$3c^2s^2$	$\frac{1}{2}\cdot(c+1)^2\cdot(2c-1)^2$	$\frac{1}{2}\cdot(c-1)^2\cdot(2c+1)^2$	$\frac{1}{2}\cdot s^2(c+1)^2$
$\tilde{\mathcal{O}}_z$					$\frac{1}{2}\cdot(c+1)^2\cdot(2c-1)^2$	$\frac{1}{2}\cdot(c-1)^2\cdot(2c+1)^2$	$s^2(c+1)^2$
$\mathcal{O}_{xz}$	$\frac{9}{32}\cdot cs\cdot(3c^2-1)$	$\frac{9}{16}\cdot cs\cdot(2c^2-1)$	$\frac{9}{32}\cdot cs^3$	$\frac{3}{2}\cdot\overline{cs^3}$	$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{4}\cdot s(c+1)^3$
$\mathcal{O}_{zx}$				$\frac{3}{2}\cdot\overline{c^3s}$	$\frac{1}{4}\cdot s(c+1)\cdot(2c-1)^2$	$\frac{1}{4}\cdot s(c-1)\cdot(2c+1)^2$	$\frac{1}{4}\cdot s(c+1)^2\cdot(c-2)$

	$\mathcal{O}''^{01}$	$\mathcal{O}''^{02}$	$\mathcal{O}''^{10}$	$\mathcal{O}''^{11}$	$\mathcal{O}''^{1\bar{1}}$	$\mathcal{O}''^{12}$
$\mathcal{O}_y{}^*$			$3cs\cdot(2c^2-1)$	$\frac{1}{2}\cdot s(c+1)\cdot(2c-1)(4c+1)$	$\frac{1}{2}\cdot s(c-1)\cdot(2c+1)(4c-1)$	$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$
$\tilde{\mathcal{O}}_y$	$\frac{9}{8}\cdot cs\cdot(2c^2-1)$	$\frac{9}{8}\cdot cs^3\cdot(2c^2-1)$		$\frac{1}{2}\cdot s(c+1)\cdot(2c-1)(4c+1)$	$\frac{1}{2}\cdot s(c-1)\cdot(2c+1)(4c-1)$	$s(c+1)^2(2c-1)$
$\mathcal{O}_{xy}$	$\frac{9}{16}\cdot cs^2$	$\frac{9}{32}\cdot cs^2$	$3c^2s^2$	$\frac{1}{4}\cdot(c+1)(2c-1)\cdot(4c^2-c-2)$	$\frac{1}{4}\cdot(c-1)(2c+1)\cdot(4c^2+c-2)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(2c-1)$
$\mathcal{O}_{yx}$	$\frac{9}{16}\cdot cs^2$	$\frac{9}{32}\cdot\overline{cs^2}$	$\frac{3}{2}\cdot\overline{c^2s^2}$	$\frac{1}{4}\cdot(c+1)(2c-1)\cdot(2c^2-2c-1)$	$\frac{1}{4}\cdot(c-1)(2c+1)\cdot(2c^2+2c-1)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(c-2)$
$\mathcal{O}_{yz}$	$\frac{9}{16}\cdot\overline{c^2s}$	$\frac{9}{32}\cdot\overline{s^3}$	$\frac{3}{2}\cdot cs^3$	$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{4}\cdot s(c+1)^3$
$\mathcal{O}_{zy}$			$\frac{3}{2}\cdot cs\cdot(2c^2-1)$	$\frac{1}{4}\cdot s(c+1)\cdot(2c-1)(4c+1)$	$\frac{1}{4}\cdot s(c-1)(2c+1)\cdot(4c-1)$	$\frac{1}{4}\cdot s(c+1)^2\cdot(2c-1)$

$$[I_s(-)+I_t(-),\ \{st\}_\nu]=C_\nu\{st\}_{\nu-1}, \tag{3·13}$$

$$\langle\{st\}_\nu\{st\}_{-\nu}\rangle_0=K_\nu I^2(I+1)^2 \tag{3·14}$$

where the values of C_ν and K_ν are given in Table 6 (a). In general the form of $[Q^r,\ \{st\}_\nu]\{st\}_{\nu'}$ is complicated and the resulting equation does not become linear in Q^r. The condition for having a resultant differential equation linear in Q^r is given by

$$(\alpha)\qquad \hbar\omega_0\ll kT\ \text{(Low frequency condition)}, \tag{3·15}$$

combinations of $\mathcal{O}^{\mu\nu}=\mathcal{O}'^{\mu\nu}+i\mathcal{O}''^{\mu\nu}$. The coefficients are tabulated using the abbreviation of interacting spin assembly.) The bar indicates a minus sign.

$\mathcal{O}'^{1\bar{2}}$	$\mathcal{O}'^{20}$	$\mathcal{O}'^{21}$	$\mathcal{O}'^{2\bar{1}}$	$\mathcal{O}'^{22}$	$\mathcal{O}'^{2\bar{2}}$
$\frac{1}{4}\cdot s^2(c-1)\cdot(c-4)$	$\frac{3}{4}\cdot c^2 s^2$	$\frac{1}{4}\cdot s^2(c+1)\cdot(2c-1)$	$\frac{1}{4}\cdot s^2(c-1)\cdot(2c+1)$	$\frac{1}{8}\cdot(c+1)^3\cdot(c-2)$	$\frac{1}{8}\cdot(c-1)^3(c+2)$
$\frac{1}{4}\cdot s^2(c-1)\cdot(4c-1)$	$\frac{3}{4}\cdot s^2(2s^2-1)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(4c+1)$	$\frac{1}{4}\cdot s^2(c-1)\cdot(4c-1)$	$\frac{1}{8}\cdot(c+1)^2\cdot(2c^2+c-1)$	$\frac{1}{8}\cdot(c-1)^2\cdot(2c^2-c-1)$
$\frac{1}{4}\cdot s^2(c-1)^2$	$\frac{3}{2}\cdot s^4$	$s^2(c+1)^2$	$s^2(c-1)^2$	$\frac{1}{4}\cdot(c+1)^4$	$\frac{1}{4}\cdot(c-1)^4$
$\frac{1}{2}\cdot s(c-1)\cdot(c^2+c-2)$	$3\,cs^3$	$s(c+1)\cdot(2c^2+c-1)$	$s(c-1)\cdot(2c^2-c-1)$	$\frac{1}{2}\cdot s(c+1)^3$	$\frac{1}{2}\cdot s(c-1)^3$
$s(c-1)\cdot(c^2+c-2)$		$\frac{1}{2}\cdot s(c+1)\cdot(2c^2+c-1)$	$\frac{1}{2}\cdot s(c-1)\cdot(2c^2-c-1)$	$\frac{1}{2}\cdot s(c+1)^3$	$\frac{1}{2}\cdot s(c-1)^3$
$\frac{1}{2}\cdot s^2(c-1)^2$	$3\,s^4$	$2s^2(c+1)^2$	$2\cdot s^2(c-1)^2$	$\frac{1}{2}\cdot(c+1)^4$	$\frac{1}{2}\cdot(c-1)^4$
$s^2(c-1)^2$		$s^2(c+1)^2$	$s^2(c-1)^2$	$\frac{1}{2}\cdot(c+1)^4$	$\frac{1}{2}\cdot(c-1)^4$
$\frac{1}{4}\cdot s(c-1)^3$	$\frac{3}{4}\cdot cs^3$	$\frac{1}{4}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{4}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{8}\cdot s(c+1)^3$	$\frac{1}{8}\cdot s(c-1)^3$
$\frac{1}{4}\cdot s(c-1)^2\cdot(c+2)$	$\frac{3}{2}\cdot cs^3$	$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{4}\cdot s(c+1)^3$	$\frac{1}{4}\cdot s(c-1)^3$

$\mathcal{O}''^{1\bar{2}}$	$\mathcal{O}''^{20}$	$\mathcal{O}''^{21}$	$\mathcal{O}''^{2\bar{1}}$	$\mathcal{O}''^{22}$	$\mathcal{O}''^{2\bar{2}}$
$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$3\,cs^3$	$s(c+1)^2(2c-1)$	$s(c-1)^2(2c+1)$	$\frac{1}{2}\cdot s(c+1)^3$	$\frac{1}{2}\cdot s(c-1)^3$
$s(c-1)^2(2c+1)$		$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{2}\cdot s(c+1)^3$	$\frac{1}{2}\cdot s(c-1)^3$
$\frac{1}{4}\cdot s^2(c-1)\cdot(2c+1)$	$\frac{3}{4}\cdot s^2(2c^2-1)$	$\frac{1}{4}\cdot s^2(c+1)\cdot(4c+1)$	$\frac{1}{4}\cdot s^2(c-1)\cdot(4c-1)$	$\frac{1}{8}\cdot(c+1)^3\cdot(2c-1)$	$\frac{1}{8}\cdot(c-1)^3\cdot(2c+1)$
$\frac{1}{4}\cdot s^2(c-1)\cdot(c+2)$	$\frac{3}{4}\cdot c^2 s^2$	$\frac{1}{4}\cdot s^2(c+1)\cdot(2c-1)$	$\frac{1}{4}\cdot s^2(c-1)\cdot(2c+1)$	$\frac{1}{8}\cdot(c+1)^3(c-2)$	$\frac{1}{8}\cdot(c-1)^3(c+2)$
$\frac{1}{4}\cdot s(c-1)^3$	$\frac{3}{4}\cdot cs^3$	$\frac{1}{4}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{4}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{8}\cdot s(c+1)^3$	$\frac{1}{8}\cdot s(c-1)^3$
$\frac{1}{4}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{3}{2}\cdot cs^3$	$\frac{1}{2}\cdot s(c+1)^2\cdot(2c-1)$	$\frac{1}{2}\cdot s(c-1)^2\cdot(2c+1)$	$\frac{1}{4}\cdot s(c+1)^3$	$\frac{1}{4}\cdot s(c-1)^3$

and

$$(\beta) \qquad I=1/2, \tag{3·16}$$

which is severer than in the case of the local field model in as much as we have the second restriction. If we confine ourselves to this special case, we are allowed to use the commutation relations

$$[\{st\}_{\nu},\ \{st\}_{\nu'}]=d_{\nu\nu'}\{I_s(\nu+\nu')+I_t(\nu+\nu')\} \tag{3·17}$$

$$[\{st\}_{-\nu},\ \{st\}_{-\nu'}]=-d_{\nu\nu'}\{I_s(\nu+\nu')+I_t(\nu+\nu')\} \tag{3·18}$$

where the numerical values of the coefficient $d_{\nu\nu'}$ are given in Table 6 (b). Retaining the terms only up to the first order in $\beta\hbar\omega$, we have

$$-\sum_{s>s'}\sum_{\mu}\sum_{\nu}\sum_{\nu'} A_{\mu\nu}A_{\bar{\mu}\nu'}\Phi^{\bar{\mu}\nu'}_{s's}\Big\{\big[[Q^{\nu},\ \{ss'\}_{\nu}],\{ss'\}_{\nu'}\big] -\beta\hbar(\mu\omega-\nu'\tilde{\omega})\langle\{ss'\}_{\nu'}[Q^{\nu},\ \{ss'\}_{\nu}]\rangle_0\Big\} \tag{3·19}$$

as the third term in (1·24), where $\langle\ \rangle_0$ indicates the operation of taking a simple trace assuming the spin density as unity. The resulting equations are linear in Q^{ν} and are written in the same form as (2·4) in the previous section, however the definition of the relaxation tensor is now different. The detailed results are given in Table 7, where the relaxation constants are expressed in terms of the parameters $\Phi^{\mu\nu}=\Phi'^{\mu\nu}+i\Phi''^{\mu\nu}$ which were defined in (1·26). In particular the condition (1·32) holds when the concept of resonance is established (i. e. $\omega_0\gg\sigma_{\mu}'$), and the expressions become simplified as given in Table 8, where $\Phi^{\mu\nu}$ should be read as an abbreviation for $\sum_{s'(\neq s)}\Phi^{\mu\nu}_{ss'}$. In this case the restoring force is found again proportional to the deviation of the polarization

Table 8. Relaxation constants (the quantities in the first row) expressed as linear combinations of $\Phi^{\mu\nu}=\Phi'^{\mu\nu}+i\Phi''^{\mu\nu}$, using the same notations as in Table 7. (Interacting spin assembly under the condition $\Phi^{10}=\Phi^{11}=\Phi^{1\bar{1}}=\Phi^{1}$, and $\Phi^{20}=\Phi^{21}=\Phi^{2\bar{1}}=\Phi^{22}=\Phi^{2\bar{2}}\equiv\Phi^{2}$.)

	Φ'^{00}	Φ'^{01}	Φ'^{02}	Φ'^{1}	Φ'^{2}	Φ'^{0}
Φ_x	$\frac{9}{32}\cdot c^2(3c^2-1)$	$\frac{9}{8}\cdot c^2s^2$	$\frac{9}{32}\cdot s^2(s^2+1)$	$\frac{5}{2}$	1	$\frac{9}{16}$
Φ_y	$\frac{9}{32}\cdot(3c^2-1)(2c^2-1)$	$\frac{9}{4}\cdot c^2s^2$	$\frac{9}{32}\cdot s^2(2c^2-1)$	$\frac{5}{2}$	1	$\frac{9}{16}$
Φ_z				1	4	
$\Phi_x{}^*$						
$\tilde{\Phi}_x$		$\frac{9}{16}\cdot c^2s$	$\frac{9}{16}\cdot s^3$	$\frac{5}{2}\cdot s$	s	$\frac{9}{16}\cdot s$
$\Phi_z{}^*$				1	4	
$\tilde{\Phi}_z$				c	4c	
Φ_{xz}	$\frac{9}{32}\cdot cs(3c^2-1)$	$\frac{9}{16}\cdot cs(2c^2-1)$	$\frac{9}{32}\cdot cs^3$			
Φ_{zx}						

	Φ''^{01}	Φ''^{02}	Φ''^{1}	Φ''^{2}	Φ''^{0}
$\Phi_y{}^*$					
$\tilde{\Phi}_y$	$\frac{9}{16}\cdot cs(2c^2-1)$	$\frac{9}{16}\cdot cs^3$	$\frac{1}{2}\cdot s$	s	$\frac{9}{16}\cdot c^3s$
Φ_{xy}	$\frac{9}{16}\cdot cs^2$	$\frac{9}{32}\cdot cs^2$	$-\frac{1}{2}$	-1	$\frac{9}{32}\cdot cs^2$
Φ_{yx}	$\frac{9}{16}\cdot cs^2$	$\frac{9}{32}\cdot cs^2$	$\frac{1}{2}$	1	$\frac{9}{32}\cdot cs^2$
Φ_{yz}	$\frac{9}{16}\cdot c^2s$	$\frac{9}{32}\cdot s^3$			$\frac{9}{32}\cdot s(c^2+1)$
Φ_{zy}					

from what is expected in the canonical distribution under the instantaneous total field. The longitudinal relaxation $\Phi_z=T_1{}^{-1}$ does not depend on the deviation from resonance, whereas the transverse relaxations Φ_x and Φ_y do depend on it, and moreover Φ_x is not equal to Φ_y in general. This situation is markedly different from what we found in case of the local field model. Φ_{xy} and Φ_{yx} stand for the shift due to the non-adiabatic

effect of the second order perturbation. In the special case in which $\omega_0 \gg \phi_\mu \gg \omega_1$, or $\omega_0 \gg \omega_1 \gg \phi_\mu$ holds, the real part of $\Phi^{\mu\nu}$ is much greater than the imaginary part, and we find that the differential equation for $\langle Q^T \rangle$ is reduced to the form of (2·6). Although Φ'^1 and Φ'^2 do not depend on time, Φ'^{00}, Φ'^{01} and Φ'^{02} do depend on it according to (1·26), so that it is not simple to solve (2·6) in general. In this situation we try to find a simplification of the equation (2·6) in two extreme cases which are discriminated according to the effect of the semi-adiabatic perturbation.

§ 3. A. The case of rapidly fluctuating lattice ($\sigma_0 \ll \phi_0$)

When the external system fluctuates more rapidly than the relative precession due to the distribution of the local field, it belongs to the situation (A), and the simplified equation (2·6) holds with time independent values of relaxation constants which are defined including both adiabatic and non-adiabatic effects. The only difference lies in the fact that here the parameter ν can assume five values 0, ± 1, ± 2 in contrast to three values 0, ± 1 in the previous section. In the case in which $\phi_0 \gg \tilde{\omega}$ the relaxation times do not depend on the power level and are given by

$$\Phi_x = \Phi_y = T_2^{-1} = \tfrac{9}{16}\Phi'^0 + \tfrac{5}{2}\Phi'^1 + \Phi'^2, \tag{3·20}$$

$$\Phi_z = T_1^{-1} = \Phi'^1 + 4\Phi'^2. \tag{3·21}$$

When $\phi_\mu \gg \omega_0$ in addition, we have $\Phi'^0 : \Phi'^1 : \Phi'^2 = 8 : 3 : 3$, owing to (1·28), (1·32), (2·6), (3·3) and (3·8), and consequently

$$\Phi_x = \Phi_y = \Phi_z, \tag{3·22}$$

which is an indication of the isotropic relaxation. All the qualitative discussions go similar to the case of the local field model, and the steady state soluton is written in the form of (2·11) to (2·13). Φ_{zz} vanishes for the case of exact resonance, or for the case in which $\phi_0 \gg \tilde{\omega}$ owing to the fact that $\Phi'^{00} = \Phi'^{01} = \Phi'^{02}$. In this case the shape of the absorption becomes completely Lorentzian, and it reduces to a Debye type relaxation formula in the limit of vanishing static field. The steady state solution can be described by "a canonical distribution in an appropriately chosen system of coordinates", and the classification and the explanation embodied in Table 3 and Figure 2 apply also here. If, on the other hand, ω_1 becomes greater than ϕ_0, i. e. the saturation goes to the extreme so as to overcome the external fluctuation, the transverse relaxation becomes dependent on the power level according to the formulae

$$\Phi_x = \frac{9}{32}\sigma_0^2\tau_0\left\{c^2(3c^2-1) + \frac{4c^2s^2}{1+(\tilde{\omega}\tau_0)^2} + \frac{s^2(2-c^2)}{1+(2\tilde{\omega}\tau_0)^2}\right\} + \frac{5}{2}\frac{\sigma_1^2\tau_1}{1+(\omega\tau_1)^2} + \frac{\sigma_2^2\tau_2}{1+(2\omega\tau_1)^2}, \tag{3·23}$$

$$\Phi_y = \frac{9}{32}\sigma_0^2\tau_0\left\{(3c^2-1)(2c^2-1) + \frac{8c^2s^2}{1+(\tilde{\omega}\tau_0)^2} + \frac{s^2(1-2c^2)}{1+(2\tilde{\omega}\tau_0)^2}\right\} + \frac{5}{2}\frac{\sigma_1^2\tau_1}{1+(\omega\tau_1)^2} + \frac{\sigma_2^2\tau_2}{1+(2\omega\tau_1)^2}, \tag{3·24}$$

although the value of Φ_z does not. At exact resonance these formulae become

$$\Phi_x = \frac{9}{32}\sigma_0^2\tau_0\left\{\frac{2}{1+(2\omega_1\tau_0)^2}\right\} + \frac{5}{2}\left\{\frac{\sigma_1^2\tau_1}{1+(\omega_0\tau_1)^2} + \frac{\sigma_2^2\tau_2}{1+(2\omega_0\tau_1)^2}\right\}, \tag{3·25}$$

$$\Phi_y = \frac{9}{32}\sigma_0^2\tau_0\left\{1+\frac{1}{1+(2\omega_1\tau_0)^2}\right\} + \frac{5}{2}\left\{\frac{\sigma_1^2\tau_1}{1+(\omega_0\tau_1)^2} + \frac{\sigma_2^2\tau_2}{1+(2\omega_0\tau_1)^2}\right\}. \tag{3·26}$$

Again it should be noted that the limiting values are different for Φ_x and Φ_y. The limiting value is one half of the unsaturated value for Φ_y, whereas for Φ_x it is of the same order as the longitudinal relaxation. The situation is shown in Figure 7.

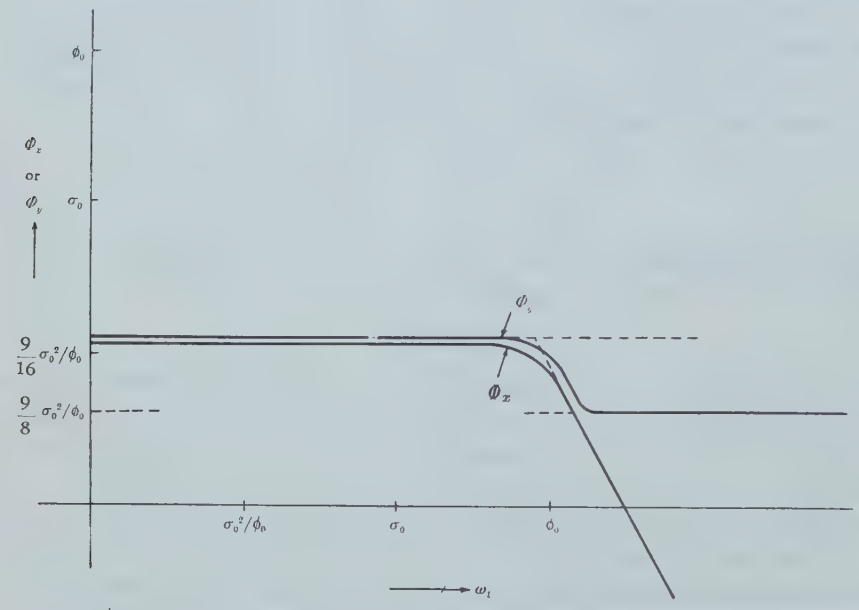

Fig. 7. Dependence of the transverse relaxation upon the degree of saturation. Case (A) in an interacting spin assembly. (logarithmic scale)

§ 3. B. The case of rigid lattice. ($\sigma_0 \gg \phi_0$)

This is the case in which the external system does not fluctuate over many periods of relative precession due to the distribution of the local field. In this case we have to treat the system more dynamically in order to have complete information about the time variation of $\langle Q^r \rangle$. However, let us again confine ourselves to the case of Gaussian shape absorption which is expected under a certain physical condition described in § 2. B. In order to obtain a qualitative insight we again replace the Gaussian line by a Lorentzian one with an equal separation between half-maximums, and concentrate our attention on the change of this width upon saturation. Then, we may use the Bloch type equation (2·6) and its solutions (2·11) to (2·13) also here. When compared with the situa-

tion (A) the main difference lies in the fact that the transverse relaxations, Φ_x and Φ_y, are now dependent on the power level of the rotating field, i. e.

$$\Phi_x = \frac{9}{32}\sigma_0'\left\{c^2(3c^2-1)+4c^2s^2\left(1-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+\tilde{\omega}^2}}\right)+s^2(2-c^2)\left(1-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+(2\tilde{\omega})^2}}\right)\right\}$$

$$+\frac{5}{2}\left(\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_1^2}\right)+\frac{\sigma_2^2\phi_2}{\phi_2^2+(2\omega)^2}, \tag{3·27}$$

$$\Phi_y = \frac{9}{32}\sigma_0'\left\{(3c^2-1)(2c^2-1)+8c^2s^2\left(1-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+\tilde{\omega}^2}}\right)+s^2(1-2c^2)\left(1-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+(2\tilde{\omega})^2}}\right)\right\}$$

$$+\frac{5}{2}\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega^2}+\frac{\sigma_2^2\phi_2}{\phi_2^2+(2\omega)^2}, \tag{3·28}$$

although the longitudinal relaxation Φ_z is not. At exact resonance we have

$$\Phi_x = \frac{9}{16}\sigma_0'\left(1-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+(2\omega_1)^2}}\right)+\frac{5}{2}\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}+\frac{\sigma_2^2\phi_2}{\phi_2^2+(2\omega_0)^2}$$

$$\equiv \sigma_0^*(1-s_2)+\sigma_1^*+\sigma_2^* \equiv \sigma'(1-\rho s_2), \tag{3·29}$$

$$\Phi_y = \frac{9}{32}\sigma_0'\left(2-e^{-\frac{\sigma_0''\phi_0}{\phi_0^2+(2\omega_1)^2}}\right)+\frac{5}{2}\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}+\frac{\sigma_2^2\phi_2}{\phi_2^2+(2\omega_0)^2}$$

$$\equiv \frac{\sigma_0^*}{2}(2-s_2)+\sigma_1^*+\sigma_2^* \equiv \sigma'\left(1-\frac{\rho}{2}s_2\right), \tag{3·30}$$

where we used the abbreviations

$$s_2 \equiv \exp\left\{-\frac{\sigma_0''\phi_0}{\phi_0^2+(2\omega_1)^2}\right\}, \tag{3·31}$$

$$\sigma_0^* \equiv \frac{9}{16}\sigma_0', \quad \sigma_1^* \equiv \frac{5}{2}\frac{\sigma_1^2\phi_1}{\phi_1^2+\omega_0^2}, \quad \sigma_2^* \equiv \frac{\sigma_2^2\phi_2}{\phi_2^2+(2\omega_0)^2} \tag{3·32}$$

and σ' and ρ are defined by

$$\sigma' \equiv \sigma_0^*+\sigma_1^*+\sigma_2^*, \tag{3·33}$$

$$\rho \equiv \sigma_0^*/\sigma'. \tag{3·34}$$

In general, terms involving σ_1 and σ_2 are of the order of the longitudinal relaxation Φ_z and are much smaller than those involving σ_0, which determine the main feature of relaxation in the case of negligible saturation. The effect of saturation becomes tangible when

$$\omega_1 \gtrsim (\sigma_0''\phi_0)^{1/2}, \tag{3·35}$$

which is contrasted with the condition $\omega_1 \gtrsim \phi_0$ in the situation (A). The behaviour of the transverse relaxations is shown in Figure 8, and it involves some significant difference

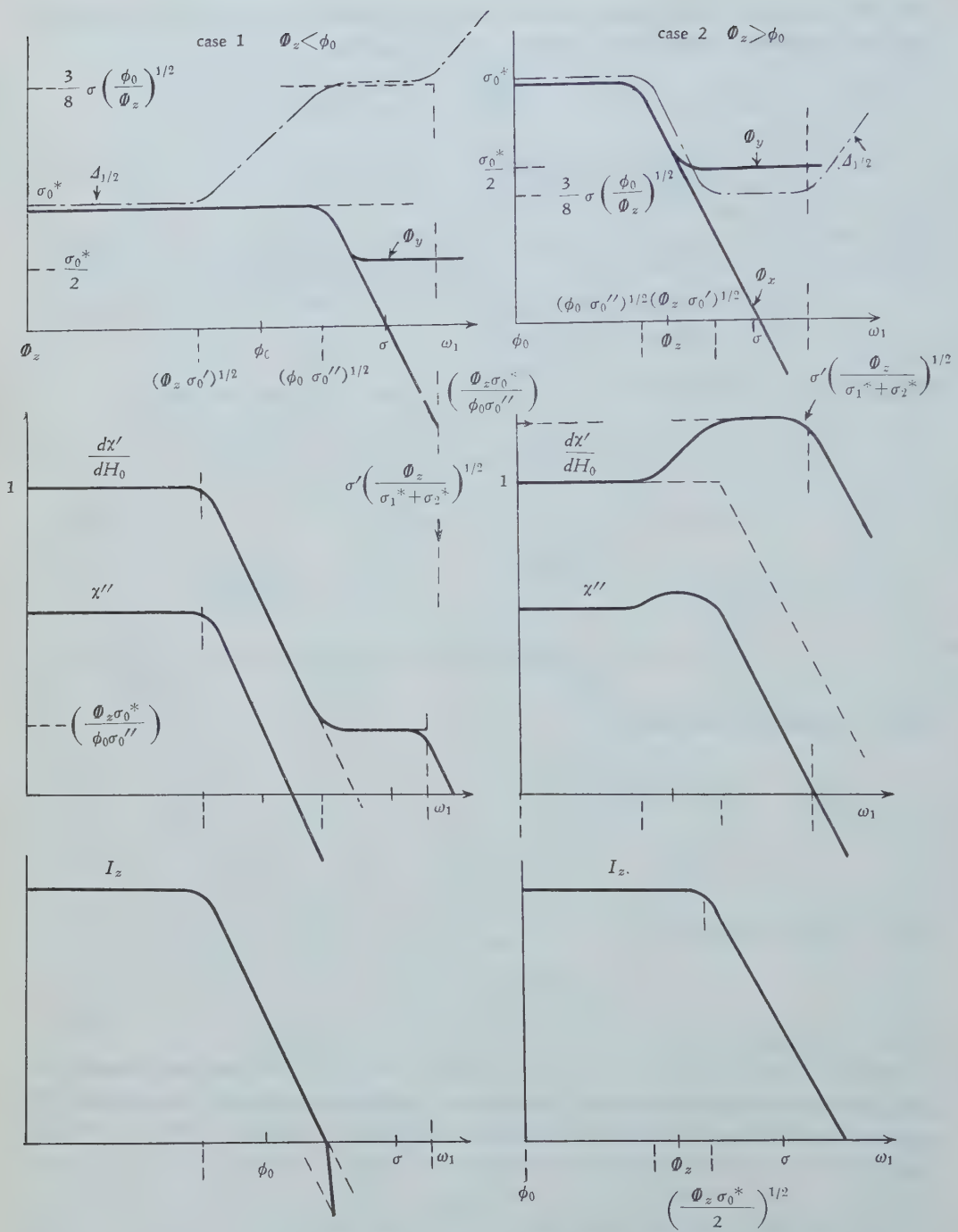

Fig. 8. Steady state solution as a function of the degree of saturation. Case (B) in an interacting spin assembly. (logarithmic scale)

as compared with the local field model. In Φ_x the term involving σ_0 declines to nought and those involving σ_1 and σ_2 survive upon saturation, which are of the same order, but not exactly the same as the longitudinal relaxation Φ_z except in the case of complete isotropy, for Φ_z is given by

$$\Phi_z=\frac{\sigma_1^2\phi_1}{\phi_1^2+\tilde{\omega}^2}+\frac{4\sigma_2^2\phi_2}{\phi_2^2+(2\tilde{\omega})^2}. \tag{3·36}$$

On the other hand in Φ_y the term involving σ_0 declines only to one half of the original value upon saturation, and remains of the same order of magnitude. Redfield anticipated this situation in a heuristic manner, but his quantitative expectation should be corrected according to our results obtained here. Now we look into the steady state solution for the case of exact resonance, which is given by

1. Dispersion derivative. (x-component)

$$\frac{dI_x}{d\Delta}=\frac{\hbar\omega}{4kT}\frac{\omega_1}{\sigma_0^{*2}}\left\{\frac{1}{(1-\rho s_2)\left(1-\frac{1}{2}\rho s_2+\omega_1^2/\sigma_0^*\Phi_z\right)}\right\}$$

$$=\left(\frac{dI_x}{d\Delta}\right)^0\left\{\frac{1}{(1-\rho s_2)\left(1-\frac{1}{2}\rho s_2+\omega_1^2/\sigma_0^*\Phi_z\right)}\right\}, \tag{3·37}$$

2. Absorption. (y-component)

$$I_y=\frac{\hbar\omega}{4kT}\frac{\omega_1}{\sigma_0^*}\left\{\frac{1}{1-\frac{1}{2}\rho s_2+\omega_1^2/\sigma_0^*\Phi_z}\right\}$$

$$=(I_y)^0\left\{\frac{1}{1-\frac{1}{2}\rho s_2+\omega_1^2/\sigma_0^*\Phi_z}\right\}, \tag{3·38}$$

3. z-polarization. (z-component)

$$I_z=\frac{\hbar\omega}{4kT}\left\{\frac{1}{1+\omega_1^2/\Phi_z\sigma_0^*(1-\frac{1}{2}\rho s_2)}\right\}$$

$$=(I_z)^0\left\{\frac{1}{1+\omega_1^2/\Phi_z\sigma_0^*(1-\frac{1}{2}\rho s_2)}\right\}. \tag{3·39}$$

The behaviour of this solution is shown in Figure 8, which is largely similar to the case of Figure 4, except the fact that we have an asymptotic slope -2 instead of -4 for the value of polarization plotted in logarithmic scale. Here, too, we have an asymptotic relation

$$\Delta_{1/2}\sim\left(\frac{d\chi'}{dH_0}\right)^{-1/2} \tag{3·40}$$

between the half-value width and the dispersion derivative. When ϕ_0 and Φ_z are not independent and are governed by single mechanism, the case (2) in Figure 8 is not realized actually. The steady state solution is again described by "a canonical distribution in an appropriately chosen system of coordinates", however we must be aware of

the fact that the definition of the relaxation constants Φ_x, Φ_y, Φ_z is now different. For a sufficiently high power level it is clear that the situation is described by a canonical ensemble in the precessing frame.

§ 3. C. The case in which two different types of mechanism are existent.

In section 2. C we have already treated the case in which the line width (spin-relaxation) and the lattice relaxation are determined by completely different mechanisms. The argument in this section runs largely parallel to that given there, however the largest difference lies in the fact that here we cannot simply superpose the effect of two mechanisms for the relaxation. Again the case (A) + (B) will be taken up as an example. Let us assume for simplicity that the process (A) is of local field type and isotropic, and the process (B) is characterized by the spin-spin interaction. The nuclear magnetic relaxation in metals belongs to this case, where the process (A) is embodied by the Fermi interaction of nuclei with the conduction electrons and is actually isotropic as was shown by Korringa.[23] On the other hand the process (B) in this case is given by the spin-spin interaction between nuclei which are situated almost rigidly in the lattice, and the resonance line width is usually determined by this process.

If we concentrate our attention on a single spin, at first it seems enough to superpose the effect of two mechanisms, however in an interacting spin assembly there exists a multiple process in which the excitation hops from one spin to another by the resonance interaction in the process (B) and then relaxes to the lattice by the process (A). In order to take complete account of multiple processes we should retain the effect of the perturbation higher in order than the fourth. Although the effect of some interplay is expected already in the second and the third order terms, it can be neglected when the first order shift is small. Let us then concentrate our attention on a representative term in the fourth order which is given by

$$\Delta_4^{baab}\sigma^{T*}=\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3}\sum_{\lambda_4}\sum_{\mu_1}\sum_{\mu_2}\sum_{\mu_3}\sum_{\mu_4}\sum_{\nu_1}\sum_{\nu_2}\sum_{\nu_3}\sum_{\nu_4}A_{\mu_1\nu_1}a_{\mu_2\nu_2}a_{\mu_3\nu_3}A_{\mu_4\nu_4}$$

$$\times\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\int_0^{t_3}dt_4\,e^{i(\omega_{\lambda_1}+\omega_{\mu_1}+\omega_{\nu_1})t_1+i(\omega_{\lambda_2}+\omega_{\mu_2}+\omega_{\nu_2})t_2+i(\omega_{\lambda_3}+\omega_{\mu_3}+\omega_{\nu_3})t_3+i(\omega_{\lambda_4}+\omega_{\mu_4}+\omega_{\nu_4})t_4}$$

$$\times\langle[I_b(\nu_1)F_b^{\bar{\mu}_1}(\lambda_1)[I_a(\nu_2)F_a^{\bar{\mu}_2}(\lambda_2)[I_a(\nu_3)F_a^{\bar{\mu}_3}(\lambda_3)[I_b(\nu_4)F_b^{\bar{\mu}_4}(\lambda_4),\,\sigma^{T*}P_F]]]]\rangle_{a,b},\qquad(3\cdot41)$$

where the suffixes a and b indicate those quantities involved in the processes (A) and (B) respectively, and $\langle\ \rangle_{a,b}$ implies the process of taking the trace after multiplying the canonical distribution with respect to the corresponding lattice degree of freedom. The following simplifications result from the conditions characterizing our example.
i) $\lambda_1+\lambda_4=0$ and $\lambda_2+\lambda_3=0$ from the independence of two lattice degrees of freedom and the definition of trace. ii) $\mu_1=\mu_4=0$ and $\omega_{\lambda_1}=\omega_{\lambda_4}=0$, consequently

$$\langle F_b^{\bar{\mu}_1}(\lambda_1)F_b^{\bar{\mu}_4}(\lambda_4)\rangle_b=\langle|F_b^0|^2\rangle,\qquad(3\cdot42)$$

because the lattice in the process (B) is rigid with respect to the precession. iii) Neglecting satellite contribution, we have $\mu_2+\mu_3=0$. Let us define a correlation function by

$$\langle F_a^{\bar{\mu}_2}(t_2) F_a^{\mu_2}(t_3)\rangle \equiv \sum_{\lambda_2} \langle F_a^{\bar{\mu}_2}(\lambda_2) F_a^{\mu_2}(\bar{\lambda}_2)\rangle e^{i\omega_{\lambda_2}(t_2-t_3)}, \tag{3·43}$$

and further assume a simple exponential decay of it as

$$\langle F_a^{\bar{\mu}_2}(t_2) F_a^{\mu_2}(t_3)\rangle = \langle |F_a^{\bar{\mu}_2}|^2\rangle e^{-\phi_{\mu_2}|t_2-t_3|}. \tag{3·44}$$

Using the above simplifications, we can write down the contribution of the term given by (3·41) to the differential equation which governs the expectation value of Q^T. This is

$$\begin{aligned}\frac{d}{dt}\mathcal{A}_4^{baab} Q^T = &\sum_{\nu_1}\sum_{\nu_4} A_{0\nu_1} A_{0\nu_4} \int_0^t dt_4' e^{-i\omega_{\nu_4} t_4'} \langle |F_b^0|^2\rangle \sum_{\mu_2}\sum_{\nu_2}\sum_{\nu_3} a_{\mu_2\nu_2} a_{\bar{\mu}_2\nu_3} \\ &\times \int_0^{t_4'} d\tau\, (t_4'-\tau)\, e^{i\omega_{\mu_2}\tau - i\omega_{\nu_2} t_2' - i\omega_{\nu_3} t_3'} \langle |F_a^{\bar{\mu}_2}|^2\rangle e^{-\phi_{\mu_2}\tau} \\ &\times [I_b(\nu_4)[I_a(\nu_3)[I_a(\nu_2)[I_b(\nu_1), Q^T]]]],\end{aligned} \tag{3·45}$$

where $t_i' \equiv t - t_i$ and $\tau \equiv t_3' - t_2' = t_2 - t_3$.

iv) By making use of the fact that the process (A) is isotropic, that is $\phi_\mu \equiv \tau_\mu^{-1} > \mathrm{Max}\ (|F_a^\mu|,\ \omega_\mu,\ \omega_\nu)$, we have

$$\begin{aligned}&\int_0^{t_4'} d\tau\, (t_4'-\tau)\, e^{i\omega_{\mu_2}\tau - i\omega_{\nu_2} t_2' - i\omega_{\nu_3} t_3'} \langle |F_a^{\bar{\mu}_2}|^2\rangle e^{-\phi_{\mu_2}\tau} \\ &\qquad \to t_4' \langle |F_a^{\bar{\mu}_2}|^2\rangle \tau_{\mu_2} + \text{constant}.\end{aligned} \tag{3·46}$$

Combining this term with the second order effect $\mathcal{A}_2^{bb}$ of the process (B), we find

$$\begin{aligned}\frac{d}{dt}(\mathcal{A}_2^{bb} + \mathcal{A}_4^{baab}) Q^T = &\sum_{\nu_1}\sum_{\nu_4} A_{0\nu_1} A_{0\nu_4} \langle |F_b^0|^2\rangle \int_0^t dt_4' e^{-i\omega_{\nu_4} t_4'} \Big\{ [I_b(\nu_4)[I_b(\nu_1), Q^T]] \\ &- t_4' \sum_{\mu_2}\sum_{\nu_2}\sum_{\nu_3} a_{\mu_2\nu_2} a_{\bar{\mu}_2\nu_2} \langle |F_a^{\bar{\mu}_2}|^2\rangle \tau_{\mu_2} [I_b(\nu_4)[I_a(\nu_3)[I_a(\nu_2)[I_b(\nu_1), Q^T]]]]\Big\},\end{aligned} \tag{3·47}$$

where the part embraced in the curly bracket can be compared with the contribution of the isolated process (A)

$$(\mathcal{A}_0 + \mathcal{A}_2^{aa}) Q^T = Q^T - t_4' \sum_{\mu_2}\sum_{\nu_2}\sum_{\nu_3} a_{\mu_2\nu_2} a_{\bar{\mu}_2\nu_3} \langle |F_a^{\bar{\mu}_2}|^2\rangle \tau_{\mu_2} [I_a(\nu_3)[I_a(\nu_2), Q^T]]. \tag{3·48}$$

Let us bear in mind that (3·48) is an approximate initial stage solution of the differential equation

$$\frac{d}{dt}\Big)_a Q^T = -\sum_{\mu_2}\sum_{\nu_2}\sum_{\nu_3} a_{\mu_2\nu_2} a_{\bar{\mu}_2\nu_3} \langle |F_a^{\bar{\mu}_2}|^2\rangle \tau_{\mu_2} [I_a(\nu_3)[I_a(\nu_2), Q^T]]. \tag{3·49}$$

If we choose the spin angular momentum $\boldsymbol{I}$ as Q^T, (3·49) can be written as

$$\frac{d}{dt}\Big)_a \boldsymbol{I} = -\mathcal{O}_z(\boldsymbol{I} - \boldsymbol{I}_0) \tag{3·50}$$

after augmenting the equilibrium polarization in the instantaneous system and taking the isotropy condition into account. Suppose we have the conditions $I=1/2$ and $\hbar\omega_0 \ll kT$, then $[I_b(\nu_4)[I_b(\nu_1), \boldsymbol{I}]]$ can be expressed simply by a linear combination of the components of vector $\boldsymbol{I}$. In this case the part in the curly bracket of (3·47) is essentially the same as (3·48). Using then the solution of (3·49) as a more complete expression, the differential equation corresponding to (3·47) is simplified to

$$\frac{d}{dt}\Big)_b Q^T = \sum_{\nu_1}\sum_{\nu_4} A_{0\nu_1} A_{0\nu_4} \langle |F_b^0|^2 \rangle [I_b(\nu_4), [I_b(\nu_1), Q^T]] \int_0^{t_4} dt_4' \, e^{-(i\omega_{\nu_4}+\Phi_z)t_4'}. \quad (3\cdot51)$$

This is nothing but an analogue of equation (1·24) involving $\Phi^{\mu\nu}$ which is defined by (1·29), except that ϕ_0 is replaced by Φ_z here. In other words ϕ_0 is no more an independent parameter, but is specified by the process (A) as

$$\phi_0 = \Phi_z. \quad (3\cdot52)$$

We now proceed to show that the terms which have not been taken up have only a negligible contribution. For example the effect of the term Δ_4^{abba} is masked by the process (A) itself. Terms of the type Δ_4^{abab} and Δ_4^{aabb} generally oscillate and give satellites only. Terms like Δ_4^{abbb} and Δ_4^{baaa} are negligible when the first order shift is small, and those like Δ_4^{aaaa} and Δ_4^{bbbb} have already been taken into account when the equation was written in a differential form. The relation (3·52) was found empirically in the previous section, whereas it has been derived theoretically here. Another difference from the local field model lies in the fact that $\Phi_x \neq \Phi_y$. For the case in which the line width is determined by the process (B) and the lattice relaxation is determined by the process (A) we may define

$$\sigma' \equiv \Phi_{2a} + \Phi_{2b}, \quad (3\cdot53)$$

$$\rho s \equiv \rho_a s_a + \rho_b s_b = (\Phi_{2a}/\sigma') s_a + (\Phi_{2b}/\sigma') s_b, \quad (3\cdot54)$$

$$\rho' s' \equiv \rho_a s_a + \tfrac{1}{2}\rho_b s_b = (\Phi_{2a}/\sigma') s_a + (\Phi_{2b}/2\sigma') s_b, \quad (3\cdot55)$$

and with them write

$$\Phi_x = \Phi_{2a}(1-s_a) + \Phi_{2b}(1-s_b) = \sigma'(1-\rho s), \quad (3\cdot56)$$

$$\Phi_y = \Phi_{2a}(1-s_a) + \Phi_{2b}(1-\tfrac{1}{2}s_b) = \sigma'(1-\rho' s'). \quad (3\cdot57)$$

By using these notations we now have the steady state solutions at exact resonance in the following forms.

1. Dispersion (x-component)

$$\frac{dI_x}{d\Delta} = \frac{\hbar\omega}{3kT} I(I+1)\omega_1 \Big/ \left\{ \Phi_x \left(\Phi_y + \frac{\omega_1^2}{\Phi_z} \right) \right\}$$

$$= \frac{\hbar\omega}{3kT} \frac{I(I+1)\omega_1}{\sigma'^2} \Big/ \left\{ (1-\rho s)\left(1-\rho' s' + \frac{\omega_1^2}{\sigma' \Phi_z} \right) \right\} \quad (3\cdot58)$$

2. Absorption (y-component)

$$I_y = -\frac{\hbar\omega I(I+1)\omega_1}{3kT}\Big/\left(\Phi_y + \frac{\omega_1^2}{\Phi_z}\right) = -\frac{\hbar\omega I(I+1)\omega_1}{3kT\sigma'}\Big/\left(1-\rho' s' + \frac{\omega_1^2}{\sigma'\Phi_z}\right) \tag{3·59}$$

3. z-polarization (z-component)

$$I_z = -\frac{\hbar\omega I(I+1)}{3kT}\Big/\left(1+\frac{\omega_1^2}{\Phi_y\Phi_z}\right) = \frac{\hbar\omega I(I+1)}{3kT}\Big/\left(1+\frac{\omega_1^2}{\Phi_z\sigma'(1-\rho' s')}\right). \tag{3·60}$$

The behaviour of these steady state solutions with respect to the saturation level is largely the same as that given in § 2. C and Figure 5. In fact qualitatively it gives the proper way of describing the experimental results obtained by Redfield,[14] Holcomb,[17] and Sugawara,[16] i. e. (1) the dispersion mode survives the absorption upon increasing the power level, (2) the line width is directly related to the dispersion derivative at resonance by

$$\Delta_{1/2} \simeq (d\chi'/dH_0)^{-1/2}. \tag{3·61}$$

When we compare the results with experimental data a further simplification is allowed by the condition $s_a \ll 1$, consequently $\rho' s' = \frac{1}{2}\rho_b s_b$. The observed characteristics are closer to Case 2 in Figure 5, although the condition $\phi_0 \sim \Phi_z$ suggests a situation somewhere between cases 1 and 2.

Again it is possible to describe the steady state solution by using a simple canonical distribution in an appropriately chosen system of coordinates and the criterion for the individual case is given in Table 3. In our special example we have the situation $\Phi_{2a} \fallingdotseq \Phi_z$ and $s_a \fallingdotseq 0$, therefore

$$\Phi_y = \Phi_z + \Phi_{2b}(1-\tfrac{1}{2}s_b) = \sigma'(1-\tfrac{1}{2}\rho_b s_b), \tag{3·62}$$

where

$$\sigma' \equiv \Phi_z + \Phi_{2b}, \quad \rho_a \equiv \Phi_z/\sigma', \quad \rho_b \equiv \Phi_{2b}/\sigma', \tag{3·63}$$

and

$$s_b \equiv \exp\left[-\sigma_0'' \Phi_z/\{\Phi_z^2 + (2\omega_1)^2\}\right]. \tag{3·64}$$

The two critical frequencies in Table 3 now become

$$\omega_0{}^* = \Phi_y{}^2/\Phi_z = (\sigma'/\rho_a)(1-\tfrac{1}{2}\rho_b s_b)^2, \tag{3·65}$$

and

$$\omega_0{}^{**} = (\Phi_y \Phi_z)^{1/2} = \sigma'[\rho_a(1-\tfrac{1}{2}\rho_b s_b)]^{1/2}, \tag{3·66}$$

so that $\omega_0{}^*$ and $\omega_0{}^{**}$ are reduced to 1/4 and $1/\sqrt{2}$ respectively, but they are not changed in the order of magnitude. Consequently, the criterions for the validity of the canonical distribution in the rotating frame are:

i) When $\omega_0 \Phi_z \gg \sigma'^2$,

$$\omega_1 \gg (\omega_0 \Phi_z)^{1/2}. \tag{3·67}$$

ii) When $(\sigma'^2/\Phi_z) \gg \omega_0 \gg (\sigma' \Phi_z)^{1/2}$,

$$\omega_1 \gg (\omega_0 \sigma' \Phi_z)^{1/3}. \tag{3·68}$$

In this way Redfield's anticipation[14] has been given quantitative criteria. The power level at which the saturational narrowing appears is estimated to be

$$(\omega_1^*)^2 \simeq \sigma' \Phi_z \tag{3·69}$$

according to (2·49), while the lattice relaxation Φ_z for the metallic nucleus is given by

$$\Phi_z = 1/T_1 = (9\pi kT/4\hbar\zeta_0^2)\,\langle \mathcal{E}(kk)\rangle_{av}^2 \tag{3·70}$$

after Korringa.[23] Therefore, we obtain the relation

$$(\omega_1^*)^2 \propto T \quad \text{(absolute temperature)}. \tag{3·71}$$

This relation coincides with the characteristic behaviour of nuclear resonance in metallic copper at low temperatures which is observed by Sugawara.[16] Concerning the absorption and dispersion derivative at exact resonance more quantitative comparisons are given in Figure 9. In the limiting cases of very low and high power levels the agreements of theory with experiment are satisfactory, however they do not really coincide in the intermediate region. The cause of this discrepancy may be traced back to several points.

i) We have already mentioned that the Bloch type equation holds rigorously for the case of $I=1/2$ only, whereas in the examples taken up for comparison I happens to be $3/2$ for both Cu^{63} and Al^{27}. Therefore, the linear theory cannot be applied in an exact sense, although it can describe the overall characteristics.

ii) The interpolation formula (2·24) connecting the Lorentzian limit to the Gaussian is, in a sense, a matter of convenience and is not actually established, although they give proper quantitative description in both limits. Point ii) may and should be improved mathematically, however point i) cannot be improved in a simple manner. It

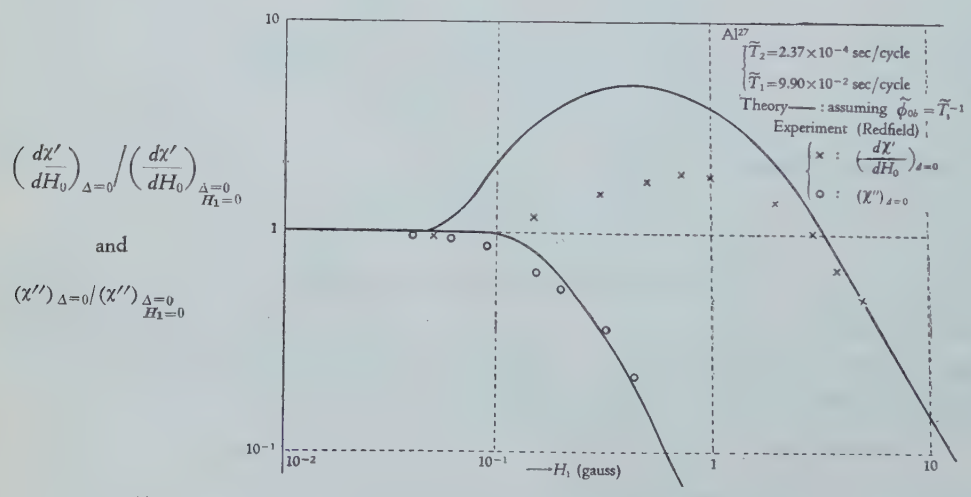

Fig. 9 (a). Nuclear magnetic dispersion and absorption as functions of the degree of saturation in metallic aluminium. Theoretical curves are based on the formulae for an interacting spin assembly. Experimental data were taken from Redfield (Reference 14).

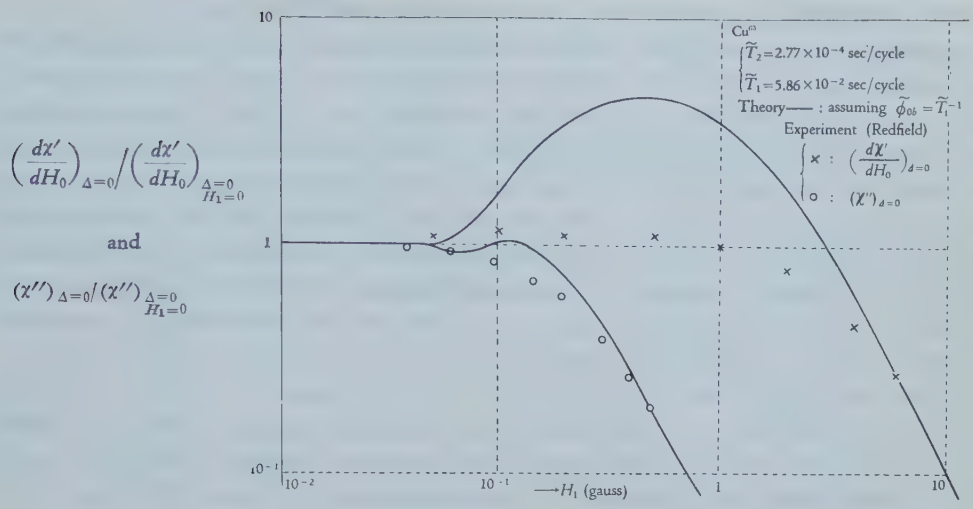

Fig. 9 (b). Nuclear magnetic dispersion and absorption as functions of the degree of saturation in metallic copper. Theoretical curves are based on the formulae for an interacting spin assembly. Experimental data were taken from Redfield (Reference 14).

should be pointed out that the local field model holds even for spins of $I \neq 1/2$ with its due reservation as was shown in the previous section.

§ 4. Discussion and summary

When we use the word " relaxation " it seems customary to anticipate an underlying linear process (simply Markoffian process), however in actual relaxation processes, we cannot neglect the role which is played by the non-Markoffian process. One of the aims of this paper is to show systematically that exactly such a process is involved as a limiting case in magnetic resonance saturation, i. e. the case called " anomalous saturation ", in which the observed characteristics do not obey the phenomenological equation of Bloch.

The situation is similar to that in the case of " motional narrowing ",[3),7)] which we shall discuss briefly at this point. When the motional frequency is greater than the frequency of the relative precession, which is caused by the distribution of the local field due to the interaction of the spin with its surrounding, the relative precession becomes moderated by the motion and is observed as an incoherent Markoffian process. This situation is characterised by the Lorentzian shape of the resonance line. When the motional frequency becomes greater than the frequency of the absolute precession, the behaviour of relaxation becomes essentially isotropic, i. e. $T_1 = T_2$.[5)]

Under an application of a strong radiofrequency field a " forced nutation ", i. e. the precession in the precessing system of coordinate, should be added to the above situation. The resulting behaviour is markedly different according to whether the

motional modulation is predominant (Situation (A)) or not (Situation (B)).

Situation (A). In order to modify the relaxation behaviour in this case appreciably we must raise the power level until the nutational frequency becomes greater than the motional, for the local field is already moderated by the motional effect. In usual examples the relaxation constant, which is given by (relative precessional frequency)^2/motional frequency, does not change upon saturation, so that once we find the behaviour of the free induction, we can deduce the saturation characteristics by simply feeding the radiofrequency field into the phenomenological equation. The above-mentioned modification of the relaxation constant at the extreme saturation has independently been treated by Bloch,[5] however his ascription of Redfield's observation to this case seems to the present author unjustified. It should rather be included in the case below.

Situation (B). In contrast to situation (A) the motional narrowing is not yet tangible and a coherent local field is observed in the absence of saturation. Therefore the forced nutation itself plays the role of modulator above a certain critical frequency, i. e. [(relative precessional frequency) × (motional frequency)]$^{1/2}$, and does disturb the coherence. In this way there appears "saturational (or forced nutational) narrowing" which is analogous to "motional narrowing", and the line shape becomes Lorentzian upon saturation irrespective of the shape of the unsaturated line. In other words the relaxation constants, in particular the transverse relaxation constants, do depend on the saturation level under experimentally attainable conditions. It has also been found that in this case dispersion survives the absorption until the nutational frequency becomes comparable with the relative precessional frequency. This phenomenon is closely related to "saturational narrowing". All of these characteristics are found in the phenomenon of so-called anomalous saturation, of which several examples have been observed by Redfield,[14] Sugawara[16] and Holcomb.[17]

In the static frame of reference the forced nutation due to radiofrequency field corresponds to the repetition of forced emission and absorption and has an effect of limiting the lifetime of spin levels. This causes the so-called "saturational broadening" which exists in the extremely high power region in both situations, (A) and (B).

When there are two different kinds of spin system interacting with each other, again the saturation behaviour is classified into two markedly different cases according to a similar criterion. The situation corresponding to Case A leads to a generalized Overhauser polarization effect,[18] while the situation corresponding to Case B leads to the phenomenon of saturational narrowing.[19] The details will be given in a separate paper.

Concerning the improvement of the present treatment we have already mentioned two points at the end of the previous section. Side by side it is highly desirable to investigate experimentally the relation between the dispersion and the line width. The case of very low constant field has been consciously excluded from our treatment, which therefore should be treated separately. The old problem of pure spin-spin relaxation[24] belongs to this case, and seems to be another example of the non-Markoffian process. As the validity of the usual perturbational concept is dubious, we must be extremely careful in the treatment of this case.

There are several other examples of anomaly in the behaviour of magnetic resonance saturation, e. g. the case of ferromagnetic resonance[25)–27)] and also the case of nuclear resonance in high polymer substances.[28),29)] For the latter case the reported evidence does not seem to be enough to prove that it is a saturational effect. If it is, we should learn much more experimentally about the specific saturation behaviour. The case of ferromagnetic resonance is outside the scope of the present treatment, which is based essentially on paramagnetic assumption, and has been treated recently from a different point of view.[30),31)] It should be remarked, however, that common to all cases anomalous saturation occurs when the situation lies outside the range of validity of the linear stochastic phenomenology. In this situation we cannot identify the width of resonance with the inverse time of spin-relaxation in general, and we should be rather careful in interpreting the experimental data. In other words, it is not enough to calculate the value of the relaxation time supposing that it exists, but the proper form of the macroscopic equation* obeyed by the system under consideration should first be derived from the microscopic standpoint. The calculation of the conventional relaxation time, then, may or may not be meaningful according to the answer to the first and more fundamental question.

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* For instance there seems to be little theoretical justification of Landau-Lifschitz's[32)] equation based on microscopic consideration.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Note on the Strange Particle Effects in S-wave Pion-Nucleon Scattering

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February 14, 1958

Although the usual π-meson theory has achieved considerable success in reproducing the large P-wave phase shifts in pion-nucleon scattering, it has not been able to explain the observed S-wave phase shifts. The purpose of this paper is to investigate the strange particle effects in S-wave pion-nucleon scattering.

If we confine our considerations within the lowest order strange particle effects, Ξ particle does not take part in. The interaction scheme that will be used here corresponds to the interaction Hamiltonian density :

$$\begin{aligned} H_I = & ig\bar{\psi}_N\gamma_5\tau_\alpha\psi_N\varphi_\alpha \\ & + ig_1\bar{\psi}_\Sigma\gamma_5 I_\alpha\psi_\Sigma\varphi_\alpha + g_2\bar{\psi}_\Lambda\psi_{\Sigma\alpha}\varphi_\alpha \\ & + if_1\bar{\psi}_N\gamma_5\psi_\Lambda\phi_K + f_2\bar{\psi}_N\tau_\alpha\psi_{\Sigma\alpha}\phi_K \\ & + h.c. \end{aligned} \tag{1}$$

where

$$(I_\alpha)^{ij} = (-i)\,\epsilon_{ijk}\delta_{k\alpha} \tag{2}$$

and (i, j, k) in the three index symbol ϵ_{ijk} are cyclic. The lowest order strange particle effects for pion-nucleon scattering are given graphically by $(a)-(d)$ in Fig. 1 and their crossed terms. In Fig. 1, the broad line denotes either Λ or Ξ, and α-line the φ_α-meson.

Using eq. (2), it is easy to derive the iso-spin factors for these processes. Apart from the numerical factor, (a), (b) type processes are expected to have the iso-spin factor of the form $\tau_\alpha\tau_\beta$, while the iso-spin factors of (d) type processes are for various (Y_1, Y_2, Y_3)-combinations as follows :

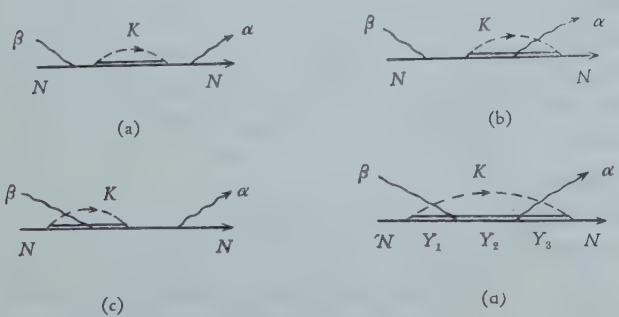

Fig. 1.

(Y_1, Y_2, Y_3)	
$\sum, \sum, \sum$	$\tau_\alpha\tau_\beta + \delta_{\alpha\beta}$
$\sum, \sum, \Lambda$	$\tau_\alpha\tau_\beta - \delta_{\alpha\beta}$
$\Lambda, \sum, \sum$	$\tau_\alpha\tau_\beta - \delta_{\alpha\beta}$
$\sum, \Lambda, \sum$	$\tau_\alpha\tau_\beta$
$\Lambda, \sum, \Lambda$	$\delta_{\alpha\beta}$
crossed term	$(\alpha \rightleftarrows \beta)$

Our interaction scheme in addition to the above iso-spin dependence makes us infer that, if there exist some remarkable features which do not appear in the case of the usual π-meson theory, they are included mainly in (d) type processes. Thus our analysis has been done for the second order terms and (d) type terms, and it has been investigated how these processes affect S-wave pion-nucleon scattering.

The effective potential for the π-meson is determined by the canonical transformation. This is then inserted in the relativistic Klein-Gordon equation for the π-meson, and the scattering in this potential is calculated exactly. This procedure is the "potential approximation", which was first proposed by S. F. Edwards and P. T. Matthews[1]. Other detailed procedures are completely parallel to their argument. (In the course of numerical calculation, we neglect the baryon recoil.) Thus, after the lengthy and tedious calculations, we can get the S-wave phase shifts for pion-nucleon scattering at the low energy limit as follows :

$$\begin{aligned} \tan\delta_1 &\simeq +0.04\,(k/\mu) \\ \tan\delta_3 &\simeq -0.30\,(k/\mu) \end{aligned} \qquad (3)$$

for $g^2/4\pi = g_1^2/4\pi = g_2^2/4\pi \simeq 10$ and $f_1^2/4\pi = f_2^2/4\pi \simeq 1$. This result shows the strong iso-spin splitting, but drastically contradicts the experimental data ($\tan\delta_1 = +0.16\,(k/\mu)$, $\tan\delta_3 = -0.11\,(k/\mu)$). As regards the various coupling constants, we cannot find the values fit for the experimental data as long as we keep $g^2/4\pi \gtrsim 10$ and (*others*) $\lesssim g^2/4\pi$. Under the same condition, we can hardly get the result which is better than eq. (3).

As there are other contributions from all sorts of the fourth order effect, some change has to be expected for the values of eq. (3). But, under our interaction scheme, it is reasonable to infer that the contributions from some of (d) type graphs are fairly large, even if the inclusion of other terms is done. So the iso-spin splitting due to the strange particle effects can be expected for the S-wave phase shifts, as long as our H_I, eq. (1), is adopted. It is also noted that there are some evidences[2] which show our interaction scheme is inadequate. Recently J. S. Langer[3] has investigated the similar effects. It must be also noted that his results depend on the use of different interaction scheme and the neglect of Λ-particle.

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On the Lateral Distribution of the Electron Component in Extensive Air Showers

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March 1, 1958

Recent experiments using the apparatus of large span have made it possible to

measure the lateral distribution of the electron component in each air shower.[1] The result shows that the shape of the lateral distribution is practically independent of shower sizes and altitudes at which observations were made, i. e., mountain elevations and sea level, and is in agreement with the Nishimura-Kamata (*N-K*) function[2] of age $\cong 1.3$. It has, therefore, been considered that the high-energy nucleonic component passing along the shower axis continues to supply its energy uniformly to the soft component as the shower develops towards lower altitudes. Thus the following speculation has prevailed : The rate of energy supplied to the latter does not strongly depend on the amount of energy carried by high-energy nuclear active particles, and the altitude variations of shower sizes are similar to each other at lower altitudes irrespective of energy of high-energy nuclear active particles.

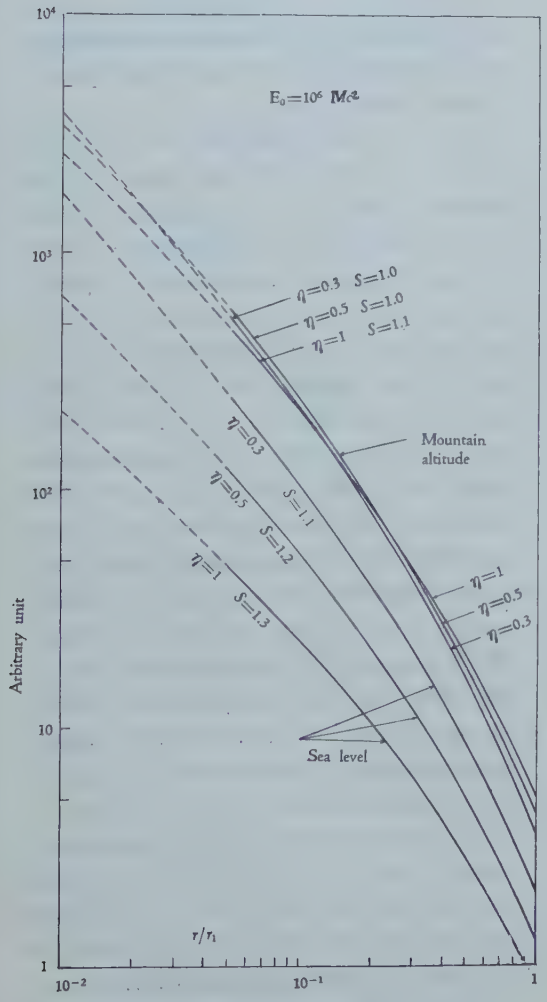

Fig. 1.

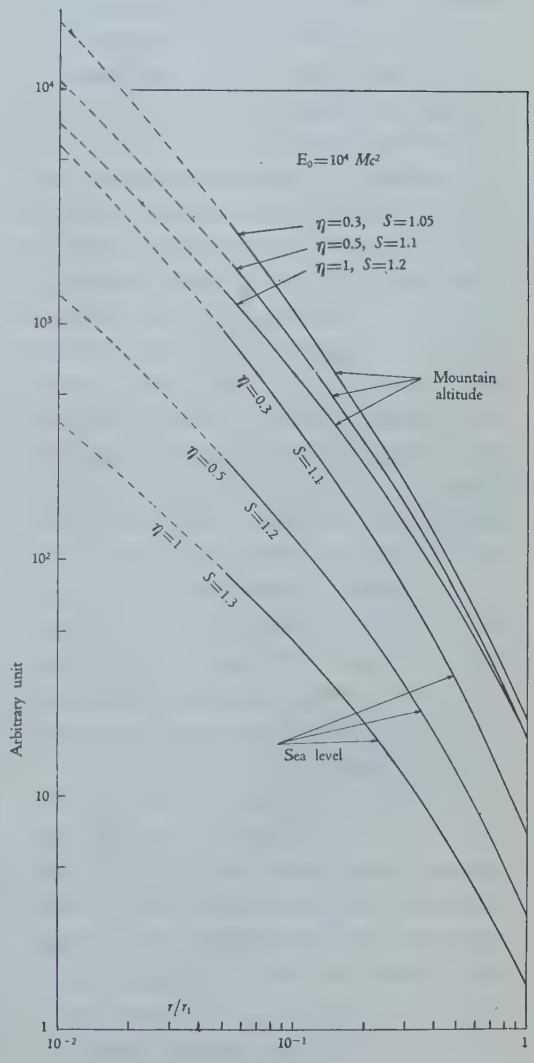

Fig. 2.

On the other hand, the altitude variation of shower sizes should be governed by the rate of the energy supplied to the soft component, because its rate is intimately related to the inelasticity at high-energy nuclear collisions. Actually we have shown in our previous paper[3] (quoted as I here) that even for the fixed value of inelasticity the altitude variation of shower sizes is affected strongly by the energy of the primary particle. It is, therefore, necessary to test quantitatively how much the inelasticity affects the lateral distribution of the electron component. This is the aim of this short note.

The model of high-energy nuclear collisions used here is the same as described in I. The calculation was made under an assumption that the lateral spread of electrons is only due to multiple Coulomb scattering of electrons and the effects of the angular spreads of nuclear active particles and two-photon-decay of neutral pions are negligible. As the structure function an approximate formula of the *N-K* function derived by Nishimura[4] was used; this formula coincides with numerically values of the *N-K* function for $s=0.6$, 1, 1.4 and 2 within errors of several percents. The calculation was made by the same procedure as in I, for primary energies of $10^4 Mc^2$ and $10^6 Mc^2$, where Mc^2 is the rest energy of the nucleon.

The results are shown in figures. s on each curve is the shower age of the *N-K* function which gives the best fit to each calculated lateral distribution and η is the inelasticity. In the dashed parts of the curves there are some errors, because of the following reason. The shower age in the three dimensional cascade theory depends not only on the primary energy E but also on the distance from the shower axis as a function of $(Er/\varepsilon_c r_1)$, where ε_c is the critical energy in air and r_1 is the Moliere unit. Accordingly, the smaller r/r_1 gets, the more showers initiated by individual neutral pions are found to have ages larger than two. The approximate formula is no longer applied to the case of such an age. For $r/r_1 \lesssim 5\cdot 10^{-2}$ the number of such showers becomes considerable, the contribution of such showers to the density of electrons is not large. For such a region, however, it becomes meaningless to evaluate the density more accurately, because of the neglect of the angular spreads as stated above.

It is interesting that the lateral distribution is insensitive to the value of the inelasticity and also to the primary energy. This fact contrasts with the fact that the altitude variation of shower sizes depends strongly both on the inelasticity and on the primary energy. Therefore, we see that the experimental result as stated above is not much related to the rate of the energy supply as has been considered, that is, to the character of high-energy nuclear collisions. Considering more finely, the ages at sea level are slightly larger than those at mountain altitude and also slightly smaller than the experimental one except for the cases of $\eta=1$ at sea level. However, this slight discrepancy will be removed automatically, if we have a model which can explain the altitude dependence of the electron component.

The authors would like to express their sincere thanks to Professor T. Inoue for his continual encouragement. They would like to express their deep gratitude to Professors J. Nishimura and S. Hayakawa for helpful discussions. One of the authors

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A Possible Size Effect on the Plasma Oscillation of Electrons in Solids

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March 3, 1958

In recent years,[1] many experimental investigations have been carrried out on the excitation of electron plasma oscillation in solids. Analyses of these experiments gave substantial support for Bohm-Pines' plasma theory of electron interaction in solids.[2]

The purpose of the present note is to point out possible manifestation of the plasma oscillation of electrons in solids. According to the theory, the frequency of the plasma oscillation is determined by the following relation,

$$\hbar\omega=\hbar\omega_p+\langle T\rangle/\hbar\omega_p\cdot(\hbar^2k^2/m), \qquad (1)$$

neglecting higher order terms of k^2. In eq. (1), $\langle T\rangle$ is the average kinetic energy of valence electrons and ω_p is the well-known plasma frequency $(4\pi e^2n/m)^{1/2}$ with the electron density n.

We call one's attention to the fact that the wave number k of the plasma oscillation is equal to momentum transfer to a valence electron from external perturbation which excites the plasma oscillation of valence electrons. Thus, in the case of electron bombardment, the k^2 term could account for the angular dependence of the eigen energy losses accompanied by the excitation of the plasma oscillation.[3]

As another possible manifestation of the effect of k^2 term, we will discuss an observable effect of it for a finite size crystal. For a finite size crystal of linear dimension d, it is possible to have only a stationary wave with maximum wave length given by $\lambda_{\max}\simeq 2d$. In other words, it gives minimum limit of the wave number k as

$$k_{\min}=2\pi/\lambda_{\max}=\pi/d.$$

Putting this into eq. (1), we get

$$\hbar\omega=\hbar\omega_p+\langle T\rangle/\hbar\omega_p\cdot(\pi^2\hbar^2/md^2). \qquad (2)$$

Watanabe has tried to observe such finite size effect for Al.[4] For the case of Al, the contribution of the second term of eq. (2) is evaluated as, assuming the valence electrons are free,

$$38/D^2\ eV \qquad (2)$$

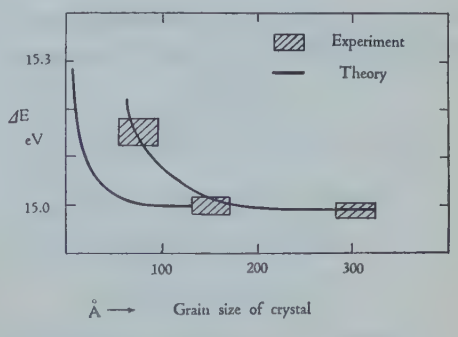

Fig. 1.

where D is measured in the unit of angstroms. Fig. 1 shows the comparison of the experimental and theoretical results. According to the above estimation, the observable effect is expected to show up itself for rather small grain size less than 50 Å, contrary to the experimental data. Contibution of the surface energy to the average kinetic energy $\langle T \rangle$ is estimated according to Swiatecky's theory.[5] The correction is in right direction, but the amount of it is extremely small so that it fails to explain the discrepancy in the data.

To get conclusive results for the effect, it is necessary to advance the experimental techniques to achieve higher energy resolution, as well as to use more tiny crystal grain.

It is a great pleasure to thank Dr. Watanabe for his kindness to inform us of his experimental data.

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Hyperon Decays and Characteristics of Interactions between Elementary Particles

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February 26, 1958

The weak interactions describing decays of elementary particles have some regularity and universality. At first, as is well known, the Fermi type weak fermion interaction with the same universal coupling constants for parity conserving and non-conserving terms has explained successfully the nuclear β-decay and π-μ-e decay. As for the pion and K-meson decays, it also seems that the universal weak boson fermion interaction of gradient type with the same amount of even and odd parts is more desirable than the direct type coupling; e. g. $R_\pi = W(\pi \to e+\nu)/W(\pi \to \mu+\nu) = 5.5$ for direct coupling, while $R_\pi = 1.3 \times 10^{-4}$ for gradient coupling,[1] and similar ratio of K_{e2} to $K_{\mu 2}$ mode is 1.1 and 2.5×10^{-5} for direct and gradient coupling, respectively. The latters in both cases are not inconsistent with experiments.[2][3] As for the hyperon decays, moreover, recent observations on the angular asymmetries have appeared to emphasize a similar tendency. We write the interaction Hamiltonian for the hyperon decays:

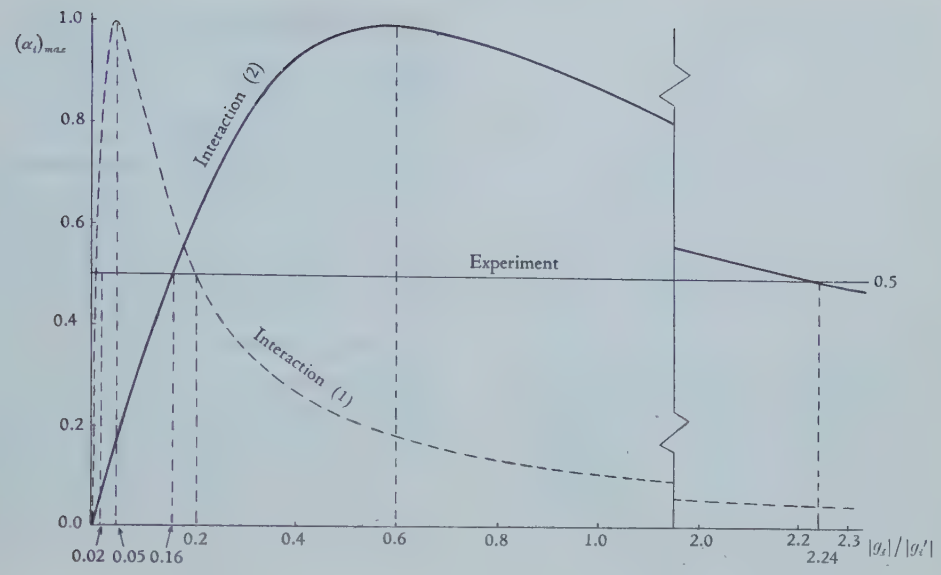
$$H_1 = g_1 \bar{\psi}_a \psi_b \phi_\pi + g'_1 \bar{\psi}_a \gamma_5 \psi_b \phi_\pi + h.c., \quad (1)$$

and

$$H_2 = i g_2 \bar{\psi}_a \gamma_\mu \psi_b \partial_\mu \phi_\pi + i g_2' \bar{\psi}_a \gamma_5 \gamma_\mu \psi_b \partial_\mu \phi_\pi + h.c.; \quad (2)$$

here suffices a and b refer respectively to a parent and daughter particle with spin 1/2. In what follows, quantities with suffix 1 and 2 correspond to the interaction (1) and (2), individually. Let θ be the angle between the direction of polarization of the hyperon and the momentum of the decay pion in the c. m. s. of the hyperon. The decay distribution function for θ in the case of (1) and (2) are given by

$$W_i(\theta)\, d\Omega = A_i \{ (|g_i|^2/4\pi) + (B_i/2)^2 (|g_i'|^2/4\pi) \}$$



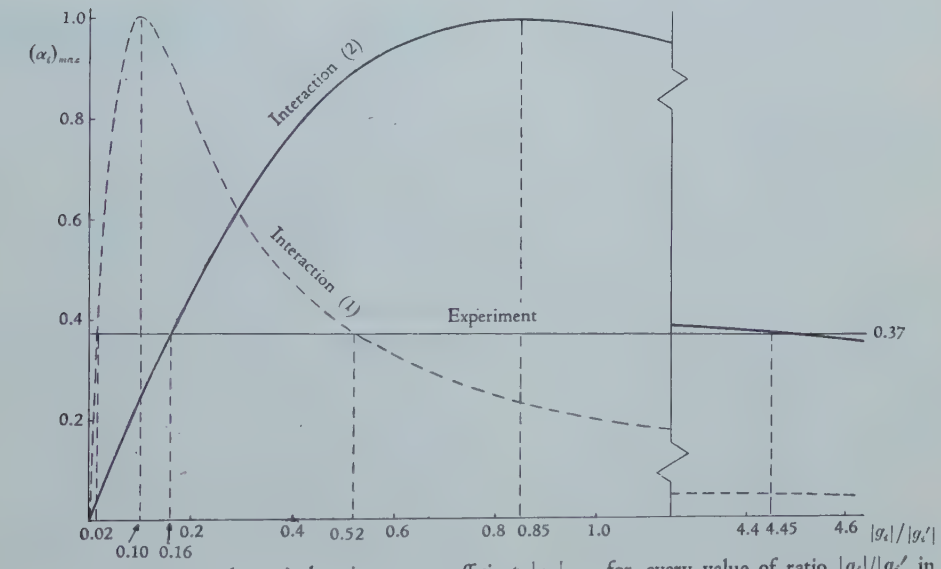


Fig. 1. Maximum theoretical anisotropy coefficient $|\alpha_i|_{\max}$ for every value of ratio $|g_i|/|g_i'|$ in Λ^0 decay.

Fig. 2. Maximum theoretical anisotropy coefficient $|\alpha_i|_{\max}$ for every value of ratio $|g_i|/|g_i'|$ in Σ^+ decay.

$$\times (1+\alpha_i P\cos\theta)\, d\Omega/4\pi \quad (i=1,\ 2), \qquad (3)$$

where P denotes the degree of polarization of the hyperons $(|P|\leq 1)$,

$$\alpha_i = B_i\, Re\,(g_i^* g_i')/[|g_i|^2+(B_i/2)^2|g_i'|^2] \qquad (4)$$

and

$$A_1 = (1/4)\,(1/m_a)^3\{(m_a+m_b)^2 - m_\pi^2\}^{3/2}\{(m_a-m_b)^2-m_\pi^2\}^{1/2},$$

$$A_2 = A_1 (m_a-m_b)^2,$$

$$B_1 = 2\{(m_a+m_b)^2 - m_\pi^2\}^{-1/2} \times \{(m_a-m_b)^2 - m_\pi^2\}^{1/2},$$

$$B_2 = B_1(m_a+m_b)/(m_a-m_b). \qquad (5)$$

Calculated results are presented in the tables and figures. The last columns in these tables are obtained from comparing $\int W_i(\theta)\, d\Omega$ with the observed lifetimes and exhibit that we may suppose the universality of coupling constants in the hyperon decay interactions. Large anisotropies have been observed in Λ^0 and Σ^+ decays, though not in Σ^- decay. For experimental values of $\alpha\bar{P}\sim 0.5$ for the Λ^0 decay and ~ -0.37 for the Σ^+ decay,[4] inequalities derived from (4):

$$|(\alpha\bar{P})_{\text{obs}}| = \bar{P}\frac{B_i Re(g_i^* g_i')}{|g_i|^2+(B_i/2)^2|g_i'|^2} \leq \frac{B_i|g_i||g_i'|}{|g_i|^2+(B_i/2)^2|g_i'|^2} = \frac{B_i|g_i|/|g_i'|}{(|g_i|^2/|g_i'|^2)+(B_i/2)^2}, \quad (i=1, 2) \qquad (6)$$

are not satisfied with the interaction (1), but consistent with the interaction (2), under the assumption $|g_i|\sim|g_i'|$, which has been also assumed for the weak fermion interaction (see the figures). The final state interaction would not change our conclusion essentially.

From above, we might expect that all the decay interactions not conserving parity come within the category of the second kind interactions (in which the coupling constants have dimension of positive power of length in natural units). The classification into the first and the second kind interactions according to the dimensional character of the coupling constants was proposed by Heisenberg[5] and disdussed by Umezawa[6] in connection with the renormalization procedure. It is interesting to note that various substantial contents of this classification are now illuminated by recent experimental development on decay and scattering phenomena. Some comment on the expectation that the strong interactions conserving parity belong to the first kind interactions will be discussed in some other place.

The authors express their cordial thanks to Prof. T. Eguchi, Prof. S. Hori, Dr. Y. Ohnuki, Dr. S. Nagata and the members of Kyusyu University for helpful discussions.

Table I

decay scheme	A_1	B_1	$\{\|g_1\|^2+(B_1/2)^2\|g_1'\|^2\}/4\pi$
$\Lambda^0\to P+\pi^-$	2.6×10^{23}	0.11	1.3×10^{-14}
$\Sigma^+\to P+\pi^0$	4.7 〃	0.20	3.1 〃
$\Sigma^+\to N+\pi^+$	4.5 〃	0.20	3.2 〃
$\Sigma^-\to N+\pi^-$	4.6 〃	0.20	1.4 〃
$\Xi^-\to\Lambda^0+\pi^-$	3.6 〃	0.39	

Table II

decay scheme	A_2/m_π^2	B_2	$\{\|g_2\|^2+(B_2/2)^2\|g_2'\|^2\}\, m_\pi^2/4\pi$
$\Lambda^0\to P+\pi^-$	4.2×10^{23}	1.2	8.6×10^{-15}
$\Sigma^+\to P+\pi^0$	16.6 〃	1.7	8.7 〃
$\Sigma^+\to N+\pi^+$	14.4 〃	1.7	10.1 〃
$\Sigma^-\to N+\pi^-$	15.6 〃	1.7	4.0 〃
$\Xi^-\to\Lambda^0+\pi^-$	7.2 〃		

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* After completion of this work, we happened to know that Umezawa, Konuma and Nakagawa have got essentially the same results. We acknowledge Prof H. Umezawa's kindness to inform us of their work.

The Spirality of the Decay Product

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March 1, 1958

It can be shown that the spiralities of the decay products have simple regularity. This stems from our previously mentioned standpoint[1] and one additional assumption that the all weak interactions must be universally V-A coupling.* Then our interaction Hamiltonians for weak processes are restricted to (1).

* From our standpoint which postulates the $I\Gamma_5 T$ invariance for interactions, the mixing of $V-A$ and $S-T-P$ type is forbidden except V). The order of four fermion fields is taken so as to allow such separation. The assumption that $(\bar{\Lambda} p)(\bar{p}\, n)$ and $(\bar{\Lambda} n)(\bar{p}\, p)$ are of the same form necessarily leads to that $\Lambda-\mathfrak{N}$ interaction is of $V-A$ type.

$$
\begin{aligned}
&\text{i)} \quad f(\bar{\psi}_p \gamma_\rho' \psi_n)\,(\bar{\psi}_e(1\mp\gamma_5)\gamma_\rho \psi_\nu) + H.C. \\
&\text{ii)} \quad f(\bar{\psi}_e(1\mp\gamma_5)\gamma_\rho \psi_\mu)\,(\bar{\psi}_\nu \gamma_\rho' \psi_\nu) + H.C. \\
&\text{iii)} \quad f(\bar{\psi}_p \gamma_\rho' \psi_n)\,(\bar{\psi}_\mu(1\mp\gamma_5)\gamma_\rho \psi_\nu) + H.C. \qquad (1) \\
&\text{iv)} \quad f(\bar{\psi}_p(1\pm\gamma_5)\gamma_\rho \psi_\Lambda)\,(\bar{\psi}_{\mu(\text{or } e)}(1\mp\gamma_5)\gamma_\rho \psi_\nu) \\
&\qquad + H.C. \\
&\text{v)} \quad f(\bar{\psi}_\Lambda(1\mp\gamma_5)\gamma_\rho \psi_p)\,(\bar{\psi}_p \gamma_\rho' \psi_n) + H.C.,
\end{aligned}
$$

where $\gamma_\rho' = \gamma_\rho$ or $\gamma_5\gamma_\rho$, ψ_e and ψ_μ are annihilation operators of e^- and μ^- respectively. For simplicity, p and n fields are transformed simultaneously under the individual γ_5 transformation (abbreviated as $I\Gamma_5 T$).[1] From the postulation of $I\Gamma_5 T$ invariance, the phases (defined in reference 1)) satisfy (2). ($\eta_a^b (b \neq a)$ is the change of phase of 'a'-field ψ_a when $\psi_b \to \eta_b^b \gamma_5 \psi_b$.)

$$
\begin{aligned}
&\text{i)} \quad \eta_e^{e*}\eta_\nu^e = \pm 1, \\
&\qquad \eta_e^{\nu*}\eta_\nu^\nu = \pm 1 \text{ and } \eta_e^{a*}\eta_\nu^a = 1 \\
&\text{(for all particles } a \text{ except } e \text{ and } \nu), \\
&\text{ii)} \quad \eta_e^{e*}\eta_\mu^e = \pm 1, \\
&\qquad \eta_e^{\mu*}\eta_\mu^\mu = \pm 1 \text{ and } \eta_e^{a*}\eta_\mu^a = 1. \\
&\text{(for all particles } a \text{ except } e \text{ and } \mu), \\
&\text{iii)} \quad \eta_\mu^{\mu*}\eta_\nu^\mu = \pm 1, \\
&\qquad \eta_\mu^{\nu*}\eta_\nu^\nu = \pm 1 \text{ and } \eta_\mu^{a*}\eta_\nu^a = 1 \qquad (2) \\
&\text{(for all particles } a \text{ except } \mu \text{ and } \nu), \\
&\text{iv)} \quad \eta_{\mathfrak{N}}^{a*}\eta_\Lambda^a\eta_\mu^{a*}\eta_\nu^a \\
&\qquad = \begin{cases} -1 \text{ for } a = \mathfrak{N},\ \Lambda,\ \mu \text{ and } \nu, \\ \quad \text{and } +1 \text{ for } a = e, \\ \quad +1 \text{ for all } a\text{'s}, \end{cases} \\
&\qquad = \begin{cases} -1 \text{ for } a = \mathfrak{N} \text{ and } \Lambda, \\ \quad \text{and } +1 \text{ for other } a\text{'s}, \\ -1 \text{ for } a = \mu \text{ and } \nu, \\ \quad \text{and } +1 \text{ for other } a\text{'s}, \end{cases}
\end{aligned}
$$

v) $\eta_{\Lambda}^{\Lambda *}\eta_{\mathfrak{N}}^{\Lambda}=\pm 1$,

$$\eta_{\Lambda}^{\mathfrak{N} *}\eta_{\mathfrak{N}}^{\mathfrak{N}}=\pm 1 \text{ and } \eta_{\Lambda}^{a *}\eta_{\mathfrak{N}}^{a}=1,$$

(for all particles a except Λ and $\mathfrak{N}$).

Double signs are of the same order as in (1).

Now, it is well established that in β-transitions the polarization is nearly $-(+)v/c$ for $e^{-}(e^{+})$.[2] Then the only allowable choice is the upper sign for i). Admitting this choice, other signs are automatically settled, e. g., $\eta_{e}^{e *}\eta_{\nu}^{e}=1$ of i) and $\eta_{\mu}^{e *}\eta_{\nu}^{e}=1$ of iii) demand $\eta_{e}^{e *}\eta_{\mu}^{e}=1$. In this way we can find a simple law for the phases of leptons.

$$\eta_{\text{lep.1}}^{a *}\eta_{\text{lep.2}}^{a}=+1, \text{ for all particles } a \qquad (2')$$

The upper signs of ii) and iii) are allowed by 2').

The restricted Hamiltonian,

ii') $f(\bar{\psi}_{e}(1-\gamma_{5})\gamma_{\rho}\psi_{\mu})\,(\bar{\psi}_{\nu}\gamma_{\rho}'\psi_{\nu})+H.C.$,

can predict the e^{+} emitted by μ^{+} is right-handed. Furthermore,

iii') $f(\bar{\psi}_{p}\gamma_{\rho}'\psi_{n})\,(\bar{\psi}_{\mu}(1-\gamma_{5})\gamma_{\rho}\psi_{\nu})+H.C.$,

predicts the spirality of μ^{+} meson produced by π^{+} decay to be left-handed.* Then the e^{+} must be emitted by μ^{+} preferentially backward to the direction of its motion. This is consistent with the experiment. The predicted behavior of $\pi^{+}-\mu^{+}-e^{+}$ chain is illustrated in Fig. 1.

By (2') and $\eta_{\Lambda}^{a *}\eta_{\mathfrak{N}}^{a}=1$ (for all a's except Λ and $\mathfrak{N}$) of v), iv) is restricted as follows.

iv') $f(\bar{\psi}_{p}(1\mp\gamma_{5})\gamma_{\rho}\psi_{\Lambda})$

$$(\bar{\psi}_{\mu(\text{or } e)}(1-\gamma_{5})\gamma_{\rho}\psi_{\nu})+H.C.$$

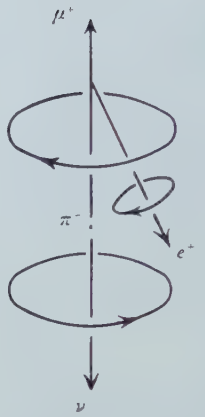

Fig. 1.

From iv'), we predict the similar behavior between $K^{+}-\mu^{+}-e^{+}$ and $\pi^{+}-\mu^{+}-e^{+}$ chains. This is consistent with the experiment.[5]

Finally we briefly note the reason why we choose the $V-A$ coupling as the universal interaction of weak processes. As we shall show in a later issue, if we choose $S-T-P$ coupling for β-decay, it is impossible to determine the phases consistently with experimental facts. There are many other reasons preferring to take up $V-A$ coupling. As is well known, the asymmetry coefficient in polarized Λ decay is large.[6] This fact necessarily requires to chose V-A coupling as far as we assume the $\Pi'_{5}T$ invariance. Furthermore, the derivative coupling is rather better to suppress the rate of β-decay of a boson than that of the direct coupling. In view of these facts, the universality of $V-A$ coupling for weak interactions and our standpoint are very natural.

The details of this letter will be published in a later issue of this journal.

* The matrix element for π^{-}-decay is assumed to be written as

$$\sim A(\bar{\psi}_{\mu}(p)\cdot(1-\gamma_{5})\gamma_{\rho}\psi_{\nu}(q))\,k_{\rho}\varphi_{\pi}(k)$$
$$\cdot\delta^{4}(p+q-k).^{4)}$$

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Reevaluation of the Electric Quadrupole Moment of the Deuteron

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February 18, 1958

The most recent experiments by the HD beam magnetic resonance method with high precision[1] have given us the following nuclear quadrupole coupling constant :

$$\frac{1}{h}\langle eQ_d q\rangle_{0,1}^{\mathrm{HD}} = 224540 \pm 60 \text{ c/s}, \quad (1)$$

which is accurate about 100 times in comparison with former experiments and also more precise than the experimental value of the pure deuterium molecule[2], where several well-known constants are included : the elementary charge e, the Planck constant h, the electric quadrupole moment of the deuteron Q_d, and the electric field gradient at the position of the deuteron $q=\langle \partial^2 V/\partial z^2\rangle_{0,1}$, which should be averaged over the vibrational $(v=0)$ and rotational $(J=1)$ quantum states.

In the present stage the most accurate value of the quadrupole moment of the deuteron evaluated through the electromagnetic interaction is considered as based upon the theoretical calculation of q obtained by the use of Newell's variational function.[3]

Although an error of 4 percent in the dissociation energy given by this electronic wave function is as much as about 10 percent improvement over that by Nordsieck's function,[4] this improvement in the dissociation energy does not always correspond exactly to that in the wave function itself[5], and thus even Newell's function would not be good enough for the evaluation of Q_d in comparison with the accuracy of the recent beam experiments. Our final evaluation of Q_d by making use of the James-Coolige (11-term) wave function[6] is now in progress.

In these calculations there are serious problems in addition to the problem of the above mentioned electronic wave function. One is the problem of the average over the vibrational and rotational states, where some ambiguities are involved in the wave function of a non-rigid vibrating rotator without exact potential and also in the interpolation method for the dependence of $q'(R)$ on the internuclear distance R $(q=-2q'/5$ for $J=1)$.

Before making the conclusive calculations we attempted to improve these two points for a preliminary verification. Thus in consideration of the special extension of the vibrational wave function we refine the interpolation of $q'(R)$ by adding a new value of $q'=0.09304$ a.u. at $R=1.6$ a.u. to Newell's results. Then we found the following value with some refinement in the vibrating rotator :

$$\langle q' \rangle_{0,1}^{\mathrm{HD}} = 0.1758 \text{ a.u.}, \qquad (2)$$

instead of Newell's value of 0.1745 a.u. And also in regard to the first excited vibrational state we obtained

$$\langle q' \rangle_{1,1}^{\mathrm{HD}} = 0.1804 \text{ a.u.} \qquad (3)$$

Our value (2) was equal to the value of $\langle q' \rangle_{0,2}^{\mathrm{D_2}}$ in the above approximation, while Newell's value for D_2 was 0.1749 a.u. Such results mean seriousness of the interpolation method mentioned above, including the problem of the interpolation function (an expansion formula).

Then taking the average of the two nuclear coupling constants of (1) and the value of $1/h\langle eQ_d q \rangle_{0,1}^{\mathrm{D_2}}$, we obtained

$$Q_d = (2.721 \pm 0.016) \cdot 10^{-27} \text{ cm}^2, \qquad (4)$$

where a probable error of 0.6 percent is to be considered barely in the above procedure.

No detailed discussions about the present calculations and the related deuteron problems can be done untill our conclusive evaluation is brought to completion.

Finally we wish to thank Professor N. F. Ramsey for sending his unpublished experimental data.

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A Covariant Interpretation of the Existence of the Parity Non-Conservation and the New Freedoms for Elementary Particles

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March 3, 1958

In the recent investigations of fundamental properties of elementary particles, two remarkable discoveries have been made. One is the foundation of the concept of strength of interactions, performed by Pais, Gell-Mann and Nishijima, leading to the classification of new particles, even though it was attained with a model in an imaginative space and there seems to remain some unsettled problems with respect to leptons. The other is the discovery of the existence of parity non-conserving terms in weak interactions, although it is not yet explained why such non-conservation occurs only in the weak interactions and not in the strong interactions; this was suggested theoretically by Lee and Yang and determined experimentally by Wu et al. This discovery was decisive because the parity non-conservation concerns with the ordinary space time, while many sorts of non-conservation occur in the former model space.

These are the most fundamental progress made in the recent years, and it seems to be one of the most important problems in the basic theoretical physics to give a comprehensive explanation to these properties of elementary particles. It seems to be possible to say that these problems should be settled simultaneously, because the non-conservation problems may be solved con-

currently with the physical nature of interactions and then it will become possible to give a reason why the parity is not conserved only in the weak interactions.

Anyhow the fact is established that the fundamental properties of particles cannot be derived from the covariance in the Minkowski space. In this note we will propose a possible covariant interpretation of these problems, by assuming a slight non-Riemann deviation from the Minkowskian structure of space time. By assuming a certain torsional deviation of space time, at least in the domain quite near the matter, of the magnitude of the order of the weak interaction coefficient, we can show that (i) a covariant interaction term can exist, approaching to the so-called parity non-conserving term in the Minkowski approximation, and can occur only in weak interactions and disappears effectively in strong interactions, (ii) two sorts of spin like new freedoms can be derived from the structure of four dimensional space time, without any fictitious space, in the same manner as the ordinary spin was derived from the Minkowskian space time structure.

Our intention is to introduce a space time structure with extra fundamental tensors yielding fundamental structural quantity with the transformation character of a pseudoscalar. Since the order of magnitude of weak interactions seems to be larger than that of the gravitation, the above stated properties should be derived even in cases with negligible gravitation. So we will introduce a slight tortion of the space time structure. We assume a space time structure with a connection

$$L^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\} + a^{\lambda}_{\mu\nu} + \Omega^{\lambda}_{\mu\nu}$$

where $\{^{\lambda}_{\mu\nu}\}$ is the Christoffel symbole (which is not a tensor), $a^{\lambda}_{\mu\nu}$ and $\Omega^{\lambda}_{\mu\nu}$ are the symmetric and antisymmetric (with respect to μ, ν) parts (which are tensors) of the connection having a magnitude of the weak interaction.

(i) We can construct a tensor antisymmetric with respect to all pairs of indices by the covariant differentiation $(,\rho)$ and the antisymmetrization ($[\ ]$) :

$$\Theta^{-}_{\lambda\mu\nu\rho} = \Omega^{\sigma}_{[\lambda\mu,\ \nu,\ \rho],\ \sigma}$$

$\Theta^{-}_{1234} \cong f_w$ (coefficient of weak interaction).

Then the terms having $\Theta^{-}_{\lambda\mu\nu\rho}$ can play an effective role only in weak interactions and can make the so-called parity non-conserving terms, in the Minskowski approximation, covariant. The covariancy of $\Theta^{-}_{\lambda\mu\nu\rho}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}$ makes the terms with β^5 covariant. The terms $a\phi_{\pi}\phi_{\pi}\phi_{\pi}$ and $b\phi_{\pi}\phi_{\pi}$ can have the same transformation character if Θ^{-} is involved in either a or b. These give a covariant interpretation of Lee-Yang-Wu et al'.s experiment and the $\theta-\tau$ puzzle.

(ii) Though in a Riemannian space we have only one fundamental tensor, i.e. $g_{\mu\nu}$, we have in our case two extra fundamental tensor $a^{\lambda}_{\mu\nu}$, $\Omega^{\lambda}_{\mu\nu}$. By these two we can construct at least two extra symmetric tensors of the second rank other than $g_{\mu\nu}$, for example, $a^{\lambda}_{\mu\nu,\lambda}$, $a^{\lambda}_{\mu(\nu}\Omega^{\mu}_{\rho)\lambda}$, $a^{\lambda}_{\lambda(\nu}\Omega^{\mu}_{\rho)\mu}$, $\cdots$. If we have a fundamental tensor $A^{\lambda\mu}$, we can introduce a spinor, by extending the method of Infeld and van der Waerden. The spin matrices $\sigma^{\lambda\dot{\alpha}\beta}$, the spin metric matrix $\epsilon^{\alpha\beta}$ and the spin connection $\Lambda^{\alpha}_{\beta\lambda}$ can be defind as follows.

$$\sigma^{\mu\alpha\dot{\beta}}\sigma^{\lambda}{}_{\dot{\alpha}\tau} + \sigma^{\lambda\dot{\alpha}\beta}\sigma^{\mu}{}_{\dot{\alpha}\lambda} = A^{\lambda\mu}\delta^{\beta}_{\tau},$$

$$\sigma^{\mu\dot{\tau}\beta}\sigma^{\tau}{}_{\dot{\alpha}\beta} + \sigma^{\lambda\dot{\tau}\beta}\sigma^{\mu}{}_{\dot{\alpha}\beta} = A^{\lambda\mu}\delta^{\dot{\tau}}_{\dot{\alpha}},$$

$$\left\{\frac{\partial\sigma^{\mu\dot{\alpha}\beta}}{\partial x^\rho}+L^\mu_{\nu\rho}\sigma^{\nu\dot{\alpha}\beta}+\Lambda^{\dot{\alpha}}_{\dot{\tau}\rho}\sigma^{\mu\dot{\tau}\beta}+\Lambda^{\sigma}_{\tau\rho}\sigma^{\mu\dot{\alpha}\nu}\right\}\sigma^{\lambda}_{\dot{\alpha}\tau}$$
$$+\sigma^{\mu\dot{\alpha}\beta}\left\{\frac{\partial\sigma^{\lambda}_{\dot{\alpha}\tau}}{\partial x^\rho}+\cdots\right\}+\left\{\frac{\partial\sigma^{\lambda\dot{\alpha}\beta}}{\partial x^\rho}+\cdots\right\}\sigma^{\mu}_{\dot{\alpha}\tau}$$
$$+\sigma^{\lambda\dot{\alpha}\beta}\left\{\frac{\partial\sigma^{\mu}_{\dot{\alpha}\tau}}{\partial x^\rho}+\cdots\right\}=\left\{\frac{\partial A^{\lambda\mu}}{\partial x^\rho}+L^\lambda_{\nu\rho}A^{\nu\mu}\right.$$
$$\left.+L^\mu_{\nu\rho}A^{\lambda\nu}\right\}\delta^\beta_\tau$$

Hence the particles in the assumed space can have the ordinary spin and two extra new spins corresponding to the metric tensor and the two extra tensors constructed by the structural tensors. Thus we have given a possibility of a covariant interpretation of the parity- and the new freedom-problems. But the precise structure must be determined after all the picture of quantum numbers of elementary particles is made clearer involving the lepton family.

Antinucleon Annihilation and Multiplicity of Emitted Pions

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The large cross section and the high multiplicity of emitted pions are the most conspicuous features of the nucleon-antinucleon annihilation experiments[1)2)]. In this note we should like to point out a semi-quantitative relation between these two facts.[3)]

Since the pair-annihilation is an exothermic reaction, its cross section at low energies* is expected to obey the $1/v$—law, if there is no resonance-like phenomenon. We shall provisionally make use of the total cross section data[1)], because a predominant fraction of the latter seems to be due to the annihilation process and not to the scattering.[4)]

Kinetic energy of $\bar{P}$ in lab. system (in Gev.)	Total cross section (in mb)	Cross section times velocity σv (in mb·c)
0.19	135±16	75±9
0.30	104±14	68±9
0.50	97±4	74±3
0.70	94±7	77±3

From these figures, it would not be unreasonable to conclude :

$$\lim_{v\to 0}\sigma\cdot v\approx 60\sim 90\ \text{mb}\cdot\text{c} \qquad (1)$$

As is well known, we can derive from (1) the rate of annihilation per unit time when a nucleon-antinucleon pair is confined in a volume V. Putting tentatively

$$V=a\left(\frac{\hbar}{m_\pi c}\right)^3, \qquad (2)$$

where a is a numerical constant of the order 1, we can estimate the effective time of an annihilation process, τ, as follows :

$$\tau=\frac{V}{\lim\sigma\cdot v}$$
$$\approx(1.1\sim 1.6)\,a\times 10^{-24}\ \text{sec}. \qquad (3)$$

On the other hand, we can derive the mean energy, $\bar{\omega}$, of the secondary pions from the experimental values of their multiplicity, 4.7 ± 0.4, and their momentum spectrum.[2)]

$$\bar{\omega}\approx 340\sim 440\ \text{Mev}, \qquad (4)$$

where $\bar{\omega}$ includes the pion rest energy.

* Compared with the available energy of the reaction,

Now it would be expected and could under certain conditions be verified that those modes of the pion field, which have frequencies of the order $1/\tau$, are most likely to be excited during the annihilation process.* Thus we have

$$\tau = b \cdot \frac{\hbar}{\bar{\omega}}$$

$$\approx (1.5 \sim 1.9) b \times 10^{-24} \text{ sec.}, \qquad (5)$$

where b is a numerical constant of the order 1.

Comparing (3) and (5) we get the ratio,

$$a/b \approx 1 \sim 2, \qquad (6)$$

which appears rather natural. This indicates that the experimental evidences for the cross sections and the mean energy of secondary pions are related to each other not unreasonably.**

A more detailed account will appear later.

The author is grateful to Professor G. Takeda for his discussions, to Dr. D. Ito and Dr. H. Miyazawa for their critical comments.

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* This way of estimation should generally give a lower limit for τ. In our case, however, it could be roughly equated to τ, because the annihilation of the pair and the emission of secondary particles are supposedly due to the same kind of strong interaction.

** When we apply this kind of reasoning to the case of electron pair annihilation, we have to introduce an effective volume $\sim (e^2/\hbar c)^2 \ (\hbar/mc)^3$, in order to obtain the actual energy of the emitted quanta. This would indicate that in the case of weaker interactions the definition of the volume is ambiguous and this approach would not make much sense.

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Cluster Expansion of the Ground State of a Bose Particle System

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Recently the properties of a hard sphere Bose gas at low density have been investigated by many researchers.[1] It is shown in this note that almost the same expression for the ground state energy can also be obtained by Jastrow's variational method[2)3)] if we take account of a part of the large cluster terms, which represents many-particle correlations caused by the mutual transfers of the same momentum among themselves.

In this method the trial function for the ground state is assumed to be a product of pair functions

$$\Psi = \Omega^{-N/2} \prod_{i<j} f(r_{ij}). \qquad (1)$$

Just as was done in the classical theory for the partition function of an imperfect gas,[4] logarithm of the normalization integral is developed in cluster terms when the particle number N is large :

$$N^{-1} \ln (\Psi, \Psi) = \sum_{l \geq 1} (k+1)^{-1} \beta_k \rho^k, \qquad (2)$$

where ρ is the density N/Ω and β_k the irreducible cluster integral of $k+1$ particles. As is well known, it is expressed as the sum of all terms, each of which corresponds to a graph of $k+1$ points connected by at least two lines. We now retain only those terms each of which corresponds to a closed chain, neglecting all the other terms which form polygons with some diagonal lines. This is the same approximation as was done in the classical theory by Montroll and Mayer.[5] In this approximation β_k is written simply as

$$\beta_k = (1/2)\int [g(q)]^{k+1} d\boldsymbol{q}/(2\pi)^3, \qquad (3)$$

where

$$g(q) = \int (f^*(r)f(r) - 1) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{r}. \qquad (4)$$

This closed chain is just the term arising from mutual transfers of the same momentum $\boldsymbol{q}$ among $k+1$ particles. After performing summation in (2) we can derive the final formula for the energy expectation value from the normalization integral in a way described in reference[3].

$$\langle H\rangle/N = (\hbar^2\rho/2m)\Big[\int F(r)\, d\boldsymbol{r} + \int \frac{\rho g(q)}{1-\rho g(q)} \{g(q)G(q) - K^2(q)\} \frac{d\boldsymbol{q}}{(2\pi)^3}\Big], \qquad (5)$$

where

$$F(r) = f^*(r)\,((m/\hbar^2)V(r) - \Delta^2)f(r),$$

$$G(q) = \int F(r) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{r},$$

$$K(q) = \int f^*(r) f'(r) j_1(qr) 4\pi r^2 dr.$$

In the last line $f'(r)$ is the derivative of $f(r)$, and $j_1(qr)$ is the usual spherical Bessel function.

In order to calculate the ground state energy of a hard sphere gas with interaction radius a, we assume the pair function to be as follows:

$$f(r) = \begin{cases} 0 & , r < a, \\ 1 - ae^{-\mu(r-a)}/r, & r > a, \end{cases} \qquad (6)$$

where μ is the variational parameter. This particular functional form is suggested from the fact that $1-a/r$ is the exact solution of the two-body problem. In this case, however, $g(q)$, $G(q)$, and $K(q)$ cannot be written in a simple form. Therefore, we expand these functions in powers of a and neglect $O(a^2)$ assuming the density of the system to be low. Then calculation of (5) gives the following energy expectation value per particle:

$$\langle H\rangle/N = (2\pi\rho a\hbar^2/m)\Big\{1 + \frac{\mu^3}{8\pi\rho} + \Big(a - \frac{\mu^2}{8\pi\rho}\Big)\sqrt{\mu^2 + 8\pi\rho a}\Big\}. \qquad (7)$$

The minimum value of (7) is found at $\mu = \sqrt{8\pi\rho a}/3$, and

$$\langle H\rangle/N = (2\pi\rho a\hbar^2/m)\Big\{1 + \frac{10}{3}\sqrt{\frac{2\pi\rho a^3}{3}}\Big\}. \qquad (8)$$

The first term in this expression, $2\pi\rho a\hbar^2/m$, was given by Lenz.[6] The coefficient of the second term is

$$\frac{10}{3}\sqrt{\frac{2\pi}{3}} = 4.824,$$

which is slightly larger than the corresponding value

$$\frac{128}{15\sqrt{\pi}} = 4.814$$

given by other methods.[1)]

The present method will be useful for the treatment of a system with interaction potentials with attractive part.

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Exact Treatment of Bardeen's Theory of Superconductivity in the Strong Coupling Limit

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In their recent paper on superconductivity, Bardeen, Cooper and Schrieffer[1)] have claimed that their theory based on the Froehlich interaction necessarily results in the energy gap for electronic excitations. There the variational method was applied making use of the trial function with total number of electrons not definite. It seems to give reasonable result for the ground state energy, but might be entirely misleading for the excited states, since the constant terms with respect to the total number have not been treated carefully. We will give here an alternative method to treat B-C-S' hamiltonian, and show that their finite energy gap follows exactly at least in the strong coupling limit.

Following B-C-S' notation, b_k means the annihilation operator of an electron pair with opposite momenta and spins, i. e. $(-\boldsymbol{k}, \downarrow)$ and $(+\boldsymbol{k}, \uparrow)$. The assembly of such pairs has the intermediate nature between the Fermi and the Bose statistics, that is to say, the commutation relations are

$$[b_k, b_{k'}^*]_- = 0 \quad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{k}', \quad \text{and} \quad \{b_k, b_{k'}^*\}_+ = 1. \tag{1}$$

As is easily shown, these relations are explicitly satisfied with

$$b_k = (\sigma_x + i\sigma_y)_k/2 = (m_x + im_y)_k, \tag{2}$$

where σ_k is the Pauli spin matrix and m_k is the angular momentum operator. Then B-C-S' hamiltonian becomes

$$H = E_f + N(0)\omega^2 - \sum_k 2\epsilon_k m_{zk} + (V - \omega_f) M_z - V(\boldsymbol{M}^2 - M_z^2), \tag{3}$$

where E_f is the Fermi energy, $N(0)$ the density of the Bloch states of one spin per unit energy at the Fermi surface, ω the average phonon frequency, ω_f the Bloch energy at the Fermi surface, ϵ_k the Bloch energy measured relative to ω_f, V the average coupling constant, and $\boldsymbol{M} = \sum_k \boldsymbol{m}_k$. Here M_z is the constant of motion related to the total electron number; solutions

with $M_c=0$ correspond to our case. We have solved the problem in the strong coupling approximation which means that $x\equiv VN(0)$ is large compared with unity. The energy of the ground state is calculated up to fifth order, and that of the first excited states up to third order. The results are as follows. The energy of the ground state is

$$E_0=E_f-\left[X-1+\frac{1}{3X}-\frac{1}{45X^3}+O\left(\frac{1}{X^5}\right)\right]N(0)\omega^2-\left[X+\frac{1}{6X}-\frac{1}{24X^3}+O\left(\frac{1}{X^5}\right)\right]\omega, \tag{4}$$

and the energy gap for pair excitation becomes

$$E.G.=\left[2X-\frac{1}{3X}+O\left(\frac{1}{X^3}\right)\right]\omega. \tag{5}$$

These values turn out to be identical with B-C-S' result except the last terms of E_0 (constant terms with respect to the total number), when the latter is expanded in the power series of $1/X$. It should be noted that they have obtained the correct energy gap in this approximation though the constant terms in the ground state have been discarded; incidentally, the same constant terms appear in the excited states. We further note that we have also obtained their energy gap for single electron excitations.

Our description of the system of electron pairs in terms of spins associated with each pair state may not give any mathematical advantage, but provides us with some valuable physical pictures about what is going on in such a system. For instance, in order to obtain B-C-S' ground state energy we have only to treat the spin as a classical quantity and to minimize the hamiltonian, eq. (3). Then all the spins are arranged in a plane which contains the z-axis and the distribution of their directional cosines is described in terms of B-C-S' distribution function h_k. In the strong coupling limit treated above, the spin-spin interactions are predominant just as in ferromagnetism. The third term in eq. (3) is interpreted as the interaction between spins and the external magnetic field which varies from spin to spin. It seems at first sight that this term may be predominant in the weak coupling limit, but this is not the case, since however weak the coupling may be, the spin-spin interactions close to the Fermi surface always surpass that interaction with $\epsilon_k\sim 0$. Thus we may get a finite energy gap also in the weak coupling case.

As a final remark, we want to note that we can sum up all the Feynman diagrams of B-C-S' hamiltonian following the technique of Gell-Mann and Brueckner[2], since the ladder type approximation is exact in this case, but we are encountered with the difficulty that the propagator for the electron pair has two imaginary poles which probably indicates that the adiabatic theorem[3] may not hold for this hamiltonian.[4]

We would like to express our sincere thanks to Professor S. Tomonaga and Professor T. Miyazima for stimulating discussions throughout this work.

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Single Particle Transition in Neutron Inelastic Scattering by Deformed Nuclei

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The contribution from the direct interaction process to the nucleon inelastic scattering changing the intrinsic configuration of the deformed nucleus is investigated. The perturbation calculation, with the zero-range two body force, is carried out using waves distorted by the collective surface interaction. The resulting cross section is small compared with the cross section for the usual rotational excitation and justifies the approximation contained in the latter calculation.

§ 1. Introduction

In a series of papers[1)-4)], the importance of the direct interaction in nuclear reactions was emphasized and its approximate treatment was discussed. In the case of nucleon inelastic scattering by a deformed nucleus, the incident nucleon interacts with the surface rotational motion and excites the target nucleus to higher rotational states. Since the interaction takes place at the nuclear surface, it seems to be a good approximation to assume a delta-function type interaction. This choice of interaction makes the calculation simple, and the collision matrix is obtained in a compact form.[2)] In the calculation, only the excited states within the same rotational band as the ground states are taken into account explicitly, while the excited states of different intrinsic configuration from the ground states are neglected. The effect of the neglected states is considered as small in view of the previous calculation[4)] in which the rotational excitation and the particle excitation for the same nucleus was compared. However the above calculation of the rotational excitation has not yet been justified until the effect of the neglected states is estimated.

In this paper the cross section of the excitation to a state with a different intrinsic configuration will be calculated. The method of calculation is similar to that of the usual particle excitation[4)]; the two-body interaction between the nucleons is taken as the perturbation to the rotational excitation problem. However, since the nucleus makes the rotational motion and the shape of it is deformed, some complexity in the calculation will arise.

§ 2. The transition matrix

The Hamiltonian of the whole system is given by

$$H=H_0+V \tag{1}$$

where H_0 represents the free Hamiltonian, a sum of the kinetic energy of the incident nucleon and the Hamiltonian of the target nucleus. V is the interaction between the incident nucleon 0 and nucleons i in the nucleus and is written as a sum of two body interactions v_{0i}

$$V=\sum_{i=1}^{A} v_{0i}. \tag{2}$$

Instead of treating the interaction V, the potential matrix U[3] is introduced. It is given in the case of the rotational excitation by

$$U=U_c-U_0 R\delta(r-R)\sum_{\mu} Y_{2\mu}(\vartheta,\varphi)(-)^{\mu}\alpha^{*}_{2,-\mu}. \tag{3}$$

The first term of the right-hand side of the above equation is the distorting potential and is assumed to be a complex square well with radius R:

$$\begin{aligned} U_c&=-V_0 \quad \text{for} \quad r<R \\ &=0 \qquad\qquad\quad r>R. \end{aligned} \tag{4}$$

The second term represents the interaction between the incident nucleon and the rotational motion of the target nucleus. The polar coordinates of the incident nucleon are r, θ and φ, and $\alpha_{2\mu}$ is the deformation parameter of the nucleus corresponding to the angular momentum 2 and its Z component (lab. coordinate) μ.

The wave function $\varphi^{(\pm)}_{\alpha_0}$ describing the direct process satisfies the following integral equation,

$$\varphi^{(\pm)}_{\alpha_0}=\Phi_{\alpha_0}+\frac{1}{E\pm i\varepsilon-H_0}U\varphi^{(\pm)}_{\alpha_0}, \tag{5}$$

where E is the energy of the whole system, ε is a small positive number insuring the outgoing (incoming) wave character of the solution $\varphi^{(\pm)}_{\alpha_0}$ and α_0 labels the incident channel. The incident wave is

$$\Phi_{\alpha_0}=\frac{2}{i}k_{\alpha_0}j_{l_{\alpha_0}}(k_{\alpha_0}r)\chi_{\alpha_0}(\xi) \tag{6}$$

which satisfies

$$(H_0-E)\Phi_{\alpha_0}=0. \tag{7}$$

In eq. (6) k_{α_0} is the wave number of the incident (outgoing) nucleon in channel α_0, l_{α_0} is the relative orbital angular momentum, $j_{l_\alpha}(x)$ is the spherical Bessel function. $\chi_\alpha(\xi)$ represents the wave function containing all variables other than r.

The integral equation (5) is equivalent to the usual Schrödinger equation and the

solution may be written as

$$\varphi_{\alpha_0}^{(+)}=\sum_{\alpha_i}\frac{1}{r}u_{\alpha_i}^{\alpha_0}(r)\,\chi_{\alpha_i}(\xi) \tag{8}$$

where $u_{\alpha_i}^{\alpha_0}(r)$ is the radial weve function of the channel α_i when the entrance channel is α_0. The possible outgoing channels α_i for the given entrance channel α_0 are determined from the potential matrix U given in eq. (3). For the component of the spin of the target nucleus along the symmetrical axis K, one has the selection rule[1]

$$\Delta K=0,\ \pm 2. \tag{9}$$

If one considers the case of the cylindrically symmetric nucleus, $\Delta K=\pm 2$ is forbidden, and the transition is restricted to the same rotational band. Although the rotation-particle perturbation mixes two levels with the same spin[5], it has no effect on the selection rule $\Delta K=0$ in the first order because the mixing occurs between the states $\Delta K=\pm 1$. For the case of the asymmetric nucleus, K is no longer a good quantum number and the states with $\Delta K=\pm 2$ mix each other. Then the transition between states with $\Delta K=\pm 2$ is allowed.

In the calculation of the direct interaction process through the potential matrix U, it is assumed that the effect of the transition forbidden by the selection rule (9) are unimportant, and it is calculated under a statistical assumption neglecting the phase relation. We now examine the above assumption by calculating explicitly the transition matrix element between the forbidden states through the interaction. According to reference 3 the exact transition matrix elements are written as follows[3] :

$$T_{\beta_0\alpha_0}=\frac{iM}{\hbar^2\sqrt{k_{\beta_0}k_{\alpha_0}}}\Big[(\Phi_{\beta_0},\,U\varphi_{\alpha_0}^{(+)})+(\varphi_{\beta_0}^{(-)},(V-U)\,\varphi_{\alpha_0}^{(+)})$$
$$+(\varphi_{\beta_0}^{(-)},(V-U)\frac{1}{E+i\varepsilon-H}(V-U)\,\varphi_{\alpha_0}^{(+)})\Big] \tag{10}$$

where the first term represents the direct process through U, the second is the direct process which is not included in the first term, and the third represents the compound process. The first and the last terms were already investigated in reference 2, and our main interest lies in the second term. Using eq. (8) the second term of eq. (10) is expressed as

$$T_{\beta_0\alpha_0}^{(2)}=\frac{iM}{\hbar^2\sqrt{k_{\beta_0}k_{\alpha_0}}}\sum_{\alpha_i\beta_j}\int u_{\beta_j}^{\beta_0}(r)\,\chi_{\beta_j}(\xi)\,V\,u_{\alpha_i}^{\alpha_0}(r)\,\chi_{\alpha_i}(\xi)\,dr\,d\xi. \tag{11}$$

§ 3. Calculation of the matrix element

The evaluation of the matrix element (11) will be given in this section. The interaction V is assumed as the simplest zero range interaction

$$v_{0i}=V_0\delta(\boldsymbol{r}_0-\boldsymbol{r}_i)$$
$$=V_0\sum_k v_k(r_0, r_i)\sum_\varkappa(-)^\varkappa C_\varkappa^{(k)}(0)\,C_{-\varkappa}^{(k)}(i) \tag{12}$$

where

$$v_k(r_0, r_i)=\frac{2k+1}{4}\,\frac{\delta(r_0-r_i)}{r_0 r_i} \tag{13}$$

and

$$C_\varkappa^{(k)}(i)=\sqrt{\frac{4\pi}{2k+1}}\,Y_{k\varkappa}(\theta_i, \varphi_i). \tag{14}$$

The wave function of the target nucleus with spin I in the strong coupling approximation is represented by

$$|IMK\Omega\rangle=\sqrt{\frac{2I+I}{16\pi^2}}\varphi_{vib}[\chi_\Omega D_{MK}^I(\vartheta_i)+(-)^{I-\Sigma j_p}\chi_{-\Omega}D_{M,-K}^I(\vartheta_i)] \tag{15}$$

where M is the Z-component of nuclear spin I, K is its component along a symmetry axis, Ω is a sum of components of the nucleon angular momenta, j_p, along the symmetry axis. φ_{vib} is the wave function of the vibrational motion and D_{MK}^I is that of the rotational motion, ϑ_i being the Euler angle.

The particle wave function χ_Ω is built up of single particle wave functions which are solutions of the deformed potential fixed to the symmetry axes. In the following we will consider a simple case where only one nucleon sits outside the core. According to Nilsson[6], the wave function χ_Ω is expressed in terms of the wave function for the spherical harmonic oscillator. However, it is more convenient to express it in terms of the wave function $|jl\Omega\rangle$ with definite total angular momentum j, its component Ω and orbital angular momentum l:

$$\chi_\Omega=\sum_{jl}B_{jl\Omega}|jl\Omega\rangle. \tag{16}$$

The coefficient $B_{jl\Omega}$ is obtained from Nilsson's $A_{l\Omega\pm}$ by

$$B_{jl\Omega}=\sum_m A_{l\Omega-m}(l\tfrac{1}{2}\,\Omega-mm|l\tfrac{1}{2}\,j\Omega), \tag{17}$$

where $(l\frac{1}{2}\,\Omega-mm|l\frac{1}{2}\,j\Omega)$ is the Clebsch Gordan coefficient. The normalization is

$$\sum|B_{jl\Omega}|^2=\sum|A_{jlm}|^2=1 \tag{18}$$

which is different from Nilsson's.

By using the wave function and the interaction described above, the matrix element may be expressed as

$$\langle IK\Omega, j_0, JM|V|I'K'\Omega', j_0', JM\rangle=\sum B_{jl\Omega}^* B_{j'l'\Omega'}$$
$$\times\langle IK\Omega jl, j_0, JM|\sum_\varkappa(-)^\varkappa C_\varkappa^{(k)}(0)\,C_{-\varkappa}^{(k)}(1)\,|I'K'\Omega' j'l', j_0', JM\rangle$$
$$\times(2k+1)^{1/2}I(jlj_0l_0\,;\,j'l'j_0'l_0') \tag{19}$$

where

$$I(jlj_0l_0\,;\,j'l'j_0'l_0') = \frac{1}{4}\int_0^\infty \phi_{jl}^*(r)\,u_{j_0l_0}(r)\,\phi_{j'l'}(r)\,u_{j_0'l_0'}(r)\,\frac{dr}{r^2}\,. \tag{20}$$

$u_{j_0l_0}(r)$ and $\phi_{jl}(r)$ are the radial wave functions of the incident nucleon and the nucleon outside the core respectively.

The calculation of the spin-angle part of the matrix element is carried out by using Racah's coefficient $W(abcd\,;\,ef)$[7] as

$$\langle IK\Omega jl,\ j_0,\ JM|\sum_\varkappa(-)^\varkappa C_\varkappa^{(k)}(0)\,C_{-\varkappa}^{(k)}(1)\,|I'K'\Omega'j'l',\ j_0',\ JM\rangle$$

$$= (-)^{I+j_0'-J}W(Ij_0I'j_0'\,;\,Jk)\,(IK\Omega jl\|C^{(k)}\|I'K'\Omega'j'l')\,(j_0\|C^{(k)}\|j_0')\,. \tag{21}$$

In the last equation the second reduced matrix element is given as before by

$$(j_0l_0\|C^{(k)}\|j_0'l_0') = (-)^{j_0'-1/2}\sqrt{\frac{(2j_0+1)\ (2j_0'+1)}{2k+1}}$$

$$\times\,(j_0j_0'\tfrac{1}{2}-\tfrac{1}{2}|j_0j_0'k0)\,\frac{1+(-)^{l_0+l_0'-k}}{2}\,. \tag{22}$$

In calculating the first reduced matrix element, the function $C^{(k)}$ written in the laboratory coordinate system should be expressed in terms of that in the symmetry axis by the irreducible representation of the rotational group $D_{\nu\varkappa}^k$. The result is

$$(IK\Omega jl\|C^{(k)}\|I'K'\Omega'j'l') = \sqrt{\frac{2I'+1}{2j+1}}\,(\varphi_{v'i'b},\ \varphi_{vib})\,(jl\|C^{(k)}\|j'l')$$

$$\times[\,(I'kK'K-K'|I'kIK)\,(j'k\Omega'\Omega-\Omega'|j'kj\Omega)$$

$$+(-)^{-I+j}(I'kK',\ -K-K'|I'kI-K)\,(j'k\Omega',\ -\Omega-\Omega'|j'kj-\Omega)\,]. \tag{23}$$

By putting (21), (22) and (23) into (19) the matrix element may be calculated.

§ 4. Discussions

First the effective strength of the nuclear force in a deformed nucleus will be considered. If the target nucleus consists of a spherical core and one nucleon moving around it, and the single particle is excited from the orbit $j'l'$ to jl, then the reduced matrix element which should be inserted in (21) is $(jl\|C^{(k)}\|j'l')$ instead of eq. (23). So one has an extra factor in eq. (23),

$$(\varphi_{v'i'b},\ \varphi_{vib})\sqrt{\frac{2I'+1}{2j+1}}\,[\,(I'kK'K-K'|I'kIK)\,(j'k\Omega'\Omega-\Omega'|j'kj\Omega)$$

$$+(-)^{-I+j}(I'kK',\ -K-K'|I'kI-K)\,(j'k\Omega',\ -\Omega-\Omega'|j'kj-\Omega)\,], \tag{24}$$

compared with the spherical case. Its first factor is an overlapping integral of the vibrational wave functions and is not greater than unity. Owing to the difficulty of the evaluation, its value is assumed as unity. The remaining factor in eq. (24) will be

found to be around $1/2$ in the example which will be presented later. Thus it may be plausible to interpret the factor (24) as a reduction factor by which the strength of the nuclear force V_0 is weakened due to the rotational motion of the nucleus.

The deformation of the nucleus causes the mixing of the total angular momentum of nucleons. Sometimes the deformation of the nuclear shape is different for different rotational bands as inferred from the difference of moment of inertia for each rotational band. In such a case the matrix element will be further reduced due to the differences of shape because the matrix element is considered as a kind of overlapping integral of two wave functions.

In the previous paper[4], magnitudes of the transition matrix element through the collective and particle interactions were compared in some detail. The transition matrix element through the particle excitation does not increase with increase of the number of nucleons outside the core but remains rather constant, while that through the rotational excitation is proportional to the intrinsic quadrupole moment which becomes very large for nuclei far from the closed shells. Thus we may expect that the matrix element for the particle excitation in the deformed nucleus is small compared with that for the particle excitation in the spherical nucleus: the latter is small compared with the rotational excitation.

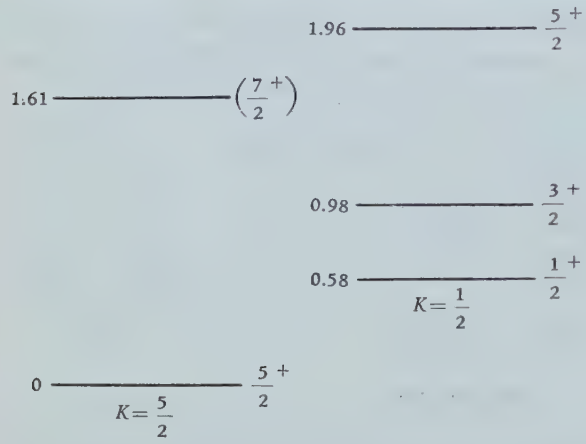

Fig. 1. The energy level diagram of ^{25}Mg. The energies are expressed in Mev in the left side and spins and parities are written in the right side for each rotational band.

In the following a numerical example will be presented to give the magnitude of the cross section due to the particle excitation in a deformed nucleus. For the target nucleus, ^{25}Mg. is chosen because two rotational bands are observed at low energies in spite of the relatively light nucleus. The energy level diagram[8] at low energy is shown in Fig. 1. The inelastic scattering of neutrons with incident energy 2.5 Mev will be calculated. The parameters for the distorting potential are chosen as

$$U_0 = 42 \text{ Mev}$$
$$R = 1.45 \times 25^{1/3} \times 10^{-13} \text{ cm} \tag{25}$$

and the shape parameter

$$\varepsilon \equiv \Delta R/R = 0.3 \tag{26}$$

is assumed. For the intrinsic motion of the target nucleus the wave function by Nilsson[6] is adopted. In his calculation the energy level $\Omega = 5/2^+$ ($d_{5/2}$ for $\varepsilon = 0$) is

higher than the level $\Omega=1/2^+$ ($s_{1/2}$ for $\varepsilon=0$) for $\varepsilon=0.3$, while in Fig. 1 the order of these levels is inverted. We use the value of parameter A_{jlm} corresponding to $\varepsilon=0.3$ despite its discrepancy about the level order because the level order is expected to change even by a small perturbation. The strength of the basic harmonic oscillator well is chosen so that the mean square of the nuclear radius may be equal to $3/5\times R^2$ where R is given by (25). For the strength of the nuclear force

$$V_0=-182\times10^{-39}\,\mathrm{cm}^3\,\mathrm{Mev} \tag{27}$$

is used. These parameters are same as those used in reference 4). First the cross sections for elastic scattering and for inelastic scattering to the 1.61 Mev level are calculated assuming rotational excitation and neglecting the residual interaction V. The results are for the elastic scattering

$$\begin{aligned} &d\sigma_{el}/d\Omega=(132\,P_0(\theta)+108\,P_1(\theta)+34\,P_2(\theta))\,\mathrm{mb/sterad}\\ &\sigma_{el}=1660\ \mathrm{mb.} \end{aligned} \tag{28}$$

and for the inelastic scattering leaving the residual nucleus at the 1.61 Mev level

$$\begin{aligned} &d\sigma_{1.61}/d\Omega=(15.6\,P_0(\theta)+8.6\,P_2(\theta))\,\mathrm{mb/sterad}\\ &\sigma_{1.61}=196\ \mathrm{mb.} \end{aligned} \tag{29}$$

For the inelastic scattering leaving the residual nucleus in the $K=1/2$ rotational band the cross sections are

$$\begin{aligned} \sigma_{0.58}&=3.1\ \mathrm{mb}\\ \sigma_{0.98}&=5.5\ \mathrm{mb}\\ \sigma_{1.96}&=1.4\ \mathrm{mb} \end{aligned} \tag{30}$$

where the suffices to σ indicate the excitation energy in Mev of the final nucleus.

By comparing (29) and (30) it is readily seen that the cross section of inelastic scattering changing K of the target nucleus is very small compared with that keeping K unchanged, and is almost forbidden through the direct interaction. Although we have only one example available, this situation is expected to occur in the general case from the argument given in the last section. Thus the approximate calculation of the rotational excitation is justified.

It is interesting to compare the above results with those of the conventional statistical theory. Assuming a black nucleus with a nuclear radius R given in (25) and the wave number inside the nucleus $1\times10^{13}\,\mathrm{cm}^{-1}$ the cross sections are calculated by using the formula of Hauser and Feshbach[9]. For quantum numbers of the levels of the target nucleus only the spin and parity are taken into account. The results are as follows,

$$\begin{aligned} \sigma_{1.61}&=175\ \mathrm{mb}\\ \sigma_{0.58}&=\ 66\ \mathrm{mb}\\ \sigma_{0.98}&=129\ \mathrm{mb} \end{aligned}$$

$$\sigma_{1.96} = 86 \text{ mb.} \tag{31}$$

By comparing (29), (30) and (31) it is seen that the two models give similar magnitude of cross section for the transition keeping K unchanged while the statistical model gives much larger cross section to the transition changing K. Even in the former case the angular distribution may be different for two models.

If one wishes to make a more realistic estimation of the cross section, one must use the distorting potential with a proper imaginary part. The method of calculation was outlined in references 2 ond 3. The average cross section has contributions from the direct process and from the compound process. The direct contributions to the transition changing K will not be increased appreciably if U has an imaginary part. On the other hand, the compound contribution depends on the magnitude of the imaginary part of U. Since the inelastic cross section changing K has contributions from the compound process, it is possible to test the importance of the compound process from the experimental cross sections.

In conclusion the authors would like to express their sincere thanks to Prof. M. Sasaki for his interest and advices. One of the authors (H.M.) is also indebted to the Yukawa Yomiuri Fellowship for the financial aid.

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A Theory of Clothed Unstable Particles

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The clothed states of unstable particles are investigated on the basis of quantum field theory, and thereby a new mathematical notion, "complex distribution", is introduced. Then the exact eigenstate of total Hamiltonian with the complex eigenvalue, whose real part represents the mass of the unstable particle and whose imaginary part is interpreted as the half reciprocal of its lifetime, can be constructed by means of the complex distribution. But this state is not observable. The physical state of the unstable particle therefore is defined as an approximate state of the exact eigenstate, which exhibits physically reasonable behaviours.

§ 1. Introduction

Recently many authors investigated the theory of unstable particles.[1)-5)] Especially, Araki *et al.*[1)] proposed two methods for naturally defining the physical mass and the lifetime of an unstable particle. The corresponding Z-factors are also defined, but they can take values larger than unity contrary to the stable case. Naito[3)] investigated the production and decay of an unstable particle in the stationary treatment, and defined its physical state, which was formally identical with the approximate eigenstate of total Hamiltonian previously proposed by Glaser and Källén.[3)] He also defined a Z-factor by the normalization constant of this state, which was interpreted as the dissociation probability (of course $0<Z<1$). This Z-factor, however, has a curious property, namely it tends to one half instead of unity at the weak coupling limit, provided that all other interactions are switched off. This fact probably gives rise to theoretical difficulties.* We therefore investigate further the clothed state of an unstable particle.

Now, as a preliminary, we briefly review the second definitions of the mass and life presented by Araki *et al.*, which seem to have the most essential meaning. For simplicity, we employ Lee's model[6)] for the time being. The total Hamiltonian is given by

$$H=m_0\psi_V^*\psi_V+m_N\psi_N^*\psi_N+\int\omega_k\alpha_k^*\alpha_k d\boldsymbol{k}$$
$$+g[\psi_V^*\psi_N\int\{G(\omega_k)/\sqrt{2\omega_k}\}\alpha_k d\boldsymbol{k}+\psi_N^*\psi_V\int\{G(\omega_k)/\sqrt{2\omega_k}\}\alpha_k^* d\boldsymbol{k}]. \quad (1\cdot1)$$

* From this standpoint, the usual perturbational approach to weak interactions would become inadequate. Further, for example, the charge independence would be violated, since proton is stable while neutron is unstable.

Here the notations are as follows.

m_0 : bare mass of a V-particle.

m_N : mass of an N-particle.

μ, $\boldsymbol{k}$ and $\omega_k=\sqrt{\boldsymbol{k}^2+\mu^2}$: mass, momentum and energy of a θ-particle, respectively.

g : unrenormalized coupling constant.

$G(\omega_k)$: cut-off function, which is assumed to be regular.

ψ_V^*, ψ_V ; ψ_N^*, ψ_N ; α_k^*, α_k : field operators of V-particle, N-particle and θ-particle, respectively.

The modified propagator of a V-particle, $S_V'(E)$, is given by the following formula :

$$[S_V'(E)]^{-1}=E-m_0+i\varepsilon+g^2\int\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{\omega_k+m_N-E-i\varepsilon},\quad(\varepsilon\to+0).\qquad(1\cdot2)$$

If the equation which is obtained by putting the right-hand side of (1·2) equal to zero has a real root, say, $E=m$, then m represents the mass of the stable V-particle because of $m<m_N+\mu$ which is evident from the integral in (1·2). So, in order to obtain the mass of the unstable V-particle, we must analytically continue $[S_V'(E)]^{-1}$ to the complex plane of E. But $[S_V'(E)]^{-1}$ has no zero point on the Riemann plane with cut along the real axis from the branching point, $m_N+\mu$, to $+\infty$. Hence we consider the analytic continuation to the lower half-plane from the side of $\mathrm{Re}\,E>m_N+\mu$. Then for E on the lower half-plane the path of ω_k-integration must be deformed as in Fig. 1 on the ω_k-plane. The integral therefore is written as the sum of the integral along the real axis and of the residue of a pole, $\omega_k=E-m_N$, that is to say,

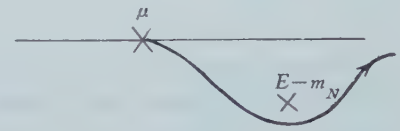

Fig. 1

$$[S_V'(E)]^{-1}=E-m_0+2\pi g^2\int_{\mu}^{+\infty}\frac{G^2(\omega)\sqrt{\omega^2-\mu^2}}{\omega+m_N-E}d\omega$$
$$+(2\pi)^2ig^2G^2(E-m_N)\sqrt{(E-m_N)^2-\mu^2},\quad(\text{for Im}\,E<0).\qquad(1\cdot3)$$

The equation which is obtained by putting the right-hand side of (1·3) equal to zero generally has complex roots. For simplicity, we assume that the equation has only one root, say, $E=m_V-i\gamma/2$, *i. e.*

$$[S_V'(m_V-i\gamma/2)]^{-1}=0.\qquad(1\cdot4)$$

m_V and γ^{-1} are interpreted as the physical mass and the life of the unstable V-particle, respectively.

Now, in the case of the stable particle, its mass is defined not only as the pole of modified propagator like (1·4) but also as the one-particle-state eigenvalue of total Hamiltonian. For the unstable particle, however, such an eigenstate does not exist, since the scattering states alone form a complete set.[1),3)] This corresponds to that (1·2) has no zero point. But as we have obtained a zero point by the analytic continuation like

(1·3), there should exist the corresponding eigenstate. To get this state needs, so to speak, the analytic continuation of state-vector, which is not a known notion. So we introduce a new notion, "*complex distribution*", in the next section. In § 3, we explicitly construct the eigenstate of total Hamiltonian with the eigenvalue $m_V - i\gamma/2$ by using the complex distribution. But this state is not contained in the conventional Hilbert space and is not observable. In § 4, we define the physical state of the unstable V-particle by an approximate one to the above eigenstate, and show that its properties are physically reasonable. In § 5, generalizations are presented. Finally, we discuss the fundamental postulate of quantum field theory in the light of the complex distribution.

§ 2. Definition of complex distribution

Consider a meromorphic function, $F(\omega)$, and two fixed points, a and b, in a domain D. For any arbitrary function, $\varphi(\omega)$, regular in D, we consider an integral,

$$F[\varphi] \equiv \int_a^b \varphi(\omega) F(\omega)\, d\omega,$$

as a linear functional of $\varphi(\omega)$. This functional is *not* well defined by $F(\omega)$ *alone*, but we must further indicate which side of each pole of $F(\omega)$ the path of integration passes through. We call the meromorphic function with such indications "complex distribution" as a generalization of Schwartz's distribution.[7]

In the following we choose μ and $+\infty$ as the end points of the path. The indications for choosing the path are denoted by the following notations.

$$\left.\begin{aligned} 1/(\omega^{(+)}-c) &: \text{ The path passes through } \textit{above} \text{ the pole } \omega=c. \\ 1/(\omega^{(-)}-c) &: \text{ The path passes through } \textit{below} \text{ the pole } \omega=c. \end{aligned}\right\} \qquad (2\cdot1)$$

Then we evidently have*

$$\left.\begin{aligned} \{1/(\omega^{(+)}-c)\}^* &= 1/(\omega^{(-)}-c^*), \\ \{1/(\omega^{(-)}-c)\}^* &= 1/(\omega^{(+)}-c^*) \end{aligned}\right\} \qquad (2\cdot2)$$

for complex conjugation. The difference between the both of (2·1) is nothing but the contribution from the pole. We therefore define complex δ-function as follows:

$$2\pi i\delta(\omega-c) \equiv 1/(\omega^{(-)}-c) - 1/(\omega^{(+)}-c). \qquad (2\cdot3)$$

From (2·3) we immediately get

$$\int_\mu^{+\infty} \varphi(\omega)\,\delta(\omega-c)\, d\omega = \varphi(c), \qquad (2\cdot4)$$

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* This is because

$$[\int d\omega\, \varphi(\omega)/(\omega^{(+)}-c)]^* = \int d\omega\, \varphi^*(\omega)/(\omega^{(-)}-c^*).$$

It must be noticed that ω is not a definite complex number but an analytic variable, and Re ω, Im ω and ω^* are meaningless.

and

$$[\delta(\omega-c)]^*=\delta(\omega-c^*) \tag{2·5}$$

because of (2·2). Hence we have

$$\int_{\mu}^{+\infty}[\delta(\omega-c)]^*\,\delta(\omega-c)\,d\omega=0, \quad (\text{for } \mathrm{Im}\, c\neq 0). \tag{2·6}$$

The definitions of the derivatives of the complex δ-function are straightforward. *Complex distributions can generally be represented by usual functions and complex δ-functions by using* (2·3).

In the above, complex distributions are defined only for the integrations from μ to $+\infty$. Though no other integrations are needed in the next section, it is necessary for the general case that complex distribiuton should be defined for the more general integrations. A convenient generalization to the integrations between two real points, a and b, is as follows: the path runs according to its indication only for the each pole whose real part is between a and b. Then the simple additivity of integrations is satisfied. In particular, consider the case in which c in (2·1) is real. The complex distribution then reduces to the well-known distribution, $1/(\omega-c\pm i\varepsilon)$, $(\varepsilon\to+0)$. The former thus is a straightforward generalization of the latter to the case in which the pole is apart from the real axis by a finite distance.

Finally, in the case of two analytic variables a complex distribution, $1/(\omega_1^{(-)}-\omega_2^{(+)})$, is defined as follows:

$$\int\frac{\varphi(\omega_1,\ \omega_2)}{\omega_1^{(-)}-\omega_2^{(+)}}d\omega_1\,d\omega_2\equiv\int\frac{\varphi(\omega_1,\ \omega_2)}{\omega_1-\omega_2-i\varepsilon}d\omega_1\,d\omega_2, \tag{2·7}$$

for any analytic *function*, $\varphi(\omega_1,\ \omega_2)$, and

$$\left.\begin{aligned}\delta(\omega_1-c)/(\omega_1^{(-)}-\omega_2^{(+)})&\equiv\delta(\omega_1-c)/(c-\omega_2^{(+)}),\\ \delta(\omega_2-c)/(\omega_1^{(-)}-\omega_2^{(+)})&\equiv\delta(\omega_2-c)/(\omega_1^{(-)}-c)\end{aligned}\right\} \tag{2·8}$$

for complex δ-functions and likewise for their derivatives. The following identities then can be easily verified.

$$\frac{1}{(\omega_1^{(-)}-c)\,(\omega_1^{(-)}-\omega_2^{(+)})}=-\frac{1}{\omega_2^{(+)}-c}\left(\frac{1}{\omega_1^{(-)}-c}-\frac{1}{\omega_1^{(-)}-\omega_2^{(+)}}\right), \tag{2·9}$$

$$\frac{1}{(\omega_1^{(-)}-c)\,(\omega_1^{(+)}-\omega_2^{(-)})}=-\frac{1}{\omega_2^{(-)}-c}\left(\frac{1}{\omega_1^{(-)}-c}-\frac{1}{\omega_1^{(+)}-\omega_2^{(-)}}\right). \tag{2·10}$$

§ 3. Exact state of the unstable V-particle

The physical state of the *stable* V-particle is defined as the exact eigenstate of H with the eigenvalue m, namely

$$|V\rangle-g\int\frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k+m_N-m}\alpha_k^*\,d\boldsymbol{k}\,|N\rangle, \tag{3·1}$$

where $|V\rangle$ and $|N\rangle$ respectively stand for the bare-V state and the N state.

In the case of the *unstable* V-particle, we obtain the state proposed by Naito[5] if we formally put $m=m_V-i\gamma/2$ in (3·1). But this is, of course, *not* the eigenstate of H. This is because the Riemann plane in (1·3) differs from that in (3·1). To get the former it is necessary to curve the path of the ω_k-integration like in Fig. 1. But it is meaningless to curve that in (3·1), since this integration is not mathematical but purely formal one because of the presence of $\alpha_k{}^*$. So we make use of the complex distribution defined in last section in order to indicate the path potentially, that is to say, we introduce a state

$$|\boldsymbol{V}\rangle\!\!\rangle=|V\rangle-g\int\frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)}+m_N-m_V+i\gamma/2}\alpha_k{}^*\,d\boldsymbol{k}\,|N\rangle. \tag{3·2}$$

Indeed, this state exactly satisfy the equation,*

$$H\,|\boldsymbol{V}\rangle\!\!\rangle=(m_V-i\gamma/2)\,|\boldsymbol{V}\rangle\!\!\rangle. \tag{3·3}$$

At first sight (3·3) seems very curious, for H is an Hermitian operator.** Actually, from (3·3) we obtain

$$i\gamma\langle\boldsymbol{V}|\boldsymbol{V}\rangle\!\!\rangle=\langle\boldsymbol{V}|(H^*-H)\,|\boldsymbol{V}\rangle\!\!\rangle=0. \tag{3·4}$$

Hence if $|\boldsymbol{V}\rangle\!\!\rangle$ had a positive norm, γ would have to vanish. In the present case we must have

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle\!\!\rangle=0 \tag{3·5}$$

because of $\gamma\neq 0$. (3·5) is verified also by the following direct calculation. From (2·2) we have

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle\!\!\rangle=1+g^2\int\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{(\omega_k^{(+)}+m_N-m_V-i\gamma/2)\,(\omega_k^{(-)}+m_N-m_V+i\gamma/2)}\,. \tag{3·6}$$

The path of the integration is shown in Fig. 2, that is, (3·6) is rewritten as

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle\!\!\rangle=1+g^2\int_{\text{real axis}}\frac{G^2(\omega_k)}{2\omega_k}\cdot\frac{d\boldsymbol{k}}{(\omega_k+m_N-m_V)^2+(\gamma/2)^2}$$
$$-2(2\pi)^2g^2\,\mathrm{Re}\big[G^2(m_V-m_N-i\gamma/2)\sqrt{(m_V-m_N-i\gamma/2)^2-\mu^2}\,\big]/\gamma. \tag{3·7}$$

Making use of (1·3) we have

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* Though the commutation relation naturally is $[\alpha_k,\ \alpha_{k'}{}^*]=\delta(\boldsymbol{k}-\boldsymbol{k}')$, this is no longer the usual δ-function, but the complex distribution defined by

$$\delta(\boldsymbol{k}-\boldsymbol{k}')=(k\omega_k)^{-1}\,\delta(\omega_k-\omega_{k'})\,\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi',\ (\mathrm{mod.}\ 2\pi)),$$

where k, θ, φ and k', θ', φ' are the polar coordinates of $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. But this expression is unnecessary in the actual calculations.

** Strictly speaking, the Hermiticity of an operator is meaningful only when its operand is designated. We have extended not the basic vectors ($|V\rangle$, $\alpha_k{}^*|N\rangle$) but the distributions as their coefficients. Since we have defined the complex distribution so as to be consistent with the usual complex conjugation, the Hermiticity of H (in the generalized sense) is not injured at all.

$$\langle V|V\rangle = -(\gamma/2)^{-1}\,\mathrm{Im}[S_V'(m_V - i\gamma/2)]^{-1}. \tag{3·8}$$

We thus obtain (3·5) because of (1·4).

Thus there has appeared a zero-norm state in our Hilbert space. Our Hilbert space is a little wider than the conventional one, since the former contains negative-norm states. But it is not represented by the well-known indefinite-metric device.[8] Incidentally, the positive-norm restriction is very additional in the mathematical definition of the abstract Hilbert space. So the general Hilbert space with indefinite norm is not necessarily represented by indefinite metric. Indeed, our Hilbert space is such an example, and we can construct a consistent theory.

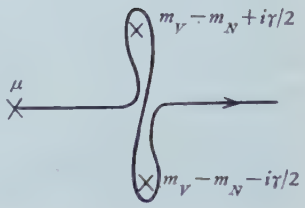

Fig. 2

In the unstable case we know that N-θ scattering states form a complete set. But as $|V\rangle$ is a new eigenstate of H, its expansions by complete sets are of interest.

(i) Incoming states

N-θ incoming-wave eigenstates are given by

$$|N\theta(\boldsymbol{p})^-\rangle = g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'^*(m_N+\omega_p^{(-)})\,|V\rangle + \int\left[\delta(\boldsymbol{k}-\boldsymbol{p}) - g\frac{G(\omega_p)}{\sqrt{2\omega_p}}S_V'^*(m_N+\omega_p^{(-)})\frac{gG(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(+)}-\omega_p^{(-)}}\right]\alpha_k^*\,d\boldsymbol{k}\,|N\rangle. \tag{3·9}$$

Here the complex-distribution notation is used instead of the usual $-i\varepsilon$ convention in the denominator of the last term.* The modified propagator, $S_V'(E)$, likewise is written as

$$[S_V'(E)]^{-1} = E - m_0 + g^2\int\frac{G^2(\omega_k)/2\omega_k}{\omega_k^{(-)}+m_N-E}\,d\boldsymbol{k} \tag{3·10}$$

in terms of complex distribution. This is equal to (1·2) for $\mathrm{Im}\,E \geqq 0$ and to (1·3) for $\mathrm{Im}\,E < 0$. A straightforward calculation of $\langle N\theta(\boldsymbol{p})^-|V\rangle$ yields

$$\langle N\theta(\boldsymbol{p})^-|V\rangle = -2\pi i g\{G(\omega_p)/\sqrt{2\omega_p}\}\,\delta(\omega_p + m_N - m_V + i\gamma/2), \tag{3·11}$$

where we have made use of (2·9), (3·10), (1·4) and (2·3). (3·11) naturally is consistent with

$$(\omega_p + m_N - m_V + i\gamma/2)\langle N\theta(\boldsymbol{p})^-|V\rangle = 0, \tag{3·12}$$

which follows from

$$H|N\theta(\boldsymbol{p})^-\rangle = (m_N+\omega_p)\,|N\theta(\boldsymbol{p})^-\rangle \tag{3·13}$$

and (3·3). From (3·11) $|V\rangle$ is expanded into

* This replacement naturally does not change the usual properties of this state, *e. g.* its normalization.

$$|V\rangle = -2\pi i g \int \frac{G(\omega_p)}{\sqrt{2\omega_p}} \delta(\omega_p + m_N - m_V + i\gamma/2)\, |N\theta(\boldsymbol{p})^-\rangle\, d\boldsymbol{p} \qquad (3\cdot 14)$$

in terms of the complete set of $|N\theta(\boldsymbol{p})^-\rangle$. $(3\cdot 14)$ is verified also by substituting $(3\cdot 9)$ in its right-hand side, where a formula,

$$S_V'^*(m_V - i\gamma/2) = -[2\pi i g^2 \int \frac{G^2(\omega_k)}{2\omega_k} \delta(\omega_k + m_N - m_V + i\gamma/2)\, d\boldsymbol{k}]^{-1}, \qquad (3\cdot 15)$$

which follows from

$$[S_V'^*(E)]^{-1} = [S_V'(E)]^{-1} - 2\pi i g^2 \int \frac{G^2(\omega_k)}{2\omega_k} \delta(\omega_k + m_N - E)\, d\boldsymbol{k} \qquad (3\cdot 16)$$

and $(1\cdot 4)$, should be taken into account.

(ii) Out-going states

The out-going states similarly are

$$|N\theta(\boldsymbol{p})^+\rangle \equiv g \frac{G(\omega_p)}{\sqrt{2\omega_p}} S_V'(m_N + \omega_p^{(+)})\, |V\rangle$$

$$+ \int \left[\delta(\boldsymbol{k} - \boldsymbol{p}) - g \frac{G(\omega_p)}{\sqrt{2\omega_p}} S_V'(m_N + \omega_p^{(+)}) \frac{g G(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)} - \omega_p^{(+)}} \right] \alpha_k^*\, d\boldsymbol{k} |N\rangle. \qquad (3\cdot 17)$$

Since $|N\theta(\boldsymbol{p})^+\rangle$'s are also the eigenstates of H, it should be

$$\langle N\theta(\boldsymbol{p})^+ | V\rangle \propto \delta(\omega_p + m_N - m_V + i\gamma/2)$$

like the above. The coefficient, however, turns out to be equal to zero. This is because the δ-singularity coincides with the singular point of $(3\cdot 17)$, and so the expansion corresponding to $(3\cdot 14)$ becomes an indefinite form, $0 \times \infty$. This is not a defect characteristic of complex distribution, but a general character of distribution. Indeed, similar singular characters often appear in the conventional scattering problems. To avoid this difficulty, the states containing a complex parameter E,

$$|V_E\rangle \equiv |V\rangle - g \int \frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k^{(-)} + m_N - E} \alpha_k^*\, d\boldsymbol{k} |N\rangle, \qquad (3\cdot 18)$$

whose norm is $-(2/\gamma)\,\mathrm{Im}\,[S_V'(E)]^{-1}$, are introduced. Then making use of $(2\cdot 10)$ we have

$$\langle N\theta(\boldsymbol{p})^+ | V_E\rangle = -g \frac{G(\omega_p)/\sqrt{2\omega_p}}{\omega_p^{(-)} + m_N - E} S_V'^*(m_N + \omega_p^{(-)}) [S_V'(E)]^{-1}, \qquad (3\cdot 19)$$

for which we obtain meaningful expansions.

Now, $|\langle N\theta(\boldsymbol{p})^- | V\rangle|^2$ should be the decay spectrum of $|V\rangle$ apart from the phase-volume factor, but it turns out to vanish from $(3\cdot 11)$ and $(2\cdot 6)$. This is obvious from the time-dependent Schrödinger equation,

$$i(\partial/\partial t)\, \Psi(t) = H \Psi(t), \qquad (3\cdot 20)$$

with $\Psi(0)=|V\rangle$ as the initial condition. From (3·3) we have

$$\Psi(t)=|V\rangle e^{-i(m_V-i\gamma/2)t} \tag{3·21}$$

which monotonously diminishes. But since the norm of $\Psi(t)$ is always equal to zero, no inconsistency appears. $|V\rangle$ gives rise to no difficulty about the conservation of probability contrary to the so-called ghost state.[9]

$|V\rangle$ thus is not observable, and therefore should not be called "physical state". So we call it "exact state of the unstable V-particle".

§ 4. Physical state of the unstable V-particle

The exact state introduced in last section is never observable, as far as one adopts the present theory of observation. Namely, $|V\rangle$ is a state beyond the "observation space" (*i. e.* conventional Hibert space). So the physical decaying state of the unstable V-particle should be regarded as an image of $|V\rangle$ projected into the observation space. Under this idea, a physical state is proposed as an approximate state to $|V\rangle$ with a positive norm in this section.

The very origin of the zero-norm is nothing but the complex δ-function in (3·2). This circumstance is caused essentially by $\int[\delta(\omega_k+m_N-m_V+i\gamma/2)]^*\cdot\delta(\omega_k+m_N-m_V+i\gamma/2)d\omega_k=0$, which would never hold if it were a function. Hence to obtain the positive-norm state we have only to replace this complex δ-function by a usual function. $2\pi i\cdot\delta(\omega_k+m_N-m_V+i\gamma/2)$ is just the contribution from the pole of $1/(\omega_k+m_N-m_V+i\gamma/2)$, but the behaviour of the both are completely different away from the pole. Hence we consider a replacement,

$$2\pi i\delta(\omega_k+m_N-m_V+i\gamma/2)\to-f(\omega_k)/(\omega_k+m_N-m_V+i\gamma/2), \tag{4·1}$$

where $f(\omega)$ is a single-valued, analytic function satisfying the following conditions.

1) $f(\omega)\approx 1$ for $|\omega+m_N-m_V|\ll\kappa$,
   where $m_V-m_N-\mu\gg\kappa\gg\gamma$.
2) $f(\omega)\approx 0$ effectively, for $|\omega+m_N-m_V|\gg\kappa$,
   where ω is real.
3) regular near the real axis.
4) vanishing far away in the lower half-plane.

Here we have introduced a new parameter, κ, so that the value of $f(\omega)$ at the very pole and that near $\omega=m_V-m_N$ on the real axis be not appreciably different. The reason for the minus sign in (4·1) is that the integral along the real axis is effectively equivalent to one round of the pole with the negative direction (cf. conditions 3) and 4)).

A typical example of $f(\omega)$ is

$$f(\omega)=\{\lambda^2/(\omega^2+\lambda^2)\}e^{-i(\omega+m_N-m_V)/\kappa}, \tag{4·2}$$

where $m_V\ll\lambda\ll\Lambda$, and Λ is the cut-off parameter in $G(\omega)$. When $G(\omega)$ is present, the factor $\lambda^2/(\omega^2+\lambda^2)$ is unnecessary. If the limits of $\gamma\to 0$, $\kappa\to 0$ and $\lambda\to\infty$ are taken

in (4·2), then the right-hand side of (4·1) reduces to $2\pi i\delta(\omega_k+m_N-m_V)$ exactly.

Now, the physical state of the unstable V-particle is defined as follows:

$$|V^f\rangle \equiv Z^{1/2}\left[|V\rangle - g\int \frac{G(\omega_k)/\sqrt{2\omega_k}}{\omega_k+m_N-m_V+i\gamma/2}\cdot(1-f(\omega_k))\,\alpha_k^*\,d\boldsymbol{k}|N\rangle\right]. \quad (4\cdot3)$$

From the normalization, Z-factor is defined by

$$Z^{-1}=1+g^2\int \frac{\{G^2(\omega_k)/2\omega_k\}\,|1-f(\omega_k)|^2}{(\omega_k+m_N-m_V)^2+(\gamma/2)^2}\,d\boldsymbol{k}. \quad (4\cdot4)$$

Z then is interpreted as the dissociation probability, as we have, of course, $0<Z<1$ and $\lim_{g\to 0} Z=1$. $|V^f\rangle$ formally reduces to $|V^\circ\rangle$ for $f(\omega)\equiv 1$, and to Naito's state for $f(\omega)\equiv 0$. $|V^f\rangle$ naturally is an approximate eigenstate of H with the eigenvalue $m_V-i\gamma/2$, as it should be.

We can easily solve the time-dependent Schrödinger equation, (3·20), under the initial condition $\Psi(0)=|V^f\rangle$, and then the time-dependence of $\langle V^f|\Psi(t)\rangle$ is $e^{-i(m_V-i\gamma/2)t}$ with a small "misty" term, which vanishes due to the interference if the production of the V-particle or the observation of the decaying process takes time longer than κ^{-1} (cf. Araki *et al.*).* $|V^f\rangle$ thus obeys an almost simple decay law.

Contrary to $|V\rangle$, $|V^f\rangle$ has decay products because of $\langle\Psi(t)|\Psi(t)\rangle=1$. They are given by the following formula.

$$\begin{aligned}\langle N\theta(\boldsymbol{p})^-|V^f\rangle = \frac{Z^{1/2}gG(\omega_p)/\sqrt{2\omega_p}}{\omega_p+m_N-m_V+i\gamma/2}\Big[& f(\omega_p)+S_V'(m_N+\omega_p)\Big\{g^2\int\frac{G^2(\omega_k)}{2\omega_k}f(\omega_k)\\ & \cdot\left((\omega_k+m_N-m_V+i\gamma/2)^{-1}-(\omega_k-\omega_p-i\varepsilon)^{-1}\right)d\boldsymbol{k}\\ & +4\pi^2 ig^2G^2(m_V-m_N-i\gamma/2)\sqrt{(m_V-m_N-i\gamma/2)^2-\mu^2}\Big\}\Big], \quad (4\cdot5)\end{aligned}$$

that is renormalizable contrary to the case of $|V\rangle$, for which we have

$$\langle N\theta(\boldsymbol{p})^-|V\rangle = g\{G(\omega_p)/\sqrt{2\omega_p}\}\,S_V'(m_N+\omega_p). \quad (4\cdot6)$$

If the coupling renormalization is defined by $g_r^2=Zg^2$ as usual, we obtain

$$\begin{aligned}\langle N\theta(\boldsymbol{p})^-|V^f\rangle = \frac{g_rG(\omega_p)/\sqrt{2\omega_p}}{\omega_p+m_N-m_V+i\gamma/2}\Big[& f(\omega_p)+S_V^{(r)}(m_N+\omega_p)\Big\{g_r^2\int\frac{G^2(\omega_k)}{2\omega_k}f(\omega_k)\\ & \cdot\left((\omega_k+m_N-m_V+i\gamma/2)^{-1}-(\omega_k-\omega_p-i\varepsilon)^{-1}\right)d\boldsymbol{k}\\ & +4\pi^2 ig_r^2G^2(m_V-m_N-i\gamma/2)\sqrt{(m_V-m_N-i\gamma/2)^2-\mu^2}\Big\}\Big], \quad (4\cdot7)\end{aligned}$$

where

$$\begin{aligned}&[S_V^{(r)}(E)]^{-1}=Z[S_V'(E)]^{-1}\\ &=(E-m_V+i\gamma/2)\Big[1+g_r^2\int\frac{G^2(\omega_k)}{2\omega_k}\,\frac{(E-m_V-i\gamma/2)\,d\boldsymbol{k}}{(\omega_k+m_N-E-i\varepsilon)\,\{(\omega_k+m_N-m_V)^2+(\gamma/2)^2\}}\end{aligned}$$

* The explicit form of $\langle V^f|\Psi(t)\rangle$ is rather lengthy, and so we refrain from writing it.

$$+g_r^2\int\frac{\{G^2(\omega_k)/2\omega_k\}\{2\,\mathrm{Re}f(\omega_k)-|f(\omega_k)|^2\}}{(\omega_k+m_N-m_V)^2+(\gamma/2)^2}\,d\boldsymbol{k}$$

$$-\frac{4\pi^2 i g_r^2 G^2(m_V-m_N-i\gamma/2)\sqrt{(m_V-m_N-i\gamma/2)^2-\mu^2}}{E-m_V+i\gamma/2}\Bigg]. \tag{4·8}$$

(4·7) is divergence-free at no cut-off limit, $\Lambda\to\infty$.*

(4·7) approximately can be written as

$$\frac{g_r G(\omega_p)/\sqrt{2\omega_p}}{\omega_p+m_N-m_V+i\gamma/2}\cdot f(\omega_p) \tag{4·9}$$

for $f(\omega)$ of (4·2). This will be verified in Appendix. It is of interest that (4·9) is obtained also by the replacement of (4·1) (and $g\to g_r$) from (3·11). The approximate decay spectrum apart from the phase-volume factor thus has the following simple and reasonable form:

$$\frac{g_r^2 G^2(\omega_p)/2\omega_p}{(\omega_p+m_N-m_V)^2+(\gamma/2)^2},\quad (\text{for } \omega_p\ll\lambda) \tag{4·10}$$

because of $|f(\omega)|^2\approx 1$.

§ 5. Generalizations

We have so far restricted ourselves to Lee's model. In this section we extend the results in last two sections to the general field theory, in which H_0 is the free Hamiltonian and H_1 is the interaction one, and $H=H_0+H_1$. The V-particle now has other freedoms, such as momentum, spin, charge, *etc.*, but we arbitrarily fix such quantum numbers. The bare states are

$$\left.\begin{aligned} H_0|V\rangle&=E_0|V\rangle,\\ H_0|n\rangle&=E_n|n\rangle,\end{aligned}\right\} \tag{5·1}$$

where $|n\rangle$'s stand for all the eigenstates of H_0 other than $|V\rangle$, namely for the projection operator to $|V\rangle$, $\Lambda\equiv|V\rangle\langle V|$, we have

$$\Lambda|n\rangle=0,\quad [H_0,\ \Lambda]=0. \tag{5·2}$$

(i) We first begin with the case of the stable V-particle. Consider a state,

$$|V_E\rangle\equiv|V\rangle-\int dn\,\frac{\langle n|U(E)|V\rangle}{E_n-E-i\varepsilon}|n\rangle \tag{5·3}$$

$$=[1-(1-\Lambda)(H_0-E-i\varepsilon)^{-1}U(E)]|V\rangle, \tag{5·4}$$

where $\int dn$ symbolically denotes sums and integrations over all $|n\rangle$'s, which especially

* Contrary to the stable case, $S_V'(E)$ can be written as a formally divergence-free form also without coupling r enormalization because of the presence of γ (cf. (3·7) and (3·5)). But $\int|\langle N\theta(\boldsymbol{p})^-|V\rangle|^2 d\boldsymbol{p}$ is divergent at $\Lambda\to\infty$ even for such a finite $S_V'(E)$.

includes the integration of E_n. The matrix $U(E)$ is defined by[10),1)]

$$U(E)=H_1-H_1(1-\Lambda)\,(H_0-E-i\varepsilon)^{-1}U(E), \tag{5·5}$$

that is,

$$U(E)=[1+H_1(1-\Lambda)\,(H_0-E-i\varepsilon)^{-1}]^{-1}H_1. \tag{5·6}$$

Now, operating $(E-H)$ to (5·4), we immediately obtain

$$(E-H)\,|V_E\rangle=\Lambda[E-E_0-U(E)]\,|V\rangle, \tag{5·7}$$

making use of (5·5). Since $\Sigma(E)\equiv\langle V|U(E)\,|V\rangle$ is the proper self-energy of the V-particle in the absence of vacuum polarization, we define its modified propagator by

$$[S_V'(E)]^{-1}=E-E_0+i\varepsilon-\Sigma(E). \tag{5·8}$$

$S_V'(E)$ does not immediately relate to the usual relativistic propagator, $S_F'(p_\mu)$, in the presence of vacuum polarization. The substitution of (5·8) in (5·7) yields

$$(E-H)\,|V_E\rangle=[S_V'(E)]^{-1}\,|V\rangle. \tag{5·9}$$

In the rest system, if m is the real root of $[S_V'(E)]^{-1}=0$, we can define the physical state of the V-particle by[11)]

$$\left.\begin{aligned} &|\boldsymbol{V}\rangle\equiv|V\rangle-\int dn\frac{\langle n|U(m)\,|V\rangle}{E_n-m}\,|n\rangle\\ \text{and then}\quad &H|\boldsymbol{V}\rangle=m\,|\boldsymbol{V}\rangle. \end{aligned}\right\} \tag{5·10}$$

A slight modification of (5·9) leads to

$$S_V'(E)=\langle V|\,(E-H+i\varepsilon)^{-1}|V\rangle. \tag{5·11}$$

Further, by multiplying (5·9) by $\langle\boldsymbol{V}|$ we obtain

$$(E-m)\langle\boldsymbol{V}\,|\boldsymbol{V}_E\rangle=[S_V'(E)]^{-1} \tag{5·12}$$

from (5·10). (5·12) becomes

$$\langle\boldsymbol{V}|\boldsymbol{V}\rangle=(\partial/\partial E)[S_V'(E)]^{-1}|_{E=m}\equiv Z^{-1} \tag{5·13}$$

at the limit of $E\to m$. This is the well-known relation between the renormalization constant and the normalization of the physical state.[6)]

(ii) We extend the above theory on the stable particle to the unstable case. As in Araki *et al.* § 4, we analytically continue $S_V'(E)$ to the lower half-plane and seek for its pole, which is denoted by $m_V-i\gamma/2$ as before. Next, such a factor as $(1-\Lambda)\cdot(H_0-E-i\varepsilon)^{-1}$ is replaced by a complex distribution like

$$(1-\Lambda)\,(H_0^{(-)}-E)^{-1}\equiv\int dn\,(E_n^{(-)}-E)^{-1}|n\rangle\langle n|. \tag{5·14}$$

The exact state of the unstable V-particle then is defined from (5·3) as follows:

$$\left.\begin{aligned} &|\boldsymbol{V}\rangle\equiv|V\rangle-\int dn\frac{\langle n|U(m_V-i\gamma/2)\,|V\rangle}{E_n^{(-)}-m_V+i\gamma/2}\,|n\rangle,\\ &H|\boldsymbol{V}\rangle=(m_V-i\gamma/2)\,|\boldsymbol{V}\rangle. \end{aligned}\right\} \tag{5·15}$$

The formula corresponding to (5·12) is

$$(E-m_V-i\gamma/2)\langle V|V_E\rangle=[S_V'(E)]^{-1}. \tag{5·16}$$

Since the coefficient of the left-hand side does not vanish at the limit of $E\to m_V-i\gamma/2$ contrary to the stable case, one finds

$$\langle V|V\rangle=0, \tag{5·17}$$

but not the relation like (5·13). The differentiation of (5·16) by E leads to

$$\langle V|V_E\rangle+(E-m_V-i\gamma/2)(\partial/\partial E)\langle V|V_E\rangle=(\partial/\partial E)[S_V'(E)]^{-1}. \tag{5·18}$$

Putting $E=m_V-i\gamma/2$ we obtain

where
$$\left.\begin{aligned} &Z_S^{-1}=-i\gamma\langle V|\partial V\rangle,\\ &|\partial V\rangle\equiv(\partial/\partial E)|V_E\rangle|_{E=m_V-i\gamma/2}. \end{aligned}\right\} \tag{5·19}$$

(iii) Finally, we must define the physical state of the unstable V-particle. For this purpose, it is not good to replace the complex distribution in (5·15) immediately as in § 4. In the actual case, the strong and weak interactions are present, and the decay is generally caused by the latters alone. Hence it is desirable for the physical state to reduce to the eigenstate for the strong interactions at the switch-off of the weak ones.

The interaction Hamiltonian is divided to the strong and weak parts : $H_I=H_s+H_w$. The physical state is almost determined by H_s. Consider a state

$$|\bar{V}_E\rangle\equiv|V\rangle-\int dn\frac{\langle n|\bar{U}(E)|V\rangle}{E_n^{(-)}-E}|n\rangle, \tag{5·20}$$

where $\bar{U}(E)$ stands for the U-matrix relating to H_s only, *i. e.*

$$\bar{U}(E)\equiv[1+H_s(1-\Lambda)(H_0^{(-)}-E)^{-1}]^{-1}H_s. \tag{5·21}$$

The superscripts $(-)$ in (5·20) and (5·21) are actually unnecessary, provided that $\mathrm{Re}\,E$ is less than the threshold energy of the strong interactions. By using (5·21),

$$U(E)=[1+(H_s+H_w)(1-\Lambda)(H_0^{(-)}-E)^{-1}]^{-1}(H_s+H_w) \tag{5·22}$$

is rewritten as

$$U(E)=\bar{U}(E)+U'(E)[1-(1-\Lambda)(H_0^{(-)}-E)^{-1}\bar{U}(E)], \tag{5·23}$$

where

$$U'(E)\equiv[1+(H_s+H_w)(1-\Lambda)(H_0^{(-)}-E)^{-1}]^{-1}H_w. \tag{5·24}$$

From (5·23) and (5·20) we obtain

$$|V_E\rangle\equiv|V\rangle-\int dn\frac{\langle n|U(E)|V\rangle}{E_n^{(-)}-E}|n\rangle=|\bar{V}_E\rangle-\int dn\frac{\langle n|U'(E)|\bar{V}_E\rangle}{E_n^{(-)}-E}|n\rangle. \tag{5·25}$$

The operation of $(E-H)$ to (5·25) yields

$$(E-H)|V_E\rangle=[E-H_0-H_s-\Lambda U'(E)]|\bar{V}_E\rangle, \tag{5·26}$$

hence from (5·9)

$$[S_V'(E)]^{-1}=[\bar{S}_V(E)]^{-1}-\langle V|U'(E)|\bar{V}_E\rangle, \tag{5·27}$$

where $\bar{S}_V(E)$ is the modified propagator in the presence of the strong interactions only. (5·27) tells us that $\langle V|U'(E)|\bar{V}_E\rangle$ represents the extra proper self-energy due to the switch-on of the weak interactions. (5·27) is nothing but the V-V element of (5·23).

Now, from (5·25) we can define the physical state as follows:

$$|V^f\rangle\equiv|\bar{V}_{m_V-i\gamma/2}\rangle-\int dn\ \frac{\langle n|U'(m_V-i\gamma/2)|\bar{V}_{m_V-i\gamma/2}\rangle}{E_n-m_V+i\gamma/2}\{1-f(E_n)\}|n\rangle. \tag{5·28}$$

Here, notice that $|\bar{V}_{m_V-i\gamma/2}\rangle$ is *not* the exact eigenstate of H_0+H_s. If we use the latter, $|V^f\rangle$ cannot be written in such a simple form as (5·28).

For instance, for a modified Lee's model, where the strong interaction $(N_1+\theta\rightleftarrows V$, coupling $g_1)$ and the weak one $(N_2+\theta\rightleftarrows V$, coupling $g_2)$ are present for two kinds of N-particles, N_1 (mass m_1, $m_1+\mu>m_V$) and N_2 (mass m_2, $m_2+\mu<m_V$), the physical state of the unstable V-particle becomes

$$\left.\begin{aligned}|\bar{V}_{m_V-i\gamma/2}\rangle&=|V\rangle-g_1\int\frac{G_1(\omega_k)/\sqrt{2\omega_k}}{\omega_k+m_1-m_V+i\gamma/2}\alpha_k^*\,d\boldsymbol{k}|N_1\rangle,\\ |V^f\rangle&=|\bar{V}_{m_V-i\gamma/2}\rangle-g_2\int\frac{G_2(\omega_k)/\sqrt{2\omega_k}}{\omega_k+m_2-m_V+i\gamma/2}(1-f(\omega_k))\alpha_k^*\,d\boldsymbol{k}|N_2\rangle.\end{aligned}\right\} \tag{5·29}$$

§ 6. Discussions

We constructed the eigenstate of H with a complex eigenvalue by introducing the new notion of complex distribution, which may seem rather unfamiliar. But this notion is not quite an additional one to the conventional theory. To see this, consider the following simple series of real distributions as an example:

$$F(\omega)\equiv\sum_{n=0}^{\infty}c^n/(\omega-a+i\varepsilon)^{n+1}, \tag{6·1}$$

where $0<\omega<+\infty$, a is a positive constant and c is complex $(|c|<a)$. If c stands for a proper self-energy, then $F(\omega)$ is just a modified propagator. For simplicity, we assume that c is a constant. The sum of series (6·1) is defined by

$$F[\varphi]\equiv\int_0^{+\infty}F(\omega)\varphi(\omega)\,d\omega=\sum_{n=0}^{\infty}\int_0^{+\infty}\frac{c^n\varphi(\omega)}{(\omega-a+i\varepsilon)^{n+1}}\,d\omega \tag{6·2}$$

as in § 2. Here $\varphi(\omega)$ is a regular function vanishing far away on the upper half-plane. We then have

$$F[\varphi]=\sum_{n=0}^{\infty}\int_0^{-\infty}\frac{c^n\varphi(\omega)}{(\omega-a)^{n+1}}\,d\omega, \tag{6·3}$$

which is absolutely convergent. Hence exchanging the order of the integration and the sum, we obtain

$$F[\varphi]=\int_{0}^{-\infty}\frac{\varphi(\omega)}{\omega-a-c}d\omega=\int_{0}^{+\infty}\frac{\varphi(\omega)}{\omega^{(+)}-a-c}d\omega, \tag{6·4}$$

and thus

$$F(\omega)=1/(\omega^{(+)}-a-c). \tag{6·5}$$

Namely, a series of real distributions generally represents a complex distribution. Feldman's difficulty for a modified propagator[12] is thus understood in terms of a complex distribution.*

Lehmann[13] pointed out that Feldman's difficulty was due to the approximation employed by him, because the exact modified propagator could always be written as

$$S_V'(\omega)=\int\frac{\rho(\kappa)}{\omega-\kappa+i\varepsilon}d\kappa, \tag{6·6}$$

and so $S_V'(\omega)$ could not have any complex pole. This is quite natural because Lehmann postulates that the eigenvalues of H are restricted to real values. But it is generally possible in quantum field theory that there appear complex distributions as mentioned above. Hence if we rather admit their appearances from the beginning, H generally has complex eigenvalues as in §§ 3, 5. *Since the states with complex eigenvalues are not observable, there is no physical reason why their presence must be forbidden.* Then $\rho(\kappa)$ becomes a complex distribution, and (6·6) is rewritten as

$$S_V'(\omega^{(+)})=\int\frac{\rho(\kappa)}{\omega^{(+)}-\kappa^{(-)}}d\kappa, \tag{6·7}$$

which generally has complex poles. For the above example, (identifying $F(\omega)$ with $S_V'(\omega)$) we have

$$\rho(\kappa)=\delta(\kappa-a-c). \tag{6·8}$$

And also for the unstable case of Lee's model we have (cf. Araki *et al.* (A·17))

$$\begin{aligned}\rho(m_N+\omega)&\equiv\int d\boldsymbol{p}\,\langle V|N\theta(\boldsymbol{p})^{\pm}\rangle\langle N\theta(\boldsymbol{p})^{\pm}|V\rangle\,\delta(\omega-\omega_p)\\&=2\pi g^2G^2(\omega)\sqrt{\omega^2-\mu^2}\,S_V'(m_N+\omega^{(+)})S_V'^{*}(m_N+\omega^{(-)}).\end{aligned} \tag{6·9}$$

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* Relativistic generalizations are straightforward.

Appendix

We derive (4·9) approximately. For simplicity we write

$$\varphi(\omega_k) \equiv 2\pi g_r^2 G^2(\omega_k) k .$$

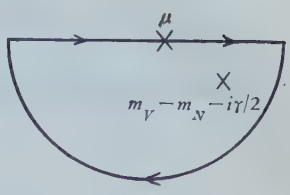

Fig. 3

As shown in Fig. 3, we analytically continue the integrands in (4·7) to the lower half-plane. We then have

$$\int_{\mu}^{+\infty} \frac{\varphi(\omega_k) f(\omega_k)}{\omega_k + m_N - m_V + i\gamma/2} d\omega_k = -2\pi i \varphi(m_V - m_N - i\gamma/2) f(m_V - m_N - i\gamma/2)$$

$$- \int_{-\infty}^{\mu} \frac{\varphi(\omega_k) f(\omega_k)}{\omega_k + m_N - m_V + i\gamma/2} d\omega_k ,$$

and

$$\int_{\mu}^{+\infty} \frac{\varphi(\omega_k) f(\omega_k)}{\omega_k - \omega_p - i\varepsilon} d\omega_k = - \int_{-\infty}^{\mu} \frac{\varphi(\omega_k) f(\omega_k)}{\omega_k - \omega_p} d\omega_k ,$$

because of the fourth condition for $f(\omega)$. Since $f(m_V - m_N - i\gamma/2) \approx 1$, the expression in the curly bracket of (4·7) approximately becomes

$$(\omega_p + m_N - m_V + i\gamma/2) \int_{-\infty}^{\mu} \frac{\varphi(\omega_k) f(\omega_k) d\omega_k}{(\omega_k + m_N - m_V + i\gamma/2)(\omega_k - \omega_p)} .$$

Here, the first factor cancels the singularity of $S_V^{(r)}(m_N + \omega_p)$ and the integral approximately vanishes because of the rapid oscillation of $f(\omega_k)$ (no pole exists near the path). Thus the second term in the square bracket of (4·7) is very small.

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Note added in proof : (4·2) is a simple example for $f(\omega)$ but not the best one, since $\int_{\mu}^{+\infty} |f(\omega)|^2 d\omega$ is divergent without the unnatural factor $\lambda^2/(\omega^2+\lambda^2)$. $f(\omega)$ generally has the form

$$\int_0^{+\infty} h(t) \exp\{-i(\omega + m_N - m_V)t\} dt$$

with $h(t) \approx 0$ for $t \approx 0$ and $\int_0^{+\infty} h(t)\, dt = 1$. A better example will be furnished by adopting as $h(t)$ the following function (ref. 7 (I. 2; 4)) :

$$h(t) = \text{const.} \exp\{-1 + \kappa^2 (t - 1/\kappa)^2\}^{-1} \qquad \text{for} \quad 0 < t < 2/\kappa$$
$$= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad t \geqq 2/\kappa .$$

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The Possible Existence of a Neutral Meson with Isotopic Spin 0

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The possible existence of a neutral meson with isotopic spin 0 is suggested. Its effects on pion physics and some ways to test its existence are discussed.

§ 1. Introduction

We now have rather good evidence for the charge independence of all strong interactions. In particular, the low energy nuclear physics has supplied us with the strongest evidence in favour of this symmetry property. Although we have many arguments in favour of accepting the symmetrical theory of pions, which of course is charge independent, direct tests of this symmetrical theory have not been sufficiently accurate so exclude all other charge independent theories of undiscovered mesons. In particular, experimental evidence does not exclude the existence of a neutral meson with isotopic spin 0 in addition to the usual charge-triplet pion (and K-mesons). Considering the existence of both charge triplet and singlet hyperons (Σ and Λ), it is quite attractive to postulate the existence of a charge singlet meson. Assuming such a meson to exist, we denote it by π', which must not be confused with the neutral member π^0 of charge-triplet pion. It is the purpose of this paper to discuss possible effects of this π' in pion physics, as well as its possible detection.

§ 2. Properties of π'

We are interested mainly in such a π' as can *affect* the *low* energy meson physics.* So we consider only two types of mesons, the usual pion which obeys the symmetrical theory and the new meson π' which obeys the neutral theory. To avoid immediate contradiction with our experience, we would like to attribute the following properties to π' :

i) The mass μ' of π' is nearly equal to the mass μ^0 of π^0, e. g., $\mu^0 \lesssim \mu' \lesssim \mu^0 + 30$ Mev.*

ii) The spin, the isotopic spin and the strangeness are all zero.

iii) It interacts with baryons as strongly as the pion does.

iv) It decays into two γ-rays with the lifetime $\lesssim 10^{-14}$ sec.

Unless we use the Panofsky effect (which we shall discuss later), we have so far

* We do not discuss here another case, $\mu' \gtrsim 2\mu^0$, which has been considered by Nambu, ref. 10).

had no accurate method to measure the mass of neutral mesons. Therefore we may conclude that the above conditions imposed on π' are sufficient to make π' and π^0 indistinguisable. As for the Panofsky effect, one can find a special model which avoids the simultaneous emission of π^0 and π' after the π^- capture. One of such models is given by a special assumption

$$\mu' \gtrsim \mu,$$

where μ is the charged pion mass. Other possibilities will be discussed later.

An interesting feature of π' is worth mentioning. Since the isotopic spin of π' is zero, the pion-nucleon (or briefly π-N) scattering with total isotopic spin $I=3/2$ has nothing to do with this π'. In fact, there is beautiful agreement[1] between the pion theory and the experiments for the π-N scattering with $I=3/2$. This is why we have assumed π' to have the isotopic spin 0. On the other hand, the π-N collision in the $I=1/2$ state can lead to a final state in which the nucleon is accompanied by a π'. The symmetrical theory of pions alone seems to be in disagreement[1] with experiment in this case. The introduction of π' may serve to remove this discrepancy (also see ref. 5)).

Let us discuss the π-N collision with $I=1/2$ in a little more detail. We have here two eigen states for each value of the total charge[2]. For example, the charge wave functions for $I=1/2$ and $I_3=-1/2$ may be written in obvious notation

$$(\cos\varepsilon)\frac{\sqrt{2}\,(\pi^- p)-(\pi^0 n)}{\sqrt{3}}+(\sin\varepsilon)\,(\pi' n) \tag{1}$$

and

$$(\cos\varepsilon)\,(\pi' n)-(\sin\varepsilon)\frac{\sqrt{2}\,(\pi^- p)-(\pi^0 n)}{\sqrt{3}}, \tag{2}$$

where ε is real. For comparison, we also notice that the charge wave function for $I=3/2$ and $I_3=-1/2$ is

$$\frac{(\pi^- p)+\sqrt{2}\,(\pi^0 n)}{\sqrt{3}}. \tag{3}$$

Let δ_1, δ' and δ_3 be the eigen phase-shifts for these three eigen states, (1), (2) and (3). Then the R-matrix elements for meson-nucleon collisions are easily found and listed in Table I.

Table I

Process	R-matrix
$\pi^+ + p \to p + \pi^+$	R_3
$\pi^- + p \to p + \pi^-$	$\frac{1}{3}(R_3 + 2\text{“}R_1\text{”})$
$\pi^- + p \to n + \pi^0$	$\frac{\sqrt{2}}{3}(R_3 - \text{“}R_1\text{”})$
$\pi^- + p \to n + \pi'$	$\frac{2}{3}\sin\varepsilon\cos\varepsilon\,(R_1 - R')$
$\pi' + p \to p + \pi'$ $\pi' + n \to n + \pi'$	$(\sin^2\varepsilon)R_1 + (\cos^2\varepsilon)R'$

In this table we have put

$$\left.\begin{aligned} &R_3 = e^{2i\delta_3} - 1,\ R_1 = e^{2i\delta_1},\ R' = e^{2i\delta'} - 1 \\ \text{and} \\ &\text{“}R_1\text{”} = (\cos^2\varepsilon)\,R_1 + (\sin^2\varepsilon)\,R'. \end{aligned}\right\} \tag{4}$$

If $\varepsilon=0$, the results in Table I will reduce to the usual symmetrical pion case. If π' exists, we see that in performing the phase shift anlysis, there are more parameters than before. Therefore, the revised phase shift analysis including π' is extremely interesting, and we may expect to obtain better agreement between theory and experiment than before.

§ 3. Tne parity of π' and the meson-nucleon scattering

We have not yet specified the parity of π'. There are two possibilities, the scalar and the pseudo-scalar cases. Let us discuss these two cases in more detail.

§ 3·1 π' : *scalar*

First we shall discuss the scalar case. The incoming s(or p)-wave pion must be converted into an outgoing p(or $s+d$)-wave π' in the π-N collision with $I=1/2$. Hence, π' will hardly come out after the collision of the slow π^- with the proton and the s-wave pion-nucleon scattering near zero energy is essentially same as before. The π^- capture by the proton from the s-state is similarly unaffected by π'. The π^- capture from the p-states may give rise to substantial π'-emission. *If* it proves necessary to predict more neutral meson emission in the Panofsky effect, one can appeal to this effect. To do this, we have to assume close equality of masses of the two neutral mesons : $\mu^0=\mu'\pm 0.2$ MeV. If we do not want the π'-emission in the Panofsky effect, one can assume, for instance that π' is heavier than the charged pion.

If $\mu'\approx\mu^0$, one can expect that the incoming p-waves of π^- on the proton will lead mostly to an outgoing s-wave of π' at low energy region. In other words, the p-wave part of the total cross-section of π^--p collision at low energies is due almost entirely to the charge exchange scattering $\pi^-+p\to n+\pi'$. This is, however, not the case, so that we are inclined to choose π' to be considerably heavier than the pion mass.

An annoying thing is the neutral meson production by proton-proton collision. If π' is scalar, we would expect sizable π'-production even near the threshold in contradiction with experiment, unless there is some accidental cancellation of matrix elements near the threshold.

On the other hand, one can check the existence of the scalar π' by means of the reactions

$$d+d\to\alpha+\pi', \qquad (5)$$

and

$$\alpha+\alpha\to 2\alpha\,(\text{or } \mathrm{Be}^8)+\pi'.$$

The π^0-production in these reactions is forbidden by the isotopic spin conservation.

From these considerations the existence of a scalar π' with $\mu'\approx\mu$ seems to be unlikely.

§ 3·2 π' : *pseudoscalar*

Next, we proceed to discuss the psedoscalar case. As is easily seen, there are no difficulties in near zero energy events like the Panofsky effct, the mesic-atom, and the

s-wave π-N scattering, *if* one assumes that the mass μ' of π' is heavier than the charged pion mass μ by, say, about 30 Mev. Therefore it is only intersting to investigate the case $\mu' \approx \mu^0$. Before going into the detail, let us define:

$$a_3 = \delta_3/\eta,\ a_1 = \delta_1/\eta, \quad \text{and} \quad a' = \delta'/\eta, \tag{6}$$

where δ_3, δ_1 and δ' are the eigen phase shifts for *s*-wave mesons (see Table I) and $\eta = (\text{meson momentum})/(\text{meson mass})$ as usual. For small η the a's are practically constant. From the low energy data of

$$\pi^+ + p \to p + \pi^+$$

and

$$\pi^- + p \to p + \pi^-$$

one finds[3]

$$\left.\begin{aligned} &a_3 = -0.11, \\ &\text{“}a_1\text{”} = (\cos^2 \mathcal{E})\, a_1 + (\sin^2 \mathcal{E})\, a' \\ &\qquad = 0.16. \end{aligned}\right\} \tag{7}$$

While the charge exchange scattering, i. e., the sum of $\pi^- + p \to n + \pi^0$ and $\pi^- + p \to n + \pi'$, gives information about the value of

$$\tfrac{2}{9}|a_3 - \text{“}a_1\text{”}|^2 + \tfrac{2}{3}(\sin \mathcal{E} \cos \mathcal{E})\,|a_1 - a'|^2. \tag{8}$$

We see that (7) and the experimental value for (8) are consistent with

$$a_1 \fallingdotseq a'$$

or

$$a_1 = a' = \text{“}a_1\text{”} = 0.16 \quad \text{and} \quad a_3 = -0.11. \tag{9}$$

These results show that the cross section of

$$\pi^- + p \to n + \pi'$$

vanishes at a very low energy. Thus the Panofsky effect is entirely due to π^0. We know that (9) seems to be compatible not only with the Panofsky effect but also with information about the a's from the π^--mesic atom. The π' can of course come out in *s*-waves at high energies and also in higher waves in π-N collisions.

We want to investigate the effect of π' on the *p*-wave π-N collisions. For the time being, it would be unnecessary to make fancy calculation. Hence, we shall use the simple *ps* (*pv*) static models[4] for both the symmetrical pions and the π', and the Born approximation to the phase shifts. We also assume for simplicity $\mu' = \mu$. Let us denote the (renormalized) coupling constants of π-N and π'-N interactions by f and f_0. Then we can describe the pion-nucleon scattering in terms of the usual phase shifts α's (which are the same as in ref. 4):

$$\left.\begin{aligned}\alpha_{11}&=-\tfrac{8}{3}\\ \alpha_{31}=\alpha_{13}&=-\tfrac{2}{3}\\ \alpha_{33}&=+\tfrac{4}{3}\end{aligned}\right\}\times\frac{f^2}{4\pi}\frac{p^3}{\omega\mu^2}. \tag{10}$$

where p is the meson momentum and $\omega=\sqrt{p^2+\mu^2}$ ($\hbar=c=1$). The process $\pi+N\to N+\pi'$ with $I=1/2$ can be described by the two phase shifts,

$$\left.\begin{aligned}\alpha_{13}'&=+\tfrac{2}{3}\\ \alpha_{11}'&=-\tfrac{4}{3}\end{aligned}\right\}\times\frac{ff_0}{4\pi}\frac{p^3}{\omega\mu^2}, \tag{11}$$

where α_{12J}' is the phase shift for the p_J-state ($J=1/2$ or $3/2$). Finally, for the process $\pi'+N\to N+\pi'$, one finds the phase shifts:

$$\left.\begin{aligned}\alpha_{13}''&=+\tfrac{2}{3}\\ \alpha_{11}''&=-\tfrac{4}{3}\end{aligned}\right\}\times\frac{f_0^2}{4\pi}\frac{p^3}{\omega\mu^2}, \tag{12}$$

where α_{12J}'' is the p_J-phase shift.

We shall assume throughout in this paper (also see § 4·3)

$$f_0^2\lesssim f^2. \tag{13}$$

The scatterings involving π' have relatively small phase shifts, α_{12J}' and α_{12J}'', which, however, can be comparable with small phase shifts like α_{11} or α_{13}. Accordingly, the discrepancy between the old theory[1] and the experiments can now be overcome by the introduction of the π'[5]. It should be noted that the total cross section $\sigma(\pi^-\to\pi')$ for $\pi^-+p\to n+\pi'$ is $\lesssim 10\%$ of the π^-p total cross section[6].

Table II

Total isotopic spin I	Eigen-state (Example only)	Eigen phase shift		
		$s_{1/2}$	$p_{1/2}$	$p_{3/2}$
3/2	(π^+p) $\dfrac{(\pi^-p)+\sqrt{2}\,(\pi^0n)}{\sqrt{3}}$	δ_3	δ_{31}	δ_{33}
1/2	$(\cos\varepsilon)\dfrac{\sqrt{2}\,(\pi^-p)-(\pi^0n)}{\sqrt{3}}$ $+(\sin\varepsilon)\,(\pi'n)$	δ_1	δ_{11}	δ_{13}
1/2	$-(\sin\varepsilon)\dfrac{\sqrt{2}\,(\pi^-p)-(\pi^0n)}{\sqrt{3}}$ $+(\cos\varepsilon)\,(\pi'n)$	δ'	δ_{11}'	δ_{13}'

Note: The ε depends on the state; ε for the $s_{1/2}$ state and ε_{12J} for the p_J-state ($J=1/2$ or $3/2$).

It is interesting to calculate the eigen phase shifts, and the parameters ε (see Table

II). It is easy to find them from eqs. (10)$\sim$(12):

$$\left.\begin{aligned}
&\varepsilon_{11}=\varepsilon_{13}=\varepsilon, \quad \tan 2\varepsilon=-2\sqrt{3}\, ff_0/(f^2+f_0^2),\\
&\delta_{33}=\alpha_{33}, \quad \delta_{31}=\alpha_{31},\\
&\delta_{13}=\left(f_0^2-f^2+\frac{2\sqrt{3}ff_0}{\sin 2\varepsilon}\right)\\
&\delta_{11}=-\left(2f^2+f_0^2+\frac{2\sqrt{3}ff_0}{\sin 2\varepsilon}\right)\\
&\delta_{13}'=\left(f_0^2-f^2-\frac{2\sqrt{3}ff_0}{\sin 2\varepsilon}\right)\\
&\delta'_{11}=-\left(2f^2+f_0^2-\frac{2\sqrt{3}ff_0}{\sin 2\varepsilon}\right)
\end{aligned}\right\}\cdot\frac{1}{12\pi}\frac{p^3}{\mu^2\omega}. \tag{14}$$

In the special case

$$f_0=\pm f,$$

one finds

$$\sin 2\varepsilon=\mp 1/2,$$

and

$$\delta_{31}=\frac{1}{2}\delta_{13}=-\delta_{11}=-\frac{1}{2}\delta'_{13}=\frac{1}{7}\delta'_{11}=-\frac{2}{3}\frac{f^2}{4\pi}\frac{p^3}{\mu^2\omega}. \tag{15}$$

In contrast to the scalar case, there is no difficulty of the neutral meson production by the proton-proton collision near the threshold, since the same reasoning as for the π^0 case can be applied to the pseudoscalar π'.

A direct test of the existence of a pseudoscalar π' is rather difficult, because the reactions (5) are now forbidden. So that we have to investigate meson production processes, where both pions and π' are involved, very carefully. For example, we can use the famous reactions

$$p+d\rightarrow\begin{cases} \mathrm{H}^3+\pi^+ \\ \mathrm{He}^3+\pi^0,\ \mathrm{He}^3+\pi' \end{cases} \tag{16}$$

and

$$\begin{aligned} &p+p\rightarrow d+\pi^+\\ &p+n\rightarrow d+\pi^0,\ d+\pi' \end{aligned} \tag{17}$$

for this purpose. An experimental accuracy better than $\sim$5% will be required in order to determine whether the reactions $p+n\rightarrow d+\pi'$, and $p+d\rightarrow\mathrm{He}^3+\pi'$ occur or not (see ref. 5) and 6)). We have so far rather poor knowledge about the positive pion to neutral meson ratios for (16) and (17). It is wise to avoid the 33 resonance region to test

the existence of the π'. Thus it is interesting to investigate experimentally the single meson production by the π-N or N-N collision at off-resonance energies.

§ 4. Other comments on the π'.

§ 4·1 *Decay processes of hyperons and K-mesons.*

If the π' exists, it is natural to suppose that one of the decay products of Σ, Λ and K-mesons could be the π'. Then, one can expect

$$\Sigma^{+}\rightarrow\begin{cases} p+\pi^{0} \\ p+\pi'. \end{cases}$$

These two processes should be easily discriminated by measuring the energy of the proton, if $\mu^{0}\neq\mu'$. Similarly, there are the following decay modes of K-mesons

$$K^{\pm}\rightarrow\pi^{\pm}+\pi', \tag{18}$$

$$K^{\pm}\rightarrow\pi^{\pm}+2\pi'$$

$$\theta_{2}^{0}\rightarrow\begin{cases} 2\pi^{0}+\pi', \\ \pi^{+}+\pi^{-}+\pi' \\ \pi^{0}+2\pi' \\ 3\pi'. \end{cases}$$

It is interesting to note that (18) is consistent with the so-called $\Delta I=1/2$ law. Evidently these decay processes are useful to check the existence of π'.

§ 4·2 *Dispersion relation.*

A few words should be added concerning the dispersion relation for the pion-nucleon scattering. Even if we introduce the π', the form of the dispersion relations (see ref. 7) for the $\pi^{\pm}+p\rightarrow p+\pi^{\pm}$ does not change at all. In other words, as far as we use experimentally measured values for the forward scatttering amplitudes and the total cross sections for $\pi^{\pm}$-p collisions, all effects due to the π' are fully taken into account. Therefore, origins of the difficulty emphasized by Puppi-Stanghellini[7], if it is real, cannot be attributed to the existence of π'. Fortunately, Lomon and Zaidi[8] have shown that the difficulty can be removed by using somewhat different values of total cross sections from those used by Puppi-Stanghellini. We believe that the proposal of Lomon-Zaidi is quite reasonable.

§ 4·3 *The meson-nucleon coupling constants.*

We have two coupling constants f and f_{0} in our theory. According to Otsuki et al[9], one can determine the strength of the meson-interaction by means of two nucleon data in the low energy region, among which the quadrupole moment of the deuteron plays the most important role. If the π' is scalar, the nuclear forces due to the π'-N interaction do not contain non-central force, so that we can say very little about

the coupling constant f_0. On the other hand, if the π' is pseudoscalar, one finds

$$\frac{f^2}{4\pi} - \frac{1}{3}\frac{f_0^2}{4\pi} \approx 0.08.$$

The accuracy of this determination is rather hard to estimate, but our feeling is that 0.08 ± 0.03. If we accept tentatively this determination we find

$$\frac{f_0^2}{4\pi} \lesssim \frac{f^2}{4\pi},$$

a condition which has already been used in § 3·2.

Moreover one can expect the resonance for the π'-N scattering. The position of this resonance ω_r' will be $\sim (f^2/f_0^2)\omega_r (\omega_r \approx 2.2\mu$ is the 33 resonance energy). This resonance can affect the π-N collision with $I=1/2$. If $f_0^2 \approx f^2$, this will be masked by the strong 33 resonance, while if $4f_0^2 \lesssim f^2$ this can partly be responsible to the second resonance of the π-N collision with $I=1/2$* around 1 Bev.

§ 4·4

We can in principle test the "symmetrical" pion theory by the multiple meson production at extremely high energy nuclear collisions. There, the ratio of the charged pions to the neutral "mesons" is only known within a substantial uncertainty. Therefore there is still room for additional π'-production.

§ 4·5

We have so far discussed the possible existence of a neutral meson π', which can affect low energy pion-phenomena.

Another possibility will be the case $\mu' \gtrsim 2\mu$, in which any explicit effects of the π' can hardly be expected on the low energy pion-phenomena. Some interesting discussions of such a case have recently been published by Nambu[10]. Gell-Mann[11] and Schwinger[12] have discussed the possible existence of a neutral meson with isotopic spin 0 from points of view entirely different from ours.

The discussion presented here may or may not be useless. We shall see in the near future which is the case. The existence of both charge triplet and singlet hyperons (Σ and Λ) is so impressive that the existence of a charge singlet meson π', whatever it may be, seems to be quite natural.

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Note added in proof Riddiford et al. (private communication) have recently measured the cross sections for single meson production in p-p and p-n collisions at 970 Mev. In particular they have found the neutral meson production cross section in p-n collisions which is 2--3 times bigger than that expected from the "symmetrical" pion theory. This is a quite interesting result for our proposal presented here, though there remain many possible effects to be considered within the framework of the usual pion theory.

* I am indebted to Dr. Miyazawa for this remark.

References

1) Cf. Chew, Goldberger, Low and Nambu, Phys. Rev. **106** (1957), 1345.
2) General considerations of π-N and π'-N scattering were published long ago by Y. Nambu and the present author, Prog. Theor. Phys. **6** (1951), 1000.
3) J. Orear, Phys. Rev. **100** (1955), 188.
4) G. F. Chew and F. E. Low, Phys. Rev. **101** (1956), 1570.
5) This is seen as follows: the s-phase shifts δ_1, δ_3 and δ' are calculated from eqs. (6) and (9), the p-phase shifts $\alpha_{2I,2J}$ from the effective range formulae given by Chew and Low, ref 4), and the remaining phases $\alpha_{12'J}$ by the Born approximation, eq. (11). Then we can calculate the differential cross sections for the ordinary scatterings

$$\pi^{\pm}+p \rightarrow p+\pi^{\pm}$$

and for the sum of two reactions:

$$\pi^{-}+p \rightarrow \begin{cases} n+\pi^0 \\ n+\pi'. \end{cases}$$

These results are in fair agreement with the observed ones in the sub-resonance region (We have assumed here $f_0^2=f^2$).
6) For $f_0^2/4\pi=f^2/4\pi=0.08$ one finds,

$$\sigma(\pi^{-}\rightarrow\pi') \fallingdotseq \begin{cases} 0.5\,\text{mb at 40 MeV} \\ 10\,\text{mb at resonance} \end{cases}$$

7) G. Puppi and A. Stanghellini, Nuovo Cimento **5** (1957), 1305
8) E. L. Lomon and N. H. Zaidi, Phys. Rev. **108** (1957), 1352 (L).
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Isotopic Spin and Strange Particles

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Studies of the isotopic spin change in various strange particle decays and of the conservation of the isotopic spin in the strong interactions have led to the introduction of a new and independent space (i'-space) in addition to the i-space. The i'-spin quantum number of baryons and heavy mesons, the property of the π and K couplings in the i'-space as well as in the i-space, and the decay selection rules of strange particles are determined in a rather unique fashion. Semi-quantitative comparison between the experiments and the predictions of the theory is made.

§ 1. Introduction*

Nishijima-Gell-Mann theory[1] (N-G theory) on the strange particles has been successful in understanding most of the qualitative features of the strange particle physics including

i) the selection rules for the production of strange particles,

ii) their long decay lifetimes,

and

iii) the presence of charge multiplets such as

$$(\Sigma'^{\pm}, \Sigma'^{0}), (K^{+}, K^{0}) \text{ and } (\bar{K}^{-}, \bar{K}^{0}).$$

There all the above features are explained phenomenologically by introducing the strangeness quantum number and the isotopic spin.

On the other hand there are some other properties of strange particles which still remain unexplained ; these are

i) some pieces of quantitative information about the isotopic spin change in their decays,[2]

ii) the mass spectrum of baryons,

and

iii) the problem of possible parity non-conservation[3] in the strange particle decays in general.

They might give us clues for advancing the theory of strange particles beyond the N-G theory at least on phenomenological ground. Particularly the study of the isotopic spin in connection with the first two problems seems to be very promising as we shall

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see in what follows.

Hereafter we assume that *the isotopic spin is conserved in every strong interaction.* The charge indepedence observed in π-N and N-N interactions and the presence of the isotopic spin multiplet seem to support this assumption. Besides there has been as yet no experiment indicating the violation of the isotopic spin conservation in the strong interactions.

Then each particle has its own isotopic spin quantum number assigned by the N-G theory :

kind of particle	$N,\ \Lambda,\ \Sigma,\ \Xi,\ \pi,\ K,\quad (\bar{K})$	(1)
isotopic spin	$1/2,\ 0,\ 1,\ 1/2,\ 1,\ 1/2,\quad (1/2)$	

In the Λ, Σ, and K decays,

$$\Lambda \to N+\pi,\ \Sigma \to N+\pi, \text{ and } K \to 2\pi,$$

the magnitudes of the isotopic spin of the final states are equal to either $1/2$ or $3/2$ for the Λ and Σ decays and 0 or 2 for the K decay*. Therefore the possible isotopic spin changes in their decays are

$$\Delta I = 1/2,\ 3/2 \text{ for the } \Lambda \text{ decay}$$

and

$$\Delta I = 1/2,\ 3/2,\ 5/2 \text{ for the } \Sigma \text{ and } K \text{ decays.}$$

Recent experiments[2] have given us some quantitative information on these isotopic spin changes :

i) $K^{\pm,0}$ decay

The lifetime of $K^{\pm}$ long compared with that of K^0 has suggested us[4] that approximately $\Delta I = 1/2$ in the K decay. The recent measurement[2] of the branching ratio of K_1^0 decay has revealed a small value of the ratio,

$$(K_1^0 \to 2\pi^0)/(K_1^0 \to 2\pi) = 0.14 \pm 0.06.$$

This ratio and the observed small ratio of the K_1^0 life time to that of $K^{\pm}$ lead us to a conclusion that the three types of interactions with $\Delta I = 1/2$, $3/2$ and $5/2$ must be present simultaneously in the K decays. Magnitudes of the $\Delta I = 3/2$ and $5/2$ interactions are of the order of 10% of the $\Delta I = 1/2$ interaction.[5] If the $\Delta I = 5/2$ decay cannot occur, the K_1^0 decay into $2\pi^0$ must be at least 28% of the total K_1^0 decay. Although this value is by about twice standard deviations far from the measured value, it is very desirable to have more accurate measurements on the ratio before we definitely conclude the presence of both the $\Delta I = 3/2$ and $\Delta I = 5/2$ decays.

ii) Σ decay

If $\Delta I = 1/2$ in Σ decays, there is almost one to one correspondence[6] between the

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* K particle is assumed to have an even spin.

ratio of τ_{Σ^+} to τ_{Σ^-} and the branching ratio of Σ^+ decay, $(\Sigma^+ \to p+\pi^0)/(\Sigma^+ \to n+\pi^+)$. The observed ratios do not satisfy the required relation and this deviation means the existence of either $\Delta I=3/2$ or $\Delta I=5/2$ interaction. Its magnitude seems to be of the order of 10% of the $\Delta I=1/2$ interaction. If the parity do not conserve in Σ decays, then $\Delta I=1/2$ could explain the experimental data without having either $\Delta I=3/2$ nor $\Delta I=5/2$ interaction.

iii) Λ decay

The observed branching ratio of Λ^0 decay, $(\Lambda^0 \to p+\pi^-)/(\Lambda^0 \to N+\pi)=0.65 \pm 0.05$[2], nicely coincides with the prediction of $\Delta I=1/2$ selection rule, although a small mixture of $\Delta I=3/2$ interaction cannot be excluded from the present experimental data.[2]

Here we would like to emphasize the importance of the above information, namely, the existence of $\Delta I=3/2$ and $5/2$ interactions. This makes a sharp contrast to the strict selection rule $\Delta I_z=\pm 1/2$, which is required by the charge conservation in the decay processes.

Since the conservation of isotopic spin in strong interactions is our essential assumption, the natural law is invariant under any rotation in the iso-space as far as we neglect the electromagnetic and weak interactions. On the other hand the observed isotopic spin changes, $\Delta I=1/2$, $3/2$, and $5/2$, in the decay processes together with the nonexistence of $\Delta I_z=\pm 3/2$ and $\pm 5/2$ mean that the weak interactions are not invariant under rotations around x or y axis of the iso-space. Otherwise we should have a process with $\Delta I_z=\pm 3/2$ or $\pm 5/2$ among the strange particle decays.

This deviation from the spherical symmetry of the i-space,* i. e., the reduction of the spherically symmetric i-space into an axially symmetric one may come from various different reasons. Let us study them by artificially switching off the electromagnetic interaction ($e^2/\hbar c > 0$) and by considering the weak interactions in thus obtained hypothetical world. Then we have two alternative possibilities: either a) $\Delta I=1/2$ or b) $\Delta I=1/2$, $3/2$, $5/2$ and $\Delta I_z=\pm 1/2$.

In the case a) the weak interactions do not specify any axis in the i-space and the world with $e^2/\hbar c=0$ has a spherically symmetric i-space. The charge conservation in the decay processes is automatically satisfied by the selection rule $\Delta I=1/2$. When we switch on the electromagnetic interaction again, there appear $\Delta I=3/2$ and $5/2$ interactions as electromagnetic corrections to the original $\Delta I=1/2$ weak interaction. On the other hand the gauge invariance of the theory keeps $\Delta I_z=\pm 1/2$ and the isotopic z axis becomes a special axis, thus violating the spherical symmetry of i-space. Rough estimation of this electromagnetic effect gives us a magnitude of $\Delta I=3/2$ or $5/2$ interaction at most of the order of 1% of the $\Delta I=1/2$ interaction.[7] We have looked for a model of the decays of strange particles which can increase the magnitude of this electromagnetic correction but it has so far been unsuccessful. Therefore the fairly large $\Delta I=3/2$ or

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* Hereafter we shall often speak of the spherical symmetry or asymmetry of the i(or i') space. When various interactions and physical properties of a system are (or are not) invariant under any rotation in the i-space, we say that the i-space is spherically symmetric (or asymmetric).

5/2 interaction required from the experiments seems to exclude the case a).*

If b) is the case, the decays in the hypothetical world ($e^2/\hbar c=0$) still shows lack of the spherical symmetry of i-space. This asymmetry must come not from the electromagnetic interaction but from some other sources. Also $\Delta I_z=\pm 1/2$, the charge conservation in the decays, cannot be due to the gauge invariance of the theory. This is required only when the electromagnetic coupling is switched on, otherwise our understanding of the electromagnetic field as an agent for keeping the charge conservation law will be lost.

One possibility for introducing the asymmetry in decays is to couple the i-spin with another vector i', which is concerned with a certain new and independent freedom in the strange particle physics. Then the spherical *asymmetry* can be brought by the coupling between i and i'-spins during the decaying processes, if the i' space itself is spherically *asymmetric*. Namely the exchange of the i-spin angular momentum with that of i' occurs during the decay processes, just as we have the exchange of ordinary spin and orbital angular momentum in nuclear reactions. If the i'-space is spherically asymmetric, it is apt to transfer the i'-spin angular momentum more in a certain direction to the i'-space than the spin angular momenta in the other directions. Thus we obtain the spherically asymmetric decays as far as the i-space is concerned. The asymmetry of the i' space must be either due to a certain kind of strong interactions or else due to an inherent nature of the i'-space. It is rather tempting to assume that the former is the case and without that strong interaction we see a spherically symmetric i' space.

There are only two kinds of strong interactions so far known to us, namely the π-interaction and K-interaction. Off-hand, it is free for us to choose either one of them as an agent for introducing the asymmetry into the i'-space. The mass spectrum of baryons and the observed isotopic spin change in the decay processes are in favour of the K interaction as the agent (§ 2). Furthermore, if the K interaction is weaker than the π interaction ($g_K^2/\hbar c < g_\pi^2/\hbar c$), this choice is consistent with the philosophy that a weaker interaction has less symmetric properties.

Thus we have finally obtained the following picture about Nature. The natural law is invariant under any rotation in the i and i'-spaces as well as in the ordinary spacetime, when only the π-coupling is switched on ($g_\pi^2/\hbar c\neq 0$, $g_K^2/\hbar c=0$, $e^2/\hbar c=0$). The decays in this super-symmetric world, which we hereafter call the primary decays, are due to the exchange of the i and i' spins with each other. Since the both spaces have no preferred direction, there is no special component of the spins more likely to be exchanged than the others. Therefore the selection rule of the decays is shown to become $\Delta I=1/2$, which contains only the magnitude but not any particular component of i-spin (§ 2). If the K interaction is introduced in addition to the π interaction ($g_\pi^2/\hbar c\neq 0$, $g_K^2/\hbar c\neq 0$, $e^2/\hbar c=0$), the law is no more invariant under any rotation in the i'-space, expect that around the i'_z axis. Then the i'-space becomes the axial symmetric one, whose axis is in

* If a K meson is really a very tiny particle and the electro-magnetic interaction inside such a small space is much stronger than we usually expect, we can still hope a) being the case. We will investigate this possibility in another place.

the z direction. During the decay processes the z component of i' spin is more (or less) frequently transferred to the i-space than the other two components of i'-spin, thus transferring the asymmetry of i'-space to the i-space to a certain extent. The electromagnetic interaction introduces an asymmetry into the i-space as is well known, but the effect of this asymmetry on the decay interactions are expected to be small and will be neglected hereafter.

In the following section (§ 2) we will show that the above explained way of thinking leads to a rather definite assignment of the i' spin to each elementary particle and to a reasonable understanding of the observed selection rules of the strange particle decays. If the types of the K interaction and the weak primary interaction are specified, we can make semi-quantitative prediction of the mass spectrum of baryons and the ΔI in the decay processes. It seems that they fit the observed values at least semi-quantitatively.

§ 2. I′ spin space and selection rule in decay interactions

Let us first study the i'-space discussed in § 1 and assign an i'-spin quantum number to each particle. One requirement for this is that if there is a particle with I' different from 0, there must be also particles with the same I' and different I_z' ($I_z'=I', I'-1, \cdots, -I'$). They together make an i'-spin multiplet and must have the samt spin and i-spin as well as the i'-spin. Moreover, if the K and electromagnetic interactions are turned off, those belonging to the same i'-multiplet must have the same mass. Observed mass differences among particles belonging to the same i'-multiplet come mainly from the K coupling and may reach even a magnitude of the order of $m_K c^2$.

If we accept the i-spin assignment (I) of strange particles given by the N-G theory, only such groups of the particles that can make an i'-spin multiplet according to the above requirments are the pairs (N, Ξ) and $(K, \bar{K})$. N and Ξ, for instance, have the same i-spin and probably the same ordinary spin $(1/2)$. Also $\bar{K}$ and K have the same i-spin $(1/2)$ and the same ordinary spin (probably 0). No other group of particles with such common properties is known to us.

Since we need the i'-space, some particles must have I' different from 0. Then we have only three alternative possibilities :

i) both (N, Ξ) and $(K, \bar{K})$ make i'-spin doulets,

ii) only (N, Ξ) makes an i'-spin doublet,

or else

iii) only $(K, \bar{K})$ makes an i'-spin doublet.

Let us take a reaction $\pi+N\rightarrow\Lambda+K$ in order to decide which one of them we should adopt. This reaction occurs through strong interactions including the K interaction. Since the K interaction introduces an asymmetry into the i'-space, this reaction may not conserve the i' spin itself. But if the asymmetry introduced there is such as to keep the axial symmetry of i'-space around its z axis, I_z' must be conserved in the above reaction. This conservation of I_z' rejects the case ii) and iii), leaving the case i) as a sole possibility.

Thus we arrive at the following assigment of I' spin to various particles. (Table I) This assignment is just the same one as given by Salam and Polkinghorne.[9] The charge of each particle is equal to the sum of I_z and I_z';

$$Q=I_z+I_z' \tag{2}$$

Table I I and I' spins of strange particles

	I	I_z	I'	I_z'
P	1/2	+1/2	1/2	+1/2
N	1/2	−1/2	1/2	+1/2
Ξ^0	1/2	+1/2	1/2	−1/2
Ξ^-	1/2	−1/2	1/2	−1/2
Λ^0	0	0	0	0
Σ^+	1	+1	0	0
Σ^0	1	0	0	0
Σ^-	1	−1	0	0
π^+	1	+1	0	0
π^0	1	0	0	0
π^-	1	−1	0	0
K^+	1/2	+1/2	1/2	+1/2
K^0	1/2	−1/2	1/2	+1/2
$\bar{K}^0$	1/2	+1/2	1/2	−1/2
$\bar{K}^-$	1/2	−1/2	1/2	−1/2

Since the π interaction is invariant under rotations in the i'-space, (P, N) and (Ξ^0, Ξ^-) have an identical mass with each other as far as we neglect the K and electromagnetic interactions. Also K and $\bar{K}$ have an identical mass. When we switch on the K interaction, it should produce the observed mass difference between Ξ and N. On the other hand K and $\bar{K}$ must keep the equality of mass, if they are to be particle conjugate with each other. It is easy to show that the local field theory for K particles satisfies this requirement irrespective of the asymmetric nature of the K interaction in the i'-space.

In the hypothetical world with only the π coupling present, we can search for a further symmetry than what we have discussed so far. Namely we can regard the i and i'-spins as two independent rotational operators in a four dimensional iso-space.[10] If we require full symmetry of the system under any rotation in this four dimensional space, then $(\Sigma^{\pm,0}, \Lambda^0)$ can be considered as making a vector in the space. And they must have an identical mass with each other. Also the two doublets (p, n) and (Ξ^0, Ξ^-) together make another four vector, represented in a form of a direct product of two spinors. Schwinger and Gell-Mann[11] have suggested this full symmetry among baryons when $g_K^2/\hbar c=e^2/\hbar c=0$.

One might attempt further to find a certain symmetric property between the two groups, $(\Sigma^{\pm,0}, \Lambda^0)$ and (P, N, Ξ^0, Ξ^-). Then all baryons could have the identical mass when only the π coupling were switched on.[11]

Now let us study the weak interactions responsible for decay processes in the hypothetical world (the primary decay interactions). Since there must occur a coupling between the i and i' spaces, the selection rule for decays contains both the i and i' spins of the system. Furthermore from the spherical symmetry of the i and i' spaces in this world, the selection rule must be expressed by only I and I' and cannot contain I_z and I_z'. Then we have

$$\Delta(nI+n'I')=0,\ 1,\ \cdots (n \text{ and } n' \text{ are non-zero integers}).$$

The compatibility with the charge conservation law (see Eq. (2)) chooses the following selection rule out of the above possibilites:

$$\Delta(I+I')=0. \tag{3}$$

This selection rule applied on the K, Λ, and Σ decays are essentially equal to the selection rule $\Delta I=1/2$:

i) K decay

We show the values of $I+I'$ and I_z+I_z' before and after the 2π decays of the various K particles in Table II.

K^+, K_2^0, and $\bar{K}^-$ make a triplet $(I+I'=1)$ and K_1^0 makes a singlet $(I+I'=0)$. The final two pion states do not have I' and their $I+I'$ are equal to I, the total i-spin of the system. Since pions obey the Bose statistics, the total i-spin is equal to either 2 or 0.

Table II. Values of $I+I'$ and I_z+I_z' in K decays

	$I+I'$	I_z+I_z'	$I+I'$	I_z+I_z'
K^+	1	$+1\to\pi^++\pi^0$	2	+1
K_1^0	0	$0\nearrow\pi^++\pi^-$ $\searrow\pi^0+\pi^0$	2, 0	0
K_2^0	1	$0\not\to 2\pi$		
$\bar{K}^-$	1	$-1\to\pi^-+\pi^0$	2	−1

Because of the selection rule $\Delta(I+I')=0$, only K_1^0 can decay into two pions. The other K's cannot decay into two pions as far as we neglect the K and electromagnetic interaction.* The branching ratio of K_1^0 decay, $(K_1^0\to 2\pi^0)/(K_1^0\to\pi^++\pi^-)$, is equal to $1/2$, because the two pions are in $T=0$ state. Thus the selection rule applied on the K decays is nothing but the $\Delta I=1/2$ selection rule when only the π coupling is switched on.

If we introduce the K interaction, the change of $I+I'$ can take some integers other than 0. The resultant selection rule would be

$$\begin{aligned}&\Delta(I+I')=0,\ 1,\ 2,\\ &\Delta(I_z+I_z')=0,\end{aligned}\tag{4}$$

because the K interaction has the axial symmetry around the z axis of i' space.

Thus $K^\pm$ would decay in general into two pions and the branching ratio of K_1^0 decay would deviate from $1/2$. If we rewrite the selection rule (4), it becomes

$$\begin{aligned}&\Delta I=1/2,\ 3/2,\ 5/2,\\ &\Delta I_z=\pm 1/2.\end{aligned}\tag{5}$$

The magnitude of the correction term with $\Delta I=3/2$ or $5/2$ depends on the actual form of the K coupling. And it could be of the order of 10% of the $\Delta I=1/2$ interaction, if the K coupling is large enough. Then the observed $K^\pm$ lifetime into two pions relative to the K_1^0 lifetime and the observed branching ratio of K_1^0 decay could be explained (§ 4).

Table III $I+I'$ and (I_z+I_z') in Λ decays

$I+I'$	I_z+I_z'	$I+I'$	I_z+I_z'
0	$0\nearrow n+\pi^0$ $\searrow p+\pi^-$	$\binom{1/2}{3/2}$	+1/2 0

ii) Λ decay

Table III shows the values of $I+I'$ and its z compenent before and after the Λ decay. The nucleon and pion system have the

* K_2^0 decay into two pions is strictly prohibited if the the theory is invariant under either time reversal or charge conjugation.

i-spin equal to either $1/2$ or $3/2$. Combining this with $I'(=1/2)$, $I+I'$ can be equal to either 0, 1, or 2. The selection rule $\Delta(I+I')=0$ leads to the nucleon and pion system with $I+I'=0$, which means the i-spin of the system must be $1/2$. Therefore we have in effect the selection rule $\Delta I=1/2$ and the branching ratio of Λ^0, $(\Lambda^0\to n+\pi^0)/(\Lambda^0\to p+\pi^-)$, is $1/2$.

The K interaction, in general, introduces an asymmetry in the i'-space and the selection rule changes into

$$\Delta(I+I')=0,\ 1,\ 2.,$$
$$\Delta(I_z+I'_z)=0.$$

This is equal to the following selection rule :

$$\Delta I=1/2,\ 3/2.,$$
$$\Delta I_z=\pm 1/2. \tag{6}$$

iii) Σ decay

The similar argument with that for the Λ decay can be made. The selection rule in the hypothetical world $(g_K^2/\hbar c=e^2/\hbar c=0)$ is essentially equal to $\Delta I=1/2$. Introduction of the K interaction produces corrections to this selection rule and we have

$$\Delta I=1/2,\ 3/2,\ 5/2,\ \text{for the } \Sigma \text{ decays.}$$
$$\Delta I_z=\pm 1/2 \tag{7}$$

The above application of the selection rule (3) on the Σ, Λ, and K decays show that we have the $\Delta I=1/2$ selection rule when we switch on only the π coupling. The K coupling makes the corrections to the original weak interaction, thus, in general introducing $\Delta I=3/2$ and $5/2$ decays of the particles. The magnitudes of these correction terms depend on the strength of the K coupling, the way of introducing an asymmetry into the i' space, and the type of primary decay interactions. Since the K coupling seems to be weaker than the π coupling and definitely stronger than the electromagnetic coupling, it will be possible for us to get the required magnitudes of the $\Delta I=3/2$ and $5/2$ decay interactions ($\sim 10\%$ of the $\Delta I=1/2$ interaction). It will be shown later that obtaining the $\Delta I=5/2$ decay is actually not easy (§ 3). If the primary decay interaction is of the Yukawa type, we have only the $\Delta I=1/2$ decay. If it is of the Fermi type, we can get both $\Delta I=1/2$ and $\Delta I=3/2$ decays but not $\Delta I=5/2$ decay. It will be required to have more complicated primary decay interactions in order to get the $\Delta I=5/2$ decay.

Let us now turn to the question of the mass spectrum of baryons. The nucleon and Ξ have the same mass if $g_K^2/\hbar c$ and $e^2/\hbar c=0$. The K interaction generally removes the degeneracy due to the i' spin multiplet and produces the observed mass difference betweeen Ξ and N. Since the K coupling connects $(\Lambda,\ \Sigma)$ with $(N,\ \Xi)$ and the way to couple them could be different for Σ and Λ, this K coupling can introduce also the mass difference between Σ and Λ. Thus the mass spectrum of baryons should depend also on the strength and the type of the K coupling. In § 4 we shall assume an explicit

form of the K interaction and discuss the selection rule for decays and the mass spectrum again.

Table IV. Conserved quantities and weak decay selection rules

	a)	b)	c)
conserved quantities	I, I_z, I', I_z'	I, I_z, I_z'	I_z, I_z'
selection rules in decay processes	$\Delta(I+I')=0$	$\Delta(I+I')=0,1,2$ $\Delta(I_z+I_z')=0$	$\Delta(I+I')=0,1,2$ $\Delta(I_z+I_z')=0$

In order to summarize the results obtained in this section (§ 2), we write down in Table IV the various conserved quantities and the weak decay selection rules in the three hypothetical worlds:

a) only the π coupling is switched on,

b) only the π and K couplings are switched on,

and c) all the couplings, π, K, and electro-magnetic, are present.

$$a) \quad g_\pi^2/\hbar c \neq 0,\ g_K^2/\hbar c = 0,\ e^2/\hbar c = 0,$$

$$b) \quad g_\pi^2/\hbar c \neq 0,\ g_K^2/\hbar c \neq 0,\ e^2/\hbar c = 0,$$

$$c) \quad g_\pi^2/\hbar c \neq 0,\ g_K^2/\hbar c \neq 0,\ e^2/\hbar c \neq 0.$$

Finally we mention here a possibility to interchange the role of the K coupling with that of the π coupling. Namely we take the π coupling as an agent for violating the spherical symmetry of the i'-space. The similar argument could be done for this case, but it is easy to see that the decay selection rules and the mass spectrum are not in agreement with this scheme.

§ 3. K interactions

Lorentz invariance including the invariance under time and space inversions, isotopic spin conservation, and locality condition restrict our choice of the interaction Hamiltonian between (N, Ξ) and (Λ, Σ) through the K meson field to the following form:

$$\begin{aligned} H_K = g_K \{ & \bar{\psi}_{N,\Xi} O (\alpha + \beta \tau_z') \psi_\Lambda \phi_{K,\bar{K}} \\ & + \bar{\psi}_{N,\Xi} O \tau (\alpha' + \beta' \tau_z') \boldsymbol{\psi}_\Sigma \phi_{K,\bar{K}} \} \\ & + \text{hermitian conjugate.} \end{aligned} \tag{8}$$

O is either 1 or $i\gamma_5$* depending on whether the relative intrinsic parity between (N, Ξ) and (Λ, Σ) is even (odd) or odd (even) when K is scalar (pseudo-scalar). If there is the super symmetry between Λ and Σ when $g_K^2/\hbar c = e^2/\hbar c = 0$, Λ and Σ must have the same intrinsic parity. Since Ξ is obtained from N by a rotation in the i'-space, Ξ and N also must have the same parity. Thus we have taken in eq. (8) the same O both for the Λ and Σ interactions.

Since we assume that H_K is invariant under the charge conjugation, α, β, α', and

* It is also possible to take the vector, pseudo-vector, or tensor coupling, if the interaction Hamiltonian contains the first derivative of ϕ. Since the arguments in this section does not depend much on the choice of O, we will leave this possibility aside.

β' are all real. Out of the five constants, α, β, α', β', and g_K, only four are independent. So we put $\alpha^2+\beta^2+\alpha'^2+\beta'^2$ equal to 1. Then g_K becomes to represent the strength of K coupling.

$\Psi_{N,\Xi}$ and $\Phi_{K,\bar{K}}$ are direct products of two spinors, one in the i-space and the other in the i'-space. We can explicitly write them as follows:

$$\Psi_{N,\Xi} \equiv \Psi_{\alpha,\beta}, \quad \begin{pmatrix} \Psi_{11} \\ \Psi_{21} \\ \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Psi_n \\ \Psi_p \\ \Psi_{\Xi^-} \\ \Psi_{\Xi^0} \end{pmatrix}, \tag{9}$$

and

$$\Phi_{K,\bar{K}} \equiv \Phi_{\alpha,\dot{\beta}}, \quad \begin{pmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{12} \\ \Phi_{22} \end{pmatrix} = \begin{pmatrix} \Phi_{K^0} \\ \Phi_{K^+} \\ \Phi_{K^-} \\ \Phi_{\bar{K}^0} \end{pmatrix}. \tag{10}$$

τ operates on the first sufix of $\psi_{\alpha,\dot{\beta}}$ and $\phi_{\alpha,\dot{\beta}}$ and τ' on the second suffix.

The β and β' terms in eq. (8) are those which introduce the asymmetry into the i'-space. Therefore if they are not zero, we have a mass difference between N and Ξ. Also the self-energies of Λ and Σ in general become different from each other. When $g_K^2/\hbar c < g_\pi^2/\hbar c$, we may be allowed to calculate their self-energies by perturbation. Although this has been already done by Gell-Mann,[11] we shall here write down the self-energies of strange particle thus calculated, for completeness sake.

$$\begin{aligned} \Delta E_N &\propto -m_K c^2 \cdot g_K^2/\hbar c \cdot \{(\alpha+\beta)^2+3(\alpha'+\beta')^2\}, \\ \Delta E_\Xi &\propto -m_K c^2 \cdot g_K^2/\hbar c \cdot \{(\alpha-\beta)^2+3(\alpha'-\beta')^2\}, \\ \Delta E_\Lambda &\propto -m_K c^2 \cdot g_K^2/\hbar c \cdot \{2(\alpha+\beta)^2+2(\alpha-\beta)^2\}, \\ \Delta E_\Xi &\propto -m_K c^2 \cdot g_K^2/\hbar c \cdot \{2(\alpha'+\beta')^2+2(\alpha'-\beta')^2\}. \end{aligned} \tag{11}$$

This result is independent of the choice of O (1 or $i\gamma_5$), although the proportionality constant is different for the two different cases.

Thus we obtain the relation

$$2(\Delta E_N+\Delta E_\Xi) = 3\Delta E_\Sigma + \Delta E_\Lambda, \tag{12}$$

which leads to the similar relation

$$2(m_N+m_\Xi) = 3m_\Sigma + m_\Lambda \tag{13}$$

if the unperturbed masses of baryons are all identical.

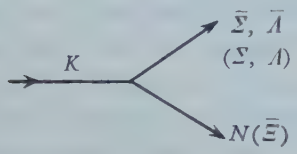
The relation (13), pointed out first by Gell-Mann, is nearly satisfied by the observed masses of baryons and the small deviation from eq. (13) would be explained by higher order terms in g_K^2.

Since the main parts of the observed mass differences among baryons should be determined by the K coupling, the three independent mass differences $m_\Xi - m_N$,

$m_\Sigma - m_\Lambda$, and $(m_N+m_\Xi)/2-(3m_\Sigma+m_\Lambda)/4$ can be in turn used to determine the parameters appeared in eq. (8). Thus in general we get three relations among the four independent parameters in H_K and two relations when the first order perturbational result (11) or (11′) is used. Because in this case $(m_N+m_\Xi)/2-(3m_\Sigma+m_\Lambda)/4$ vanishes irrespective of the values of the parameters.

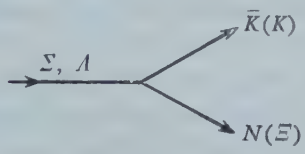
The magnitudes of decay interactions with various different isotopic spin changes depend also on the values of the parameters in H_K. Thus this together with the mass spectrum of baryons will over-determine the parameters in H_K and we can check the validity of our theory. Unfortunately, calculation of the decay interactions with $\Delta I=3/2$ and $5/2$ involves calculation of higher orders in g_K and we have to be satisfied only with a semi-quantitative study of the decay interactions at present.

It has been already pointed out that if there is no K coupling our selection rule (3) gives $\Delta I=1/2$ to the strange particle decays. There are three types of the K interaction as shown in Fig. 1, out of which only the second can change $I+I'$ by ± 1. Since the primary weak decay interaction as well as the π interaction do not change $I+I'$, only the virtual processes, Σ' and Λ going into (K, N) or (K, Ξ), are sources for changing $I+I'$. Therefore the selection rule can be different from $\Delta(I+I')=0$ or $\Delta I=1/2$, only when the decay proceeds at least once through the virtual process, Fig. 1_B.



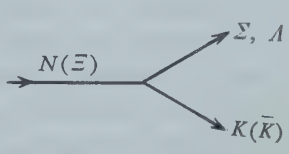
$\Delta(I+I')=0$

Fig. 1_A



$\Delta(I+I')=\pm 1,\ 0$

Fig. 1_B



$\Delta(I+I')=0$

Fig. 1_C

An actual decay process is a very complicated one and contains as a part of it a slow process due to the primary decay interaction, which is usually taken into consideration. The one is the Yukawa type interaction, giving directly the Σ' or Λ decay into a nucleon (or Ξ) and a pion. We include also the weak interaction K (or $\bar{K}$) going into two pions as belonging to this type. As we have seen before, these primary decay interactions must satisfy the $\Delta(I+I')=0$ selection rule, which equals to the $\Delta I=1/2$ selection rule in effect. Since the strong interactions do not change the i-spin of any system at all, the i-spin change of the actual K, Λ, and Σ' decays is equal to that of the primary decay interactions irrespective of how complicated they are. Therefore, if we had the primary decays of Yukawa type, we could not obtain decays with $\Delta I=3/2$ and $5/2$.

The other type of primary decay interactions is the Fermi type, which is biquadratic in Fermion fields. This gives, for example, Λ or Σ' going into N, N, $\bar{N}$. This combined with some complicated processes due to strong interactions could lead to the observed K, Λ, and Σ decays.

If we apply the $\Delta(I+I')=0$ selection rule to the primary Fermi type decay interactions, we obtain both $\Delta I=1/2$ and $3/2$. The reason is as follows: three $N(\bar{N})$ or $\Xi(\bar{\Xi})$ can make both $I'=1/2$ and $I'=3/2$ states and this combined with $\Delta(I+I')=0$

Table V

	$\Lambda(\Sigma) \rightarrow$	$N(\Xi)$	$+N(\Xi)$	$+\bar{N}(\bar{\Xi})$
I'	0	1/2	1/2	1/2

gives $\Delta I=1/2$ and $\Delta I=3/2$. (See Table V)

On the other hand, in the Yukawa type decays only one particle with $I'=1/2$, either $K(\bar{K})$, N, or Ξ, participates and so we have only $\Delta I'=1/2$. This was the reason why we had $\Delta I=1/2$ from $\Delta(I+I')=0$.

The primary decay interactions with $\Delta I_z=\pm 3/2$ do not lead to the actual decays, since the strong interactions conserve I_z as well as I of any system and cannot connect between the primary decay $\Delta I_z=\pm 3/2$ and the actual decay $\Delta I_z=\pm 1/2$.* If the K and electro-magnetic couplings are turned off, there should be no prefered axis of i and i' spins even for the decay processes. Therefore if the primary decay interaction with $\Delta I=3/2$ and $\Delta I_z=\pm 3/2$ does not lead to the actual decays, then it is also true with $\Delta I=3/2$ and $\Delta I_z=\pm 1/2$.

On the other hand when the K coupling is switched on, there appear differences between the intetactions with different I_z. Thus we can have the decays with $\Delta I=3/2$ and $\Delta I_z=\pm 1/2$ contrary to the absence of decays with $\Delta I=3/2$ and $\Delta I_z=\pm 3/2$. In the K^+ decay (Table II) $\Delta(I+I')$ must be at least equal to 1 and in the K_1^0 decay into 2π with $I=2$ it is equal to 2. Thus from the previous consideration on the K coupling (Fig. 1) there must be virtual processes, Σ or Λ going into $(\bar{K}, N)$ or $(K, \bar{\Xi})$, at least once for the former case and twice for the latter case. Furthermore, the above consideration on ΔI requires that the decay with $\Delta I=3/2$ must proceed through the primary Λ or Σ decay of the Fermi type. Thus during the decay process of the K virtual Λ or Σ must be emitted and reabsorbed (or decayed) at least twice in the K^+ and three times in the K_1^0 decay into 2π with $I=2$.

Then the matrix elements of the K^+ and K_1^0 decays into 2π with $I=2$ must be at least of the order of g_K^3 and g_K^5 respectively. In just the same way the matrix elements of the Λ and Σ decays with $\Delta I=3/2$ are at least of the order of g_K^2. Since these decays with $\Delta I=3/2$ involve higher order terms in g_k, any quantitative estimation of their magnitudes is very difficult. But the required magnitude of $\Delta I=3/2$ K decays ($\sim 10\%$ of that of $\Delta I=1/2$) could be obtained in this way since the K coupling is considered as much stronger than the electromagnetic coupling.

The decay with $\Delta I=5/2$ has not been obtained by our method. It will require the introduction of more complex primary decay interactions such as those containing sixth powers of Fermion fields. Since the necessity for requiring both $\Delta I=3/2$ and $\Delta I=5/2$ interactions depends not only on the fairly confirmed ratio of τ_{K^+} to $\tau_{K_1^0}$ but also on the branching ratio of K_1^0 decay, it is highly desirable to have more precise measurement on the latter before we make any modification of the primary decay intera-

* I_z' is equal to one half of the sum of the strangeness S and the nucleon number N. The decay with $\Delta I_z'=\pm 3/2$ ($\Delta I_z=\mp 3/2$) means $\Delta S=\pm 3$. Although there is no reason to prohibit weak interactions giving $\Delta S=\pm 3$, there is no such decay energetically possible. For instance, the weak interaction $\Lambda^0 \rightarrow N+N+\bar{\Xi}^0$ can be present virtually and gives $\Delta I_z'=3/2$, but the real such decay of Λ^0 is energetically prohibited.

ctions. If it turns out that the $\Delta I=5/2$ interaction is much smaller than the $\Delta I=3/2$ interaction, this may be taken as an indication that the violation of the selection rule $\Delta I=1/2$ is really due to the reason considered here and not principally due to the electromagnetic interaction. The latter will give rise to a $\Delta I=5/2$ interaction comparable with $\Delta I=3/2$ interaction.

§ 4. Discussions

Throughout the course of this work we have assumed that the isotopic spin is conserved in strong interactions and to each strange particle is assigned its own isotopic spin given by the N-G theory. Then the observation of the fairly large mixture of $\Delta I=3/2$ or $5/2$ decay interaction ($\sim 10\%$ of the $\Delta I=1/2$ decay interaction) has forced us to the introduction of the asymmetric i'-space in a rather unique way.

If the asymmetry is due to the K interaction, the switching off of the K and electromagnetic interactions leads us to have the complete symmetry under rotations in the i, i', and Lorentz spaces. There the selection rule of strange particle decays should be $\Delta I=1/2$ and the masses of baryons could be taken among each other. At least N and Ξ and very probably Σ and Λ would have the same mass.

The K interaction introduces the mass difference among baryons and violates the selection rule $\Delta I=1/2$ in the decays. Inversely the observed mass spectrum of baryons and the observed isotopic spin change in strange particle decays can determine the explicit form of the K interaction. Although only semi-quantitative arguments have been done in § 3, it seems probable that the K interaction (eq. (8)) could explain the experimental data.

On the other hand if more quantitative discussions reveal any discrepancy between the theory and the experiments, some of our assumptions should be checked. First, the isotopic spin conservation in strong interactions and the isotopic spin assignment to strange particles by the N-G theory may not be correct. These will be checked experimentally in near future.

Secondly the independence of i and i'-spaces from the ordinary spacetime should be examined. According to the present concept of the i and i'-space the entire world is a direct product of the i space, i'-space, and the ordinary space-time. If we rotate the coordinate system of the i or i' space at a space-time point $(\boldsymbol{x}, t)$, we have also to rotate those at every other space-time points by the same amount.

The opposite extreme is to allow arbitrary and independent rotations of the i or i' coordinate at $(\boldsymbol{x}, t)$ from those at other points including infinitesimally distant points from $(\boldsymbol{x}, t)$ and to require the invariance of the physical law under such rotations. Then we have to introduce a vector field with i (or i')-spin equal to 1.[8] There is no such field so far known experimentally.

If the i or i'-spin of a particle at a space-time point $(\boldsymbol{x}, t)$ should determine the coordinate system of the i or i'-space at that point, an independent rotation of i and i' spaces at any two ordinary space-time points distant, say, by less than the size of the particle may not be allowed. This interesting possibility, which is an intermediate case

between the above two extremes, is outside our scope in this paper but should be considered if necessary.

Thirdly the question of parity non-conservation should be taken into our theory. This can be done in different ways. For example, the primary decay interactions can be made parity violating. The other possibility may be to regard K, Σ, and Λ as particles not having a definite parity.[12] Since we cannot measure the intrinsic parity by strong interactions,[13] there we need only the conservation of parity attached with the orbital motion of anticipating particles. The K interaction (eq. (8)) satisfies this requirement, if we take Ψ_{Λ}, Ψ_{Σ}, and $\Phi_{K,\bar{K}}$ as field operators with no definite parity. Therefore we can incorporate the parity non-conservation of decay processes into our theory.

We should mention here that the estimation on the magnitudes of $\Delta I=3/2$ and $5/2$ interactions depends on whether the parity is conserved or not in the decay processes.[3),14)] If it is not conserved, our estimation will turn out to be wrong for Σ and Λ decays. But the estimation for K decays is independent of this problem and, therefore, our theory will not be altered in its essential parts.

Fourthly we have neglected the presence of any strange particle other than observed. If a strange particle with double charge is found, it might mean the presence of a particle multiplet with $I'=3/2$ or $I=3/2$. This makes easier for us to get the $\Delta I=3/2$ and $\Delta I=5/2$ decay interactions.

Also we have not considered the effect of the light particles such as μ, e, and ν upon the strange particle decays. There has been neither experimental nor theoretical reason to believe the effect being appreciable.

Finally let us turn to the question of the electromagnetic effect on the ΔI in strange particle decays. Although we have assumed that this effect is very small, this point should be studied more carefully. If the possible parity non-conservation is due to a certain property of the very inner part of strange particles, there we might also expect unusually large electromangetic effects on decay interactions. We shall investigate this possibility in some other place.

Acknowledgements

After completing our work we received a manuscript by Espagnat, Prentki, and Salam,[15] in which the same decay selection rule with ours $\Delta(I+I')=0$, is discussed. The points where emphases are made are somewhat different from ours.

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Long-Range Spin Interaction and the Meissner-Ochsenfeld Effect

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It has been suggested,[1] from a perturbation calculation, that an electron gas exhibits the Meissner-Ochsenfeld effect if a special type of spin-dependent interaction between electrons, $ge^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2/r$, is assumed. We show that in fact such a system exhibits only a modified paramagnetism.

§ 1. Introduction

Recently, Z. Mikura has suggested[1] that an electron gas, under the influence of a spin-dependent electron-electron interaction,

$$\mathcal{H}_i = ge^2 \frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}, \tag{1}$$

exhibits a Meissner-Ochsenfeld (M—O) effect. Using the apparatus of Dirac four-vector wave-functions for the electrons, and treating the interaction (1) to first order, Mikura derives the London equation. The coefficient differs in form from London's original one, but is numerically of the same order, if the parameter g is of order unity.

It may be seen that the M—O effect under discussion arises from the electron spin term in the Hamiltonian, that is to say, it is an effect due to alignment of electron spins by a spin-dependent interaction.

However, it is difficult to see how spin alignment could possibly lead to a M—O effect; such a "paramagnetic"—type alignment does not have the long range order property[2] which is required for a M—O. Consequently one is led to suspect that the M—O effect under discussion is spurious, and due to neglect of higher order terms in the perturbation treatment of the interaction. This paper verifies that suspicion.

In section 2, a perturbation treatment of the current density, taking into account electron spin terms in the Hamiltonian and the current density, but not employing relativistic wave-functions, is used to rederive Mikura's result and to extend it to next higher order.

In section 3, we give a "molecular field" treatment of the problem. Such a treatment is justified by the fact that the interaction (1) is long-range, and thus the dominant

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contribution to the interaction energy at any point comes from far distant electrons; neighbouring contributions are outweighed by distant ones. The result obtained shows that there is in fact no M—O effect, but instead a modified Pauli paramagnetism. The result also shows how the terms derived in section 2 occur in a perturbation treatment.

§ 2. Perturbation expansion

We first recall the usual criterion[3] for the occurrence of a M—O effect: The current density, $\boldsymbol{i}$, in a weak inhomogeneous magnetic field, with vector potential $\boldsymbol{A}$, may be expressed in wave-number space as

$$i_\mu(\boldsymbol{q}) = \sum_{\nu=1}^{3} K_{\mu\nu}(q)\, A_\nu(\boldsymbol{q}). \tag{2·1}$$

Using translational and gauge invarience, we may write, in an isotropic medium

$$K_{\mu\nu}(\boldsymbol{q}) = (q_\mu q_\nu - \delta_{\mu\nu} q^2)\, K(q^2). \tag{2·2}$$

The criterion for a (perfect) M—O effect is that $K(q^2)$ should have a first order pole at the origin; that is

$$K(q^2) = K_0/q^2 + K_r(q^2) \tag{2·3}$$

where K_0 is a positive constant, and $K_r(q^2)$ is a regular function of q.

Mikura obtains such a singularity, with

$$K_0 = \frac{ge^4}{\pi\hbar^2 c}\left(\frac{3n}{\pi}\right)^{2/3} \tag{2·4}$$

with n, the number density of electrons.

We wish to investigate the second and higher order terms in the perturbation expansion of the current density in powers of the interaction (1). To this end we firstly rederive the first order term, using a simplified technique which avoids the complications of Dirac wave-functions and negative energy states.

If we enclose our system in a cubical box, volume V, with periodic boundary conditions, the appropriate eigenfunctions, in second quantized form, are

$$\left.\begin{aligned} \psi_q(\boldsymbol{x}) &= V^{-1/2} e^{iq\cdot x} a_q \\ \psi_q^*(\boldsymbol{x}) &= V^{-1/2} e^{-iq\cdot x} a_q^* \end{aligned}\right\}. \tag{2·5}$$

a_q and a_q^* are the usual absorption and emission operators obeying:

$$\left.\begin{aligned} [a_q^*,\ a_{q'}] &= \delta_{qq'} \\ a_q^* a_q &= n(\boldsymbol{q}) \end{aligned}\right\}. \tag{2·6}$$

The matrix elements of the Hamiltonian, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}' + \mathcal{H}_i + \mathcal{H}_s$, where $\mathcal{H}_s$ is the electron spin term, are

$$
\begin{aligned}
(\boldsymbol{p}|\mathcal{H}_0|\boldsymbol{p}') &= \frac{\hbar^2 p^2}{2m}\,\delta(\boldsymbol{p}-\boldsymbol{p}')\,n(\boldsymbol{p}) \\
(\boldsymbol{p}|\mathcal{H}'|\boldsymbol{p}') &= -\frac{e\hbar}{2mc}\,(\boldsymbol{p}+\boldsymbol{p}')\cdot\boldsymbol{A}(\boldsymbol{p}-\boldsymbol{p}')\,a_p{}^* a_{p'} \\
(\boldsymbol{p}|\mathcal{H}_s|\boldsymbol{p}') &= -\frac{ie\hbar}{2mc}\,\sigma\cdot(\boldsymbol{p}-\boldsymbol{p}')\times\boldsymbol{A}(\boldsymbol{p}'-\boldsymbol{p})\,a_p{}^* a_{p'} \\
(\boldsymbol{p}_1\boldsymbol{p}_2|\mathcal{H}_i|\boldsymbol{p}_1'\boldsymbol{p}_2') &= \frac{2\pi e^2 g}{V}\,\frac{\delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_1'-\boldsymbol{p}_2')\,\sigma_1\cdot\sigma_2}{(\boldsymbol{p}_1-\boldsymbol{p}_1')^2}\,a_{p_1}{}^* a_{p_1'} a_{p_2}{}^* a_{p_2'}
\end{aligned}
\qquad (2\cdot 7)
$$

Corresponding terms for the current density elements may be readily written down. In particular, for the spin current density we have ($\mu_0 = e\hbar/2mc$ being the Bohr magneton)

$$
\begin{aligned}
(\boldsymbol{p}|\boldsymbol{j}_s|\boldsymbol{p}') &= c\mu_0\,\mathrm{curl}.\{\psi_p{}^*(\boldsymbol{x})\,\sigma\psi_{p'}(\boldsymbol{x})\} \\
&= \frac{ie\hbar}{2mV}\,\sigma\times(\boldsymbol{p}-\boldsymbol{p}')\,e^{i(p'-p)\cdot x}\,a_p{}^* a_{p'}\,.
\end{aligned}
$$

For the orbital current densities

$$
\begin{aligned}
(\boldsymbol{p}|\boldsymbol{j}_0|\boldsymbol{p}') &= \frac{e\hbar}{2mV}\,(\boldsymbol{p}+\boldsymbol{p}')\,e^{i(p'-p)\cdot x}\,a_p{}^* a_{p'} \\
(\boldsymbol{p}|\boldsymbol{j}'|\boldsymbol{p}') &= -\frac{e^2}{2mV}\,\boldsymbol{A}(\boldsymbol{x})\,e^{i(p'-p)\cdot x}\,a_p{}^* a_{p'}\,.
\end{aligned}
\qquad (2\cdot 8)
$$

The mean current density in thermal equilibrium is given by statistical mechanics as

$$
\boldsymbol{i}(\boldsymbol{x}) = \mathrm{Trace}\,\{\boldsymbol{j}(\boldsymbol{x})\,\mathcal{F}_0(\mathcal{H})\} \qquad (2\cdot 9)
$$

where $\mathcal{F}_0(E)$ is the distribution function for the grand canonical ensemble,

$$
\mathcal{F}_0(E) = \exp\{\beta(\Omega_0+\zeta N-E)\}\,. \qquad (2\cdot 10)
$$

Ω_0 is the thermodynamical potential for the unperturbed system, and ζ the chemical potential.

Equation (2·9) may be expanded in powers of the magnetic field ($\boldsymbol{A}$) and the interaction (g) by a technique precisely analogous to that employed in expanding it without spin terms.[4] We expand only to first order in $\boldsymbol{A}$, and (for the moment) to first order in g. Moreover we know that

(i) in the absence of an interaction, $\mathcal{H}_i$, the current density vanishes, and

(ii) if we include $\mathcal{H}_i$, but neglect electron spin terms, the current density again consists only of terms regular at $q=0$.[4] Therefore we consider only terms in the expansion which contain one or both of $\mathcal{H}_s$ and $\boldsymbol{j}_s$, together with $\mathcal{H}_i$.

Then we get

$$\boldsymbol{i}'(\boldsymbol{x}) = \sum_{1,1',2,2'} (12|\mathcal{H}_s|1'2') \left\{ \begin{array}{l} (1'|\mathcal{H}_s|1)\,(2'|\boldsymbol{j}_s|2) + (2'|\mathcal{H}_s|2)\,(1'|\boldsymbol{j}_s|1) \\ + (1'|\mathcal{H}_s|1)\,(2'|\boldsymbol{j}_0|2) + (2'|\mathcal{H}_s|2)\,(1'|\boldsymbol{j}_0|1) \\ + (1'|\mathcal{H}'|1)\,(2'|\boldsymbol{j}_s|2) + (2'|\mathcal{H}'|2)\,(1'|\boldsymbol{j}_s|1) \end{array} \right\}$$
$$\times F_1(E_1,\ E_{1'}) F_1(E_2,\ E_{2'}) \qquad (2\cdot11)$$

where $\mathcal{H}_s$ conserves momentum as it scatters a pair of electrons 1, 2 with wave-numbers $\boldsymbol{p}_1-\boldsymbol{q}/2$, $\boldsymbol{p}_2+\boldsymbol{q}/2$, into the pair 1′, 2′ with wave-numbers $\boldsymbol{p}_1+\boldsymbol{q}/2$, $\boldsymbol{p}_2-\boldsymbol{q}/2$. The elements $(1|\mathcal{H}'|1')$, etc., are the matrix elements given by (2·7) and (2·8), with the creation and absorption operators a, a^* eliminated. Also we have defined

$$F_1(E,\ E') = \frac{F_0(E) - F_0(E')}{E - E'} \qquad (2\cdot12)$$

with $F_0(E)$ the Fermi distribution function

$$F_0(E) = \{e^{\beta(E-\zeta)}+1\}^{-1}. \qquad (2\cdot13)$$

Substituting from (2·7) and (2·8) into (2·11), one has

$$\boldsymbol{i}'(\boldsymbol{x}) = -\frac{g\pi e^4\hbar^2}{m^2 c V^2} \sum_{\sigma_1\sigma_2} \sum_{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{q}} \left(\frac{1}{q^2} - \frac{1}{(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}\right) \left\{ \begin{array}{l} \sigma_1\cdot\boldsymbol{q}\times\boldsymbol{A}(\boldsymbol{q})\,\sigma_2\times\boldsymbol{q} \\ -2i\boldsymbol{p}_1\cdot\boldsymbol{A}(\boldsymbol{q})\,\sigma_2\times\boldsymbol{q} \\ -2i\sigma_1\cdot\boldsymbol{q}\times\boldsymbol{A}(\boldsymbol{q})\,\boldsymbol{p}_2 \end{array} \right\}$$
$$\times \sigma_1\cdot\sigma_2\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} F_1(E_1,\ E_{1'}) F_1(E_2,\ E_{2'}). \qquad (2\cdot14)$$

The second two terms in the bracket vanish on summing over spins; also the second term in the expression for the interaction, the "exchange term", obviously does not yield the required singularity. In addition, one may show that for the Fermi distribution

$$\frac{1}{V}\sum_{\boldsymbol{p}} F_1\left\{ \frac{\hbar^2}{2m}\ (\boldsymbol{p}+\boldsymbol{q}/2)^2,\ \frac{\hbar^2}{2m}\ (\boldsymbol{p}-\boldsymbol{q}/2)^2 \right\} = -\frac{m}{2\pi\hbar^2}\left(\frac{3n}{\pi}\right)^{1/3} + \mathcal{O}(q^2). \qquad (2\cdot15)$$

Then (2·13) is simplified to

$$\boldsymbol{i}'(\boldsymbol{x}) = -\frac{ge^2}{4\pi\hbar^2 c}\left(\frac{3n}{\pi}\right)^{2/3} \sum_{\sigma_1\sigma_2\boldsymbol{q}} \frac{1}{q^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \sigma_1\cdot\sigma_2\, \sigma_1\cdot\boldsymbol{q}\times\boldsymbol{A}(\boldsymbol{q})\ \sigma_2\times\boldsymbol{q} + \mathcal{O}(q^2\boldsymbol{A}(\boldsymbol{q})). \qquad (2\cdot16)$$

The Pauli spin matrices, σ, possess the property that

$$(\sigma\cdot\boldsymbol{A})\,(\sigma\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + i\sigma\cdot\boldsymbol{A}\times\boldsymbol{B}. \qquad (2\cdot17)$$

So we finally get Mikura's result,

$$i_\mu'(\boldsymbol{q}) = \frac{ge^4}{\pi\hbar^2 c}\left(\frac{3n}{\pi}\right)^{2/3} \sum_{\nu=1}^{3} \frac{(q_\mu q_\nu - \delta_{\mu\nu}q^2)}{q^2} A_\nu(\boldsymbol{q}). \qquad (2\cdot18)$$

This perturbation expansion may now be extended to second order in the interaction parameter g, by following along the lines indicated above for the first order calculation.

After a rather involved calculation, one obtains for the contribution with the highest singularity at $q=0$

$$i_\mu''(\boldsymbol{q}) = -\frac{4g^2e^6m}{\pi\hbar^4c}\left(\frac{3n}{\pi}\right)\sum_{\nu=1}^{3}\frac{(q_\mu q_\nu-\delta_{\mu\nu}q^2)}{q^4}A_\nu(\boldsymbol{q}). \tag{2·19}$$

It would thus appear that the perturbation expansion diverges for small q; this point is clarified by the "molecular field" approach.

§ 3. "Molecular Field" approach

We may rewrite equations (2·1) and (2·2) in terms of the magnetic field $\boldsymbol{H}$ ($\boldsymbol{H}=\operatorname{curl}\boldsymbol{A}$), and the magnetization, $\boldsymbol{M}$, which is related to the current density by

$$\boldsymbol{i}=c\operatorname{curl}\boldsymbol{M}. \tag{3·1}$$

Then one has the equation

$$c\boldsymbol{M}(\boldsymbol{q}) = -K(q^2)\boldsymbol{H}(\boldsymbol{q}). \tag{3·2}$$

In the absence of a spin-dependent interaction, the Fermi electron gas in a magnetic field exhibits the well-known Pauli paramagnetism.[5] The magnetization is given by

$$\boldsymbol{M}=\frac{3n\mu_0^2}{2\zeta}\boldsymbol{H}. \tag{3·3}$$

However, this result is modified if we include the spin-dependent interaction (1)

$$\mathcal{H}_s=ge^2\frac{\sigma_1\cdot\sigma_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{3·4}$$

The total interaction energy due to a given electron, spin σ_1 (which we take to be at the origin) is then the quantum-mechanical sum over all the individual interactions $\sigma_1\cdot\sigma_2(\boldsymbol{r})/r$, where $\sigma(\boldsymbol{r})$ represents the spin of an electron at the point $\boldsymbol{r}$. We approximate this total interaction energy as being the product of σ_1 and some average "field", which we call the "molecular field" in analogy with that employed in the Weiss theory of ferromagnetism.

This treatment involves replacing the quantum-mechanical average $\langle\sigma_1\cdot\sigma_2(\boldsymbol{r})/r\rangle$, by $\langle\sigma_1\rangle\cdot\langle\sigma_2(\boldsymbol{r})\rangle/r$. The approximation thus means treating σ_1 and $\sigma_2(\boldsymbol{r})$ as uncorrelated: we expect this to be the case when they are far apart (i. e. r large). Hence the "molecular field" approximation is justified if the dominating contribution to the "field" at the origin comes from far distant electrons; that is, from the region where σ_1 and σ_2 are essentially uncorrelated.

The "field", σ_2/r, at the origin due to an electron at distance r decreases like r^{-1}; but the total number of electrons in a spherical shell of thickness Δr contributing to the "field" is proportional to $r^2\,\Delta r$. Thus the net effect at the origin of the field due to electrons at distance r increases as $r\Delta r$— the dominating contribution indeed comes from far distant electrons, where σ_1 and σ_2 are uncorrelated.

We note that $\boldsymbol{M}(\boldsymbol{r})=\mu_0 n(\boldsymbol{r})\langle\sigma_z(\boldsymbol{r})\rangle$, where $n(\boldsymbol{r})$ is the number density of electrons at $\boldsymbol{r}$. Then the total field acting on a spin at a point $\boldsymbol{x}$ is

$$\mu_0 \boldsymbol{H}_{\text{eff.}}(\boldsymbol{x})=\mu_0 \boldsymbol{H}(\boldsymbol{x})-\sum_{\boldsymbol{x}'}\frac{ge^2\langle\sigma(\boldsymbol{x}')\rangle}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{3·5}$$

$$=\mu_0 \boldsymbol{H}(\boldsymbol{x})-\frac{ge^2}{\mu_0}\int\frac{d\boldsymbol{x}'\,\boldsymbol{M}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{3·6}$$

where the integral term is the "molecular field" produced by (3·4).

After a Fourier transform, this reads

$$\boldsymbol{H}_{\text{eff.}}(\boldsymbol{q})=\boldsymbol{H}(\boldsymbol{q})-\frac{4\pi ge^2}{\mu_0^2}\frac{1}{q^2}\boldsymbol{M}(\boldsymbol{q}). \tag{3·7}$$

Instead of (3·3), one now has to write

$$\boldsymbol{M}=\frac{3n\mu_0^2}{2\zeta}\boldsymbol{H}_{\text{eff.}} \tag{3·3^1}$$

from which one gets

$$\boldsymbol{M}(\boldsymbol{q})=\frac{3n\mu_0^2/2\zeta}{1+6\pi ge^2 n/\zeta\cdot 1/q^2}\boldsymbol{H}(\boldsymbol{q}). \tag{3·8}$$

Or, comparing with (3·2)

$$\boxed{K(q^2)=\frac{-3n\mu_0^2 c/2\zeta}{1+6\pi ge^2 n/\zeta\cdot 1/q^2}}. \tag{3·9}$$

The kernel is perfectly regular. As $q\to 0$, $K(q^2)\to 0$: there is no singularity and consequently no M—O effect.

An expansion in powers of g (i. e. a perturbation expansion in the interaction) can be seen from (3·9) to be in fact an expansion in powers of g/q^2. As such it is invalid in the limit $q\to 0$.

Making this expansion,

$$K(q^2)=-\frac{3n\mu_0^2 c}{\zeta}+\frac{9\pi n^2 e^2\mu_0^2 c}{\zeta^2}\frac{g}{q^2}-\frac{54\pi^2 n^3 e^4\mu_0^2 c}{\zeta^3}\frac{g^2}{q^4}+\cdots . \tag{3·10}$$

The coefficients of the $1/q^2$ and $1/q^4$ terms (K_0 and K_1 respectively) may be rewritten

$$\begin{aligned} K_0&=\frac{ge^4}{\pi\hbar^2 c}\left(\frac{3n}{\pi}\right)^{2/3} \\ K_1&=-\frac{4g^2 e^6 m}{\pi\hbar^4 c}\left(\frac{3n}{\pi}\right). \end{aligned} \tag{3·11}$$

Comparing these to equations (2·4) and (2·19), we see that these terms correspond exactly to the first and second order terms of the perturbation series considered in section

2. Mikura's result (the $1/q^2$ term) thus arises as the first term of a series which diverges in the limit $q \to 0$.

We conclude that a spin-dependent interaction (1) does not create a M—O effect in the electron gas.

Our thanks are due to Professor H. Messel for his constant interest and encouragement. One of us (R.M.M.) is also indebted to General Motors Holden for a research fellowship.

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Cluster Theory of Condensing Systems*

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In this paper we develop a detailed and rigorous theory of vapour-liquid coexistence in the condensation of a system consisting of a very large number (N) of interacting molecules (in a volume Nv). We start from the Ursell-Mayer expansion of the partition function and make certain ordinary assumptions about the cluster integrals (b_l). First we take a function $l^*(N)$ which satisfies certain conditions, and we define "small" clusters [for which $l \leqq l^*(N)$] and "large" clusters [for which $l > l^*(N)$]; thus all clusters in the system (for each N) are classified into "small" and "large" clusters, and the partition function (Ω_N) of the system (for each N) is expressed in terms of the factors (φ_N) which contain "small" clusters only and the factors (Φ_N) which contain "large" clusters only. Then in Theorem I it is proved that an assembly of molecules divided into any number of "large" clusters may be replaced by only one "larger" cluster composed of all the molecules of the assembly with errors as small as we please if N is sufficiently large. Then in Theorem II it is proved that as soon as the density of the system exceeds that of the saturated vapour, the isotherm becomes horizontal and there appears one "huge" cluster, which is defined as a cluster for which l/N is as near to some positive constant as we please if N is sufficiently large. Thus the "huge" cluster is of a macroscopic size and should be considered to represent the liquid phase coexisting with the saturated vapour.

In Appendix A, we show that Mayer's theory of condensation has some defects, and we propose an improved approximate theory in which these defects are avoided and which is substantially similar to the rigorous theory given in the text of this paper.

In Appendix B, we compare the present theory with the theory of Bose-Einstein condensation, and remark that the present theory is quite free from the faults contained in the saddle-point method.

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§ 7. Concluding remarks.

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* A summary of this paper was published in 1954 (reference 1).

The contents of Appendix A of this paper were read at the 21st regular meeting of the Kyusyu branch of the Physical Society of Japan held at Kyusyu University on March 15, 1952, and were published in 1952 (reference 2) in Japanese. The contents of the text of this paper were read at the 9th annual meeting of the Physical Society of Japan held at Osaka University on November 1, 1954, and were published in 1955 and 1957 (reference 3) in Japanese.

Appendix A. *Approximate theory of condensing systems.*
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Appendix B. *Comparison with other theories of condensation.*
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Introduction

In this paper, on the basis of orthodox statistical mechanics and by means of a rigorous mathematical method, we discuss in detail a condensing system, especially in the state of vapour-liquid coexistence (i. e. in the horizontal part of an isotherm).

We consider a macroscopic system (or thermodynamic system) composed of N interacting molecules in a volume V and at a temperature T, and we start from the Ursell-Mayer expansion[4),5),6),7)] of the partition function into "cluster integrals". As to the definition and the physical interpretation of a "cluster (of molecules)" we follow Mayer,[5)] Kahn and Uhlenbeck,[6)] and others.[7)]

We regard a macroscopic system (for which N is of the order of 10^{23} and V is of a macroscopic size) as an infinitely large system (for which $N\to\infty$ and $V\to\infty$ with $v=V/N$ fixed), and treat our problem from the point of view of mathematical analysis and by means of rigorously defined mathematical concepts; the equalities and inequalities and words, such as "every", "any", "almost every (any)", and "infinite", appearing in this paper† should be read in the pure mathematical sense.††

In this paper, assuming that the cluster integrals are volume-independent and positive, and making a few other ordinary assumptions about the cluster integrals, we have obtained the following results.

(i) By a purely mathematical method we have defined the concepts "small", "large", and "huge" for the sizes of clusters:—"small" is defined as "smaller than or equal to $l^*(N)$" and "large" is defined as "larger than $l^*(N)$" [where $l^*(N)$ is a function which satisfies certain conditions] and "huge" is defined as "of the order of magnitude of N"; "huge" implies "large", the converse being not true.

(ii) We have proved that the situation where a set of molecules are grouped into an arbitrary number of "large" clusters is (if N is very large) equivalent to the situation where the set of molecules together composes only one "large" cluster; thus in a thermodynamic system one cannot decide how many "large" clusters are present, and so

† Except the appendices.
†† Cf. the theory of sets.

one may for convenience consider that only one "large" cluster is present. [Theorem I.]

(iii) For $v \geqq v_s$ (where $v[=V/N]$ is the specific volume of the system and v_s is the v for the saturated vapour) we have proved that in thermodynamic equilibrium there is no "large" cluster. [Theorem III.]

(iv) For $v < v_s$ we have proved that in thermodynamic equilibrium a "huge" cluster appears and coexists with the saturated vapour part which consists of "small" clusters only. According to the physical interpretation of a cluster, the "huge" cluster should be regarded as representing the liquid phase which has appeared in the condensation process; thus we have explained by a rigorous mathematical method, the vapour-liquid coexistence in the condensation range. [Theorem II]

(v) We have proved that the isotherm is horizontal for $v < v_s$, i. e. in the condensation range. [Corol. 1 of Theorem II.]

Because of the above-mentioned results (i)—(v), especially (ii) and (iv), we have made an advance over Mayer's[5] and Kahn-Uhlenbeck's[6] theories of condensation. Mayer's mathematical methods used for his theory of condensation, that is, the maximum-term method and the saddle-point method, are valid only for $v > v_s$ (i. e. up to the saturated vapour density i. e. up to the condensation point) but are invalid for $v < v_s$ (i. e. in the range of vapour-liquid coexistence i. e. in the condensation range) [cf. §§ A1 & A4 of Appendix A and § B2 of Appendix B], and so Mayer's theory does not mathematically rigorously prove for $v < v_s$ the horizontality of the isotherm or the appearance of the huge or very large cluster representing the liquid phase. Kahn-Uhlenbeck's theory of condensation (though it is rigorous in mathematical treatment) does not contain such an analysis of the sizes of clusters as given in the present paper, and so is incapable of deriving the huge or very large cluster representing the liquid phase.

While the present paper, as well as the theories of Mayer, Kahn & Uhlenbeck, and others, have treated a system with volume-independent cluster integrals,* the author's previous paper[8] has treated a system with volume-dependent cluster integrals.** In the previous paper,[8] however, we have discussed, using a rigorous mathematical method, the behaviour of the system up to the condensation point and the analytical properties of the condensation point itself, but have not gone into the problem of vapour-liquid coexistence (for $v < v_s$) which is discussed in the present paper. We wish in future to attempt to discuss this problem for a system with volume-dependent cluster integrals.

§ 0. Partition function and the cluster integrals

Let us consider a system composed of N identical molecules (particles) [each of mass m] in a vessel of volume V, and at a temperature T. Then the partition function Q_N of the system is given by

$$Q_N = (1/\lambda^{3N})\, \Omega_N, \qquad (0 \cdot 1)$$

* In the previous paper[8] such a system was called the "(0)-system" or "ideal system".

** In the previous paper[8] such a system was called the "real system".

where $$\lambda = h/\sqrt{2\pi mkT} \tag{0·2}$$

(k and h being Boltzmann's and Planck's constants respectively) and

$$\Omega_N = (1/N!) \int_V \cdots \int_V W_N\{N\}\, d\{N\}, \tag{0·3}\dagger$$

where

$$W_N\{N\} = N!\, \lambda^{3N} \sum_n \varphi_n^*\{N\}\, e^{-E_n/kT} \varphi_n\{N\}, \tag{0·4}$$

the φ_n's ($n=1, 2, \cdots$) being the normalized energy-eigenfunctions of the system, and E_n being the eigenvalue for the n-th φ_n. Then the Helmholtz free energy A of the system is given by

$$A = -kT \ln Q_N = NkT \ln \lambda^3 - kT \ln \Omega_N, \tag{0·5}$$

from which all the other thermodynamic properties can be derived; in particular, the equation of state can be obtained:

$$p = -\partial A/\partial V = kT \partial \ln \Omega_N/\partial V, \tag{0·6}$$

p being the pressure of the system.

If we assume the validity of classical theory, then (0·4) may be replaced by

$$W_N\{N\} = \exp(-U_N\{N\}/kT), \tag{0·4'}$$

$U_N\{N\}$ being the total potential energy of the system.

Now we have for Ω_N the Ursell-Mayer expansion[4),5),6),7)]

$$\Omega_N = \sum_{m_l \geqq 0} \Big(\sum_{l=1}^{N} l m_l = N\Big) \prod_{l=1}^{N} \frac{(V b_l)^{m_l}}{m_l!}, \tag{0·7}$$

where the sum extends over all possible sets $\{m_l\}_{l=1}^{N}$ of non-negative integers m_l which satisfy the condition stated in the parentheses, and where b_l is the "cluster integral" with respect to l molecules ($l=1, 2, \cdots$), defined by

$$b_l = (1/l!\, V) \int_V \cdots \int_V S_l\{l\}\, d\{l\}, \tag{0·8}\dagger\dagger$$

$S_l\{l\}$ being defined by

$$S_l\{l\} = \sum \{\kappa\{\nu_i\}_l\}_{unc} (-)^{\kappa-1} (\kappa-1)! \prod_{i=1}^{\kappa} W_{\nu}\{\nu_i\} \tag{0·9}*$$

where the symbol $\sum\{\kappa\{\nu_i\}_l\}_{unc}$ indicates summation over all possible sets $\{\{\nu_1\}, \{\nu_2\}, \cdots, \{\nu_\kappa\}\}$ (for various values of κ) of unconnected** sets $\{\nu_1\}, \{\nu_2\}, \cdots, \{\nu_\kappa\}$ whose

† $\{N\}$ denotes the set of the coordinates of the N particles, that is, $\{N\} \equiv (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$; and $d\{N\} \equiv d\tau_1\, d\tau_2 \cdots d\tau_N$.

†† Especially $b_1 \equiv 1$.

* Especially $S_1\{1\} \equiv W_1\{1\} \equiv 1$.

** If $\{E\}$ is a set of sets, the sets of the set $\{E\}$ are *unconnected* if no element belongs to more than one of the sets of the set $\{E\}$.

union is the set $\{l\}$. Thus each b_l depends on the volume (V) and shape of the vessel and the temperature (T) as well as on the intermolecular forces. The group of molecules which are concerned in a cluster integral (b_l) is called a "cluster" (of l molecules).

If we assume (i) the validity of classical theory, (0·4′), and (ii) the additivity of the intermolecular forces, e. i.

$$U_N\{N\}=\sum_{N\geqq i>}\sum_{j\geqq 1} u(r_{ij}), \tag{0·10}$$

$u(r_{ij})$ being the pair potential, then we have in place of (0·9)

$$S_l\{l\}=\sum_{l\geqq i>}\prod_{j\geqq 1} f_{ij} \tag{0·9′}$$

(sum over all products consistent with single *cluster*) [cf. ref. 5]

where

$$f_{ij}=\exp\{-u(r_{ij})/kT\}-1. \tag{0·11}$$

However, in this paper (except § A5 and the latter half of § A6 of Appendix A), both these assumptions are unnecessary.

In this paper we make the following assumptions about the cluster integrals. [Mayer, Kahn & Uhlenbeck, and others have made almost the same set of assumptions as given here.]

(a) The b_l's are independent of the volume and shape of the vessel. That is, we treat a system whose Ω_N is

$$\Omega_N^{(0)}=\sum_{m_l\geqq 0}\Big(\sum_{l=1}^{N} l m_l=N\Big)\prod_{l=1}^{N}\frac{(Vb_l^{(0)})^{m_l}}{m_l!}, \tag{0·12}$$

where the $b_l{}^{(0)}$'s are the volume-independent cluster integrals* [cf. ref. 8] defined by

$$b_l^{(0)}=b_l^{(0)}(T)=(1/l!)\int_{\infty}\cdots\int_{\infty} S_l\{l\}\,d\{l-1\}. \tag{0·13}$$

The l-th particle being fixed at a point, the integration, with respect to each of the other $l-1$ particles, is extended over the whole space (with no boundary).

Or

$$b_l^{(0)}=b_l^{(0)}(T)=\lim_{V\to\infty} b_l(V,T). \tag{0·14}$$

[Here we assume that the range of the intermolecular forces is sufficiently short, so that $S_l\{l\}$ tends sufficiently rapidly to zero for spatial separation of the molecules $\{l\}$, so that $b_l^{(0)}(T)$ is definite and finite.] That is, in this paper we treat the "(0)-system" i. e. the "ideal system" —to use the terms defined in the previous paper[9]—but we shall drop the superscript "(0)" on Ω_N and b_l in the following sections.

(b) $b_l^{(0)}(T)>0$ for every l.

This is considered to hold for all sufficiently low temperatures, i. e., for all the temperatures lower than some temperature which may be a little lower than the critical temperature.

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* In ref. 8, the $b_l{}^{(0)}$'s were called the non-boundary cluster integrals.

The assumption (b) is essential for the physical interpretation of a "cluster" as a group of molecules which are really linked together and so are close to each other in space.*

(c) $\lim_{l\to\infty}\{b_l^{(0)}(T)\}^{1/l}=b_0(T)$ exists and $0<b_0(T)<+\infty$.

Hence $1/b_0(T)\,[\equiv z_s(T)$ say$]$ is the radius of convergence R of the power series $\sum_{l=0}^{\infty} l b_l^{(0)}(T)z^l$. [For, by the Cauchy-Hadamard theorem, $1/R=\limsup_{l\to\infty}|b_l^{(0)}(T)|^{1/l}$.] Hence [with the assumption (b)] $z_s(T)$ is the first singularity (on the positive real axis of z) of the analytic function defined by that power series. The assumption (c) means that the contribution per particle to an infinitely large cluster integral should be definite, finite, and positive; especially the assumption that $0<b_0(T)<+\infty$ is necessary in order that the condensation may occur at a finite and positive fugacity [cf. § 5].

(d) The series $\sum_{l=1}^{\infty} l b_l^{(0)}(T)\{z_s(T)\}^l$ is convergent. This assumption is necessary in order that the condensation may occur at a finite density (i. e. at a v larger than zero) [cf. § 5].

Every argument in this paper is given for the "(0)-system" and for any fixed temperature for which (b), (c) and (d) hold.

Starting from the arguments in this section, we shall develop our theory in the following sections.

§ 1. Preliminary mathematical argument

Lemma 1. *Let $\lambda(N)$ be a function of N $(=1, 2, 3, \cdots)$ such that:*

1° *$\lambda(N)$ assumes a positive integral value for every N,*

2° *$\lambda(N)\leqq N$ for almost** every N,* (1·1)

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* As far as we are concerned with *configurations of molecules* in a thermodynamic system, we may regard a "cluster" (which has originally been defined mathematically) as a physical reality, that is, a group of linked molecules. For *only* the configurations in which a group of molecules are close together make contributions to a cluster integral (which then makes a contribution to the partition function); this means that only such configurations are physically realized (in equilibrium) from the statistical thermodynamical point of view. This is the basis on which we regard the appearance of a "huge" cluster (in § 5) as the two-phase separation, which is a state of configuration of molecules. [Of course, the physical image of a "cluster" is different in some points from a *real* group of linked molecules; for example, in calculating the gas pressure, we must consider that a "cluster" behaves as if it were a molecule of a perfect gas, i. e., as if it had neither its own volume nor interactions with other clusters. But the above physical interpretation related to configurations of molecules is nevertheless true.]

However, if there are negative cluster integrals, the above interpretation may be irrelevant, since for such cluster integrals the configurations in which a group of molecules are close together will (on an average) make negative contributions to the partition function, which may mean that such configurations are more unlikely to be realized than in the absence of intermolecular forces. This corresponds to the case that, in the function f_{ij} in the integrand of a cluster integral [eq. (0·11)], the contribution of the negative part (i. e. the effect of repulsion) is large compared with the contribution of the positive part (i. e. the effect of attraction). This may occur at relatively high temperatures (probably, near or higher than the critical temperature).

** By "almost every N" we mean "every value of N except at most a finite number of values of N", i. e., "every value of N larger than some value (N_0 say) of N", i. e., "all sufficiently large values of N". Similarly for "almost any N".

$$3^\circ \quad \lim_{N\to+\infty} \lambda(N) = +\infty. \tag{1·2}$$

[*It is obvious that there exist such functions* $\lambda(N)$.] *And, let* $\{\nu, m_{\lambda(N)}, m_{\lambda(N)+1}, \cdots, m_\nu\}_N$ *be a set of integers* $\nu, m_{\lambda(N)}, m_{\lambda(N)+1}, \cdots, m_\nu$ *(for a given* N*) such that:*

$$\text{(i)} \quad \lambda(N) \leqq \nu \leqq N, \tag{1·3}$$

$$\text{(ii)} \quad m_l \geqq 0 \quad \textit{and} \quad \sum_{l=\lambda(N)}^{\nu} l m_l = \nu. \tag{1·4}$$

[*By* (1·1) *there exist such sets for almost any* N.] *Then, given any* $\epsilon > 0$, *we have the following five relations* (I)–(V) *for all sets* $\{\nu, m_{\lambda(N)}, m_{\lambda(N)+1}, \cdots, m_\nu\}_N$, *for almost every* N.

$$\text{(I)} \quad \left|(1/N)\sum_{l=\lambda(N)}^{\nu} \eta(l)\, m_l\right| < \epsilon \tag{1·5}$$

where $\eta(l)$ *is any function of* $l(=1, 2, 3, \cdots)$ *such that*

$$\lim_{l\to\infty} \{\eta(l)/l\} = 0, \tag{1·6}$$

$$\text{(II)} \quad 0 < (1/N)\sum_{l=\lambda(N)}^{\nu} m_l < \epsilon, \tag{1·7}$$

$$\text{(III)} \quad 0 \leqq (1/N)\sum_{l=\lambda(N)}^{\nu}{}' \ln m_l < \epsilon, \tag{1·8}$$

$$\text{(IV)} \quad 0 < (1/N)\sum_{l=\lambda(N)}^{\nu}{}' 1 < \epsilon, \tag{1·9}$$

$$\text{(V)} \quad 0 \leqq (1/N)\sum_{l=\lambda(N)}^{\nu}{}' m_l \ln (N/m_l) < \epsilon, \tag{1·10}$$

where the prime of the summation symbol $\sum'$ *means that in the summation the terms for which* $m_l = 0$ *are excluded.**

[*Proof of* (I)] From (1·6) we see that, for any given $\epsilon > 0$, there exists a positive integer $M(\epsilon)$ such that

$$|\eta(l)|/l < \epsilon \quad (\text{whenever } l \geqq M(\epsilon)). \tag{1·11}$$

By (1·2) we have for almost every N

$$\lambda(N) \geqq M(\epsilon). \tag{1·12}$$

By (1·11) and (1·12), we have for almost every N

$$\left|(1/N)\sum_{l=\lambda(N)}^{\nu} \eta(l)\, m_l\right| \leqq (1/N)\sum_{l=\lambda(N)}^{\nu} |\eta(l)|\, m_l < (1/N)\sum_{l=\lambda(N)}^{\nu} \epsilon l m_l = (\nu/N)\,\epsilon \leqq \epsilon \tag{1·13}$$

for all sets $\{\nu, m_{\lambda(N)}, m_{\lambda(N)+1}, \cdots, m_\nu\}_N$. Here we have used (1·4) and (1·3).

[*Proof of* (II)] In (I) we put $\eta(l) = 1$, since this satisfies (1·6).

[*Proof of* (III)] Since

$$0 \leqq \ln m_l < m_l \quad (m_l \geqq 1), \tag{1·14}$$

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* Throughout the present paper, $\sum'$ has the same meaning that is given here, unless otherwise stated.

we have by (II)

$$0 \leqq (1/N) \sum_{l=\lambda(N)}^{\nu}{}' \ln m_l < (1/N) \sum_{l=\lambda(N)}^{\nu} m_l < \epsilon. \tag{1·15}$$

[*Proof of* (IV)] By (II)

$$0 < (1/N) \sum_{l=\lambda(N)}^{\nu}{}' 1 \leqq (1/N) \sum_{l=\lambda(N)}^{\nu} m_l < \epsilon. \tag{1·16}$$

[*Proof of* (V)] For almost every N we have for all sets $\{\nu, m_{\lambda(N)}, m_{\lambda(N)+1}, \cdots, m_\nu\}_N$

$$\begin{aligned} 0 &\leqq \sum_{l=\lambda(N)}^{\nu}{}' m_l \ln (N/m_l) \leqq \max_{\{m_l\}_{\lambda(N)}^{N}} \Big(\sum_{l=\lambda(N)}^{N} l m_l = N \Big) \Big\{ \sum_{l=\lambda(N)}^{N}{}' m_l \ln (N/m_l) \Big\} \\ &\leqq \max_{\{x_l\}_{\lambda(N)}^{N}} \Big(\sum_{l=\lambda(N)}^{N} l x_l = N \Big) \Big\{ \sum_{l=\lambda(N)}^{N} x_l \ln (N/x_l) \Big\}. \end{aligned} \tag{1·17}$$

Here $\max_{x}(\cdots) Y$ means the maximum of Y with respect to X subject to the condition stated in the parentheses. And $\{m_l\}_p^q$ means a set $\{m_p, m_{p+1}, \cdots, m_q\}$ of non-negative integers m_l's, and $\{x_l\}_p^q$ means a set $\{x_p, x_{p+1}, \cdots, x_q\}$ of positive real numbers x_l's. We shall now calculate the last maximum in (1·17). With the use of the undetermined multiplier $\ln \zeta$, the maximizing set $\{x_l\}_{\lambda(N)}^{N}$ must satisfy

$$(\partial/\partial x_l) \Big\{ \sum_{l=\lambda(N)}^{N} x_l \ln (N/x_l) + \Big(\sum_{l=\lambda(N)}^{N} l x_l \Big) \ln \zeta \Big\} = 0, \tag{1·18}$$

that is,

$$x_l = N\zeta^l/e \quad (l=\lambda(N), \lambda(N)+1, \cdots, N), \tag{1·19}$$

where ζ is determined by

$$\sum_{l=\lambda(N)}^{N} l\zeta^l/e = 1. \tag{1·20}$$

Hence the required maximum is

$$\sum_{l=\lambda(N)}^{N} N\zeta^l/e + N \ln (1/\zeta). \tag{1·21}$$

Substituting this in (1·17), we have

$$0 \leqq (1/N) \sum_{l=\lambda(N)}^{\nu}{}' m_l \ln (N/m_l) \leqq \sum_{l=\lambda(N)}^{N} \zeta^l/e + \ln (1/\zeta). \tag{1·22}$$

Now by (1·2), for any given $\epsilon > 0$ we have for almost every N

$$\lambda(N) > 2/\epsilon. \tag{1·23}$$

Hence for almost every N

$$0 < \sum_{l=\lambda(N)}^{N} \zeta^l/e < \sum_{l=\lambda(N)}^{N} (\epsilon/2) l\zeta^l/e = \epsilon/2. \tag{1·24}$$

Here we have used (1·20).

Now, for any given $\epsilon > 0$, the infinite series

$$\sum_{l=1}^{\infty} l e^{-(\epsilon/2)l} \tag{1·25}$$

is convergent. Hence by (1·2) we have for almost every N

$$\sum_{l=\lambda(N)}^{N} l e^{-(\epsilon/2)l}/e < \sum_{l=\lambda(N)}^{\infty} l e^{-(\epsilon/2)l}/e < 1. \tag{1·26}$$

Hence by (1·20) we have for almost every N

$$\sum_{l=\lambda(N)}^{N} l\zeta^l/e > \sum_{l=\lambda(N)}^{N} l e^{-(\epsilon/2)l}/e. \tag{1·27}$$

Hence we have for almost every N

$$\zeta > e^{-(\epsilon/2)}, \tag{1·28}$$

since, if N is fixed, $\sum_{l=\lambda(N)}^{N} l\zeta^l/e$ is a monotonic increasing function of ζ. On the other hand we have for almost every N

$$\sum_{l=\lambda(N)}^{N} l/e > 1 \left[= \sum_{l=\lambda(N)}^{N} l\zeta^l/e\right]. \tag{1·29}$$

Hence for almost every N

$$\zeta < 1. \tag{1·30}$$

By (1·28) and (1·30) we have for almost every N

$$0 < \ln(1/\zeta) < \epsilon/2. \tag{1·31}$$

From (1·22), (1·24), and (1·31), we obtain (1·10). Thus the proof is completed.

§ 2. Discrimination between "large" and "small" clusters

Let us now put $V=Nv$, so that v (>0) is the volume per molecule, i. e. the specific volume. Then eq. (0·7) becomes

$$\Omega_N = \sum_{m_l \geqq 0} \left(\sum_{l=1}^{N} l m_l = N\right) \prod_{l=1}^{N} \{(Nvb_l)^{m_l}/m_l!\}. \tag{2·1}$$

Then we shall consider *an infinite sequence* (for $N=1, 2, \cdots$ to ∞) of Ω_N (and consequently, of systems defined by Ω_N) for any fixed value of v (>0) (of course, for a given temperature and for given intermolecular forces).

Now by the assumptions (b) and (c) for b_l in § 0, we have

$$b_l = f(l)\, b_0^{\,l} \ , \ f(l) > 0 \ , \ b_0 > 0 \quad (l=1, 2, \cdots), \tag{2·2}$$

where

$$\lim_{l\to\infty} (1/l) \ln f(l) = 0. \tag{2·2a}$$

Now it is very easily proved that we can find a function $l^*(N)$ which satisfies the following three conditions :†

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† The conditions for $l^*(N)$ in the present paper are not so exacting as those for $l^*(N)$ in the previous paper[8] (cf. Lemma 4 of ref. 8), since in the previous paper the volume dependence of the cluster integrals were discussed. Every function $l^*(N)$ which satisfies the conditions (1°–5°) stated in Lemma 4 of ref. 8, satisfies the conditions (1°–3°) stated here; but the converse is not true.

1° $l^*(N)$ assumes a positive integral value for every N.

2° $\lim_{N\to\infty}\{l^*(N)/N^{1/2}\}=0.$ (2·3)

3° There is a positive constant K (independent of N) such that

$$(\ln N)/l^*(N)\leqq K \quad \text{for every } N. \tag{2·4}$$

In general there exist many such functions $l^*(N)$. But we take *any one* of these $l^*(N)$'s, and denote it by $l^*(N)$. We shall use this $l^*(N)$ to discriminate between "large" and "small" clusters [large $l>l^*(N)$, small $l\leqq l^*(N)$]. In a word, $l^*(N)$ is the border-line between "large" and "small".

Then by (2·3) we have for almost every N

$$l^*(N)<N. \tag{2·3a}$$

Consequently, by using $l^*(N)$, we may give the following expressions to Ω_N for almost every N:

$$\Omega_N=\sum_{\boldsymbol{i}[N]}\varphi_N(\boldsymbol{i}[N])+\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N]) \tag{2·5}$$

$$=\sum_{\boldsymbol{i}[N]}\varphi_N(\boldsymbol{i}[N])+\sum_{N_2=l^*(N)+1}^{N}\sum_{\boldsymbol{i}[N-N_2]}\varphi_N(\boldsymbol{i}[N-N_2])\sum_{\boldsymbol{k}[N_2]}\Psi_N(\boldsymbol{k}[N_2]). \tag{2·6}$$

Here we have defined

$$\chi_N(\boldsymbol{j}[N])\equiv\prod_{l=1}^{N}\{(Nvb_l)^{m_l}/m_l!\}, \tag{2·7}$$

where $\boldsymbol{j}[N]$ represents a set $\{m_1, m_2, \cdots, m_N\}$ of non-negative integers m_l such that

1° $\sum_{l=1}^{N} l m_l=N$ (2·7a)

2° $m_l\geqq 1$ for at least one l such that $N\geqq l\geqq l^*(N)+1$;

and the m_l's given on the right-hand side of (2·7) are the set of m_l represented by $\boldsymbol{j}[N]$ on the left-hand side. The symbol $\sum_{\boldsymbol{j}[N]}$ indicates summation over all such sets $\boldsymbol{j}[N]$ for a given N. And we have defined

$$\varphi_N(\boldsymbol{i}[M])\equiv\prod_{l=1}^{l^*(N)}\{(Nvb_l)^{m_l}/m_l!\}, \tag{2·8}$$

where $\boldsymbol{i}[M]$ represents a set $\{m_1, m_2, \cdots, m_{l^*(N)}\}$ of non-negative integers m_l such that

$$\sum_{l=1}^{l^*(N)} l m_l=M \tag{2·8a}$$

(where M is a non-negative integer); and the m_l's given on the right-hand side of (2·8) are the set of m_l represented by $\boldsymbol{i}[M]$ on the left-hand side. The symbol $\sum_{\boldsymbol{i}[M]}$ indicates summation over all such sets $\boldsymbol{i}[M]$ for a given M. [In (2·6) we have put $M=N$ and $M=N-N_2$.] Especially for $M=0$ we have

$$\varphi_N(\boldsymbol{i}[0])\equiv 1, \quad \boldsymbol{i}[0]\equiv\{0, 0, \cdots, 0\}. \tag{2·9}$$

We have also defined

$$\Phi_N(\boldsymbol{k}[N_2]) \equiv \prod_{l=l^*(N)+1}^{N} \{(Nvb_l)^{m_l}/m_l!\}, \tag{2·10}$$

where $\boldsymbol{k}[N_2]$ represents a set $\{m_{l^*(N)+1}, m_{l^*(N)+2}, \cdots, m_N\}$ of non-negative integers m_l such that

$$\sum_{l=l^*(N)+1}^{N} lm_l = N_2 \tag{2·10a}$$

(where N_2 is an integer such that $l^*(N)+1 \leqq N_2 \leqq N$ or $N_2=0$) ; and the m_l's on the right-hand side of (2·10) are the set of m_l represented by $\boldsymbol{k}[N_2]$ on the left-hand side. The symbol $\sum_{\boldsymbol{k}[N_2]}$ indicates summation over all such sets $\boldsymbol{k}[N_2]$ for a given N_2. Especially for $N_2=0$ we have

$$\Phi_N(\boldsymbol{k}[0]) \equiv 1, \quad \boldsymbol{k}[0] \equiv \{0, 0, \cdots, 0\}. \tag{2·11}$$

[*Remark*] In the previous paper[8] [cf. eq. (12) on page 355 of reference 8*c*] we gave Ω_N the expression (2·5), in which the terms (φ_N) containing no "large" cluster have been collected in the first part and the terms (χ_N) containing at least one "large" cluster have been collected in the second part, and we showed (in Lemma 6 of the previous paper[8]) that, if $v > v_s$, the second part is negligible in calculating the thermodynamic properties. In the present paper, however, we go into the range $v \leqq v_s$, and so the second part is not necessarily negligible, but demands a more detailed analysis, and so we express the second part as a sum of products of a sum of terms (φ_N) containing "small" clusters only and a sum of terms (ψ_N) containing "large" clusters only, as is shown in (2·6).

An expression which is substantially similar to (2·6) is given in § A3 of Appendix A [see (A3·2)], although, there, for discriminating between "large" and "small" clusters, the rigorous method [by means of the border-line function $l^*(N)$], used here and in the previous paper,[8] is not used.

§ 3. The "large" clusters

Theorem I.† *For any given* $\epsilon > 0$, *we have for almost every* N

$$|(1/N) \ln \Phi_N(\boldsymbol{k}[N_2]) - (N_2/N) \ln b_0| < \epsilon \tag{3·1}$$

for every $\boldsymbol{k}[N_2]$ *for every* N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$.

[*Proof*] According to Stirling's theorem, we have

$$\ln m_l! = m_l(\ln m_l - 1) + \tfrac{1}{2} \ln m_l + \gamma(m_l) \tag{3·2}$$

for every positive integer m_l, where $\gamma(m_l)$ is bounded, that is, there exists a positive absolute constant K_1 such that

† The thermodynamic properties are determined by the limit of $(1/N) \ln \Omega_N$ (when N is very large), while we have [by (4·1)] $(1/N) \ln \Omega_N \simeq (1/N) \max \ln \tau_N(N_2, \boldsymbol{i}[N-N_2])$ (when N is very large) and we have [by (4·2)] $(1/N) \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) = (1/N) \ln \varphi_N(\boldsymbol{i}[N-N_2]) + (1/N) \ln \sum \Phi_N(\boldsymbol{k}[N_2])$. Therefore $(1/N) \ln \sum \Phi_N(\boldsymbol{k}[N_2])$ plays a part in determining the thermodynamic properties. This is the reason why we investigate $(1/N) \ln \sum \Phi_N(\boldsymbol{k}[N_2])$ in this theorem.

$$|\gamma(m_l)| \leqq K_1 \quad \text{for every } m_l. \tag{3·3}$$

Now from (2·10) we have by (2·2) and (3·2)

$$\ln \Phi_N(\boldsymbol{k}[N_2]) = \sum_{l=l^*(N)+1}^{N_2}{}' \{l m_l \ln b_0 + m_l \ln f(l) + m_l(1+\ln v) - \tfrac{1}{2}\ln m_l - \gamma(m_l) + m_l \ln(N/m_l)\}. \tag{3·4}$$

Now from (2·4) we have

$$\lim_{N\to\infty} l^*(N) = +\infty. \tag{3·5}$$

Hence, in Lemma 1 we can put $\lambda(N) = l^*(N)+1$ and $\nu = N_2$, since (2·3a), (3·5), and (2·10a) correspond to (1·1), (1·2), and (1·4) respectively. And, in (I) of Lemma 1 we can put $\eta(l) = \ln f(l)$, since (2·2a) corresponds to (1·6). Consequently, by (I), (II), (III), (IV), and (V) of Lemma 1, for any given $\epsilon > 0$ we have the following relations for every $\boldsymbol{k}[N_2]$ for every $N_2 (l^*(N)+1 < N_2 \leqq N)$, for almost every N.

$$\begin{aligned}
&\text{(I)} && |(1/N) \sum_{l=l^*(N)+1}^{N_2}{}' m_l \ln f(l)| < \epsilon/5 \\
&\text{(II)}^{\dagger} && 0 < (1/N) \sum_{l=l^*(N)+1}^{N_2}{}' m_l < \epsilon/(5|1+\ln v|) \\
&\text{(III)} && 0 \leqq (1/N) \sum_{l=l^*(N)+1}^{N_2}{}' \ln m_l < 2\epsilon/5 \\
&\text{(IV)} && 0 < (1/N) \sum_{l=l^*(N)+1}^{N_2}{}' 1 < \epsilon/(5K_1) \\
&\text{(V)} && 0 \leqq (1/N) \sum_{l=l^*(N)+1}^{N_2}{}' m_l \ln(N/m_l) < \epsilon/5.
\end{aligned} \tag{3·6}$$

By (2·10a), (3·3), and (3·6), we obtain (3·1) from (3·4). Thus the proof is completed.

Corollary 1. *For any given $\epsilon > 0$, we have for almost every N*

$$|(1/N)\ln \Phi_N(\boldsymbol{k}[N_2]) - (1/N)\ln \Phi_N(\boldsymbol{k}'[N_2])| < \epsilon \tag{3·7}$$

for every $\boldsymbol{k}[N_2]$ and $\boldsymbol{k}'[N_2]$ for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$.

[*Proof*] In (3·1), ϵ is an arbitrary positive number, and so we may take $\epsilon/2$ in place of ϵ. Hence

$$\begin{aligned}
|(1/N)\ln \Phi_N(\boldsymbol{k}[N_2]) - (1/N)\ln \Phi_N(\boldsymbol{k}'[N_2])| &\leqq |(1/N)\ln \Phi_N(\boldsymbol{k}[N_2]) - (N_2/N)\ln b_0| \\
&\quad + |(1/N)\ln \Phi_N(\boldsymbol{k}'[N_2]) - (N_2/N)\ln b_0| < \epsilon/2 + \epsilon/2.
\end{aligned} \tag{3·8}$$

[*Note*] Here $\boldsymbol{k}'[N_2]$ means any set which satisfies the same conditions as $\boldsymbol{k}[N_2]$ satisfies. ($\boldsymbol{k}[N_2]$ and $\boldsymbol{k}'[N_2]$ may be identical or not.)

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† If $\ln v = -1$, then (II) is unnecessary.

Corollary 2. *For any given* $\epsilon>0$, *we have for almost every* N

$$|(1/N)\ln\Phi_N(\boldsymbol{k}[N_2])-(1/N)\ln b_{N_2}|<\epsilon \tag{3·9}$$

for every $\boldsymbol{k}[N_2]$ *for every* N_2 $(l^*(N)+1\leqq N_2\leqq N)$.

[*Proof*] By (2·2) we have

$$(1/N)\ln b_{N_2}-(N_2/N)\ln b_0=(N_2/N)(1/N_2)\ln f(N_2). \tag{3·10}$$

Given any $\epsilon>0$, we have by (2·2a)

$$|(1/N_2)\ln f(N_2)|<\epsilon \tag{3·11}$$

for almost every N_2; hence by (3·10), we have for almost every N

$$|(1/N)\ln b_{N_2}-(N_2/N)\ln b_0|<\epsilon \tag{3·12}^{\dagger}$$

for every[††] $N_2(l^*(N)+1\leqq N_2\leqq N)$. From (3·1) [in which ϵ is replaced by $\epsilon/2$] and (3·12) [in which ϵ is replaced by $\epsilon/2$], we obtain (3·9). Thus the proof is completed.

[*Other proof*] In Corollary 1, take for $\boldsymbol{k}'[N_2]$ the set of m_l for which $m_l=0$ (for $l\neq N_2$) and $m_{N_2}=1$, and replace ϵ by $\epsilon/2$, and use the fact that $|(1/N)\ln(Nv)|<\epsilon/2$ for almost every N.

Corollary 3. *For any given* $\epsilon>0$, *we have for almost every* N

$$|(1/N)\ln\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])-(N_2/N)\ln b_0|<\epsilon \tag{3·13}$$

for every N_2 $(l^*(N)+1\leqq N_2\leqq N)$.

[*Proof*] Since $b_l>0$ for every l [assumption (b) in § 0], we have

$$\Phi_N(\boldsymbol{k}[N_2])>0 \tag{3·14}$$

for every $\boldsymbol{k}[N_2]$. Hence, if we let $q(N, N_2)$ be the number of terms in the sum $\sum_{\boldsymbol{k}[N_2]}$, we have

$$\Phi_N(\bar{\boldsymbol{k}}[N_2])\leqq\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])\leqq q(N, N_2)\,\Phi_N(\bar{\boldsymbol{k}}[N_2]) \tag{3·15}$$

where $\bar{\boldsymbol{k}}[N_2]$ means the $\boldsymbol{k}[N_2]$ for which Φ_N is largest; that is, $\Phi_N(\bar{\boldsymbol{k}}[N_2])=\max_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])$. Now let P_N be the partitio numerorum of N. Then we have

$$q(N,\ N_2)<P_N \tag{3·16}$$

for every N and N_2. And we have asymptotically

$$\ln P_N\sim\pi(2N/3)^{1/2} \tag{3·17}$$

as $N\to\infty$. Consequently, for any given $\epsilon>0$, we have for almost every N

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† If, in Theorem I, we take for $\boldsymbol{k}[N_2]$ the set of m_l for which $m_l=0$ (for $l\neq N_2$) and $m_{N_2}=1$, and replace ϵ by $\epsilon/2$, and use the fact that $|(1/N)\ln(Nv)|<\epsilon/2$ for almost every N, then we obtain (3·12) directly.

†† If a condition holds for almost every N_2, then it holds for every N_2 (such that $l^*(N)+1\leqq N_2\leqq N$) for almost every N, since $l^*(N)\to\infty$ (as $N\to\infty$) by (3·5).

$$0 \leqq (1/N) \ln q(N, N_2) < \epsilon/2 \tag{3·18}$$

for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$. By (3·15) and (3·18), for any given $\epsilon > 0$ we have for almost every N

$$(1/N) \ln \sum_{\boldsymbol{k}[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - (1/N) \ln \Phi_N(\bar{\boldsymbol{k}}[N_2]) | < \epsilon/2 \tag{3·19}$$

for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$.

Now put $\boldsymbol{k}[N_2] = \bar{\boldsymbol{k}}[N_2]$ in Theorem I. Then, for any given $\epsilon > 0$, we have for almost every N

$$| (1/N) \ln \Phi_N(\bar{\boldsymbol{k}}[N_2]) - (N_2/N) \ln b_0 | < \epsilon/2 \tag{3·20}$$

for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$. From (3·19) and (3·20) we obtain (3·13). Thus the proof is completed.

Corollary 4. *For any given $\epsilon > 0$, we have for almost every N*

$$| (1/N) \ln \sum_{\boldsymbol{k}[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - (1/N) \ln \Phi_N(\boldsymbol{k}'[N_2]) | < \epsilon \tag{3·21}$$

for every $\boldsymbol{k}'[N_2]$ for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$.

[*Proof*] Put $\boldsymbol{k}[N_2] = \bar{\boldsymbol{k}}[N_2]$ in Corollary 1. Then, for any given $\epsilon > 0$, we have for almost every N

$$| (1/N) \ln \Phi_N(\bar{\boldsymbol{k}}[N_2]) - (1/N) \ln \Phi_N(\boldsymbol{k}'[N_2]) | < \epsilon/2 \tag{3·22}$$

for every $\boldsymbol{k}'[N_2]$ for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$. From (3·19) and (3·22) we obtain (3·21).

Corollary 5. *For any given $\epsilon > 0$, we have for almost every N*

$$| (1/N) \ln \sum_{\boldsymbol{k}[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - (1/N) \ln b_{N_2} | < \epsilon \tag{3·23}$$

for every N_2 $(l^*(N)+1 \leqq N_2 \leqq N)$.

[*Proof*] From (3·13) [in which ϵ is replaced by $\epsilon/2$] and (3·12) [in which ϵ is replaced by $\epsilon/2$], we obtain (3·23).

[*Other proof*] In Corollary 4, take for $\boldsymbol{k}'[N_2]$ the set of m_l for which $m_l = 0$ (for $l \neq N_2$) and $m_{N_2} = 1$, and replace ϵ by $\epsilon/2$, and use the fact that $| (1/N) \ln (N\nu) | < \epsilon/2$ for almost every N.

[*The physical meaning of Theorem I and its corollaries*]

Theorem I shows that, if we take N large enough, the difference between $(1/N) \cdot \ln \Phi_N(\boldsymbol{k}[N_2])$ and $(N_2/N) \ln b_0$ becomes as small as we please, irrespective of the total number (N_2) of particles composing "large" clusters, and irrespective of the way $(\boldsymbol{k}[N_2])$ of grouping the N_2 particles into "large" clusters. Corollary 1 shows that, if we take N large enough, the change of $(1/N) \ln \Phi_N(\boldsymbol{k}[N_2])$ caused by an arbitrary change $(\boldsymbol{k}[N_2] \to \boldsymbol{k}'[N_2])$ of the way $(\boldsymbol{k}[N_2])$ of grouping the N_2 particles into "large" clusters, becomes as small as we please, irrespective of the total number (N_2) of particles composing "large" clusters. Corollary 2 results if we replace the general term $\Phi_N(\boldsymbol{k}'[N_2])$ in Corollary 1 by the particular term for which $m_l = 0$ (for $l \neq N_2$) and $m_{N_2} = 1$,

that is, the term for which the N_2 particles together compose one "large" cluster and there are no other "large" clusters. Corollaries 3, 4, and 5 show that, in Theorem I, Corollary 1, and Corollary 2, we may replace the general term $\Phi_N(\boldsymbol{k}[N_2])$ by the sum of $\Phi_N(\boldsymbol{k}[N_2])$ over $\boldsymbol{k}[N_2]$ for any fixed N and N_2 if N is large enough. In particular, Corollary 4 shows that the operation $\sum_{\boldsymbol{k}[N_2]}$ upon $\Phi_N(\boldsymbol{k}[N_2])$ for any fixed N and N_2 is practically of no effect if N is large enough.

Thus, Theorem I and its corollaries show that the results for the system under consideration are practically independent of the way of grouping the N_2 particles into "large" clusters for any fixed N and N_2 if N is large enough, and so we may, most conveniently, adopt the particular way of grouping in which the N_2 particles together compose only one "large" cluster $(\ln b_{N_2})$.

This is shown also in § A3 of Appendix A by an argument which is not so rigorous as the present argument. Theorem I and Corollaries 1 and 2 correspond to the equations (A3·3) and (A3·4) in Appendix A; and Corollaries 3, 4, and 5 correspond to (A3·7) in Appendix A.

§ 4. Maximum term and the Δ-system

Lemma 2. *For any given* $\epsilon>0$, *we have for almost every* N

$$|(1/N)\ln \Omega_N-(1/N)\max_{N_2,\boldsymbol{i}[N-N_2]}\ln \tau_N(N_2,\ \boldsymbol{i}[N-N_2])|<\epsilon \tag{4·1}$$

where

$$\tau_N(N_2,\ \boldsymbol{i}[N-N_2])=\varphi_N(\boldsymbol{i}[N-N_2])\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2]) \tag{4·2}$$

in which $N_2=0$ or $l^*(N)+1\leqq N_2\leqq N$.

[*Proof*] Since $b_l>0$ for every l [assumption (b) in § 0], we have

$$\tau_N(N_2,\ \boldsymbol{i}[N-N_2])>0 \tag{4·3}$$

for every $\boldsymbol{i}[N-N_2]$ for every N_2. Hence, if we let r_N be the number of terms in the sum $\Omega_N=\sum_{N_2,\boldsymbol{i}[N-N_2]}\tau_N(N_2,\ \boldsymbol{i}[N-N_2])$ which extends over $N_2=0$ and N_2 $(l^*(N)+1\leqq N_2\leqq N)$ and over $\boldsymbol{i}[N-N_2]$, then we have

$$\max_{N_2,\boldsymbol{i}[N-N_2]}\tau_N(N_2,\ \boldsymbol{i}[N-N_2])\leqq\Omega_N\leqq r_N\max_{N_2\ \boldsymbol{i}[N-N_2]}\tau_N(N_2,\ \boldsymbol{i}[N-N_2]). \tag{4·4}$$

Now let P_N be the partitio numerorum of N. Then we have

$$r_N\leqq P_N \tag{4·5}$$

for every N. Hence by (3·17), we have for any given $\epsilon>0$

$$0\leqq(1/N)\ln r_N<\epsilon \tag{4·6}$$

for almost every N. From (4·4) and (4·6) we obtain (4·1). Thus the proof is completed.

Lemma 3. (i) *For any given* $\epsilon>0$, *we have for almost every* N

$$|(1/N)\ln \tau_N(N_2,\ \boldsymbol{i}[N-N_2])-(1/N)\ln \tau_N^{\Delta}(N_2,\ \boldsymbol{i}[N-N_2])|<\epsilon \tag{4·7}$$

for every $\boldsymbol{i}[N-N_2]$ *for every* N_2 $(l^*(N)+1 \leqq N_2 \leqq N$ or $N_2=0)$, *where*

$$\ln \tau_N^{\Delta}(N_2,\ \boldsymbol{i}[N-N_2]) \leqq \sum_{l=1}^{l^*(N)}{}' \{m_l \ln(Nvb_l) - m_l \ln m_l + m_l\} + N_2 \ln b_0. \tag{4·8}$$

(ii) *For any given* $\epsilon > 0$, *we have for almost every* N

$$|(1/N) \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N(N_2,\ \boldsymbol{i}[N-N_2]) - (1/N) \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2,\ \boldsymbol{i}[N-N_2])| < \epsilon. \tag{4·9}$$

[*Proof of* (i)] By (4·2), (2·8), and (3·2), we have

$$(1/N) \ln \tau_N(N_2,\ \boldsymbol{i}[N-N_2]) = (1/N) \sum_{l=1}^{l^*(N)}{}' \{m_l \ln(Nvb_l) - m_l \ln m_l + m_l - (1/2) \ln m_l - \gamma(m_l)\} + (1/N) \ln \sum_{\boldsymbol{k}[N_2]} \Phi_N(\boldsymbol{k}[N_2]). \tag{4·10}$$

By (4·10) and (4·8), we have

$$|(1/N) \ln \tau_N(N_2,\ \boldsymbol{i}[N-N_2]) - (1/N) \ln \tau_N^{\Delta}(N_2,\ \boldsymbol{i}[N-N_2])| \leqq (1/2N) \sum_{l=1}^{l^*(N)}{}' \ln m_l + (1/N) \sum_{l=1}^{l^*(N)}{}' |\gamma(m_l)| + |(1/N) \ln \sum_{\boldsymbol{k}[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - (N_2/N) \ln b_0|. \tag{4·11}$$

By (2·8a) we have

$$\sum_{l=1}^{l^*(N)} l m_l = N - N_2 \tag{4·12}$$

for every $\boldsymbol{i}[N-N_2]$. Hence

$$m_l \leqq N \quad (l=1, 2, \cdots, l^*(N)). \tag{4·13}$$

Therefore

$$(1/2N) \sum_{l=1}^{l^*(N)}{}' \ln m_l \leqq \{l^*(N)/2N\} \ln N. \tag{4·14}$$

By (3·3) we have

$$(1/N) \sum_{l=1}^{l^*(N)}{}' |\gamma(m_l)| \leqq \{l^*(N)/N\} K_1. \tag{4·15}$$

By (2·3), for any given $\epsilon > 0$ we have for almost every N

$$0 < l^*(N) (\ln N)/N < \epsilon \tag{4·16}$$

$$0 < l^*(N)/N < \epsilon, \tag{4·17}$$

since we have for almost every N

$$0 < l^*(N)/N < l^*(N) (\ln N)/N < l^*(N)/N^{1/2}. \tag{4·18}$$

Using (4·14), (4·16) [in which ϵ is replaced by $2\epsilon/3$], (4·15), (4·17) [in

which ϵ is replaced by $\epsilon/(3K_1)$], and (3·13) [in which ϵ is replaced by $\epsilon/3$], we obtain (4·7) from (4·11). Thus the proof is completed.

[*Proof of* (ii)] This proof is similar to part of the proof of Lemma 5 in the previous paper.[8]

Let $\bar{N}_2$ and $\bar{\boldsymbol{i}}[N-\bar{N}_2]$ mean the N_2 and $\boldsymbol{i}[N-N_2]$ for which $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ is largest. And let $\bar{N}_2^{\Delta}$ and $\bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}]$ mean the N_2 and $\boldsymbol{i}[N-N_2]$ for which $\tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2])$ is largest. Then we have

$$\tau_N(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) \geqq \tau_N(\bar{N}_2^{\Delta}, \bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}]) \tag{4·19a}$$

$$\tau_N^{\Delta}(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) \leqq \tau_N^{\Delta}(\bar{N}_2^{\Delta}, \bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}]). \tag{4·19b}$$

Applying (4·7) to the term for which $N_2=\bar{N}_2$, $\boldsymbol{i}[N-N_2]=\bar{\boldsymbol{i}}[N-\bar{N}_2]$, and to the term for which $N_2=\bar{N}_2^{\Delta}$, $\boldsymbol{i}[N-N_2]=\bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}]$, then we have for almost every N

$$|(1/N)\ln\tau_N(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) - (1/N)\ln\tau_N^{\Delta}(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2])| < \epsilon \tag{4·20a}$$

$$|(1/N)\ln\tau_N(\bar{N}_2^{\Delta}, \bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}]) - (1/N)\ln\tau_N^{\Delta}(\bar{N}_2^{\Delta}, \bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}])| < \epsilon. \tag{4·20b}$$

By (4·19a, b) and (4·20a, b), we have for any given $\epsilon>0$

$$|(1/N)\ln\tau_N(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) - (1/N)\ln\tau_N^{\Delta}(\bar{N}_2^{\Delta}, \bar{\boldsymbol{i}}^{\Delta}[N-\bar{N}_2^{\Delta}])| < \epsilon \tag{4·21}$$

for almost every N. [For, if $a_1 \geqq a_2$, $b_1 \leqq b_2$, $|a_1-b_1|<\epsilon$, and $|a_2-b_2|<\epsilon$, then we have $-\epsilon < a_2-b_2 \leqq a_1-b_2 \leqq a_1-b_1 < \epsilon$.] (4·21) is identical with (4·9). Thus the proof is completed.

[*The physical meaning of Lemmas 2 and 3*]

Lemma 2 shows that, if we define the terms $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ [each of which is determined by the total number (N_2) of particles composing "large" clusters and the way $(\boldsymbol{i}|N-N_2|)$ of grouping the remaining particles $(N-N_2)$ into "small" clusters], then $(1/N)\ln\Omega_N$ may be replaced by the largest of $(1/N)\ln\tau_N(N_2, \boldsymbol{i}[N-N_2])$ with as small error as we please if N is large enough.

Lemma 2 corresponds to (A3·11) in Appendix A.

Lemma 3 shows that the assumptive system denoted by Δ and the system under consideration are practically equivalent to each other if N is large enough.

Lemma 3 corresponds to (A3·10) in Appendix A, where, however, the difference between the system under consideration and the Δ-system is left out of account from the beginning.

Using Theorem I and Lemmas 2 and 3, we have the following theorem.

§ 5. Appearance of a "huge" cluster (The two-phase separation)

Theorem II. *If* $v < v_s[\equiv(\sum_{l=1}^{\infty} l b_l z_s^l)^{-1}$ *where* $z_s \equiv b_0^{-1}]$,* *then, for any given* $\epsilon>0$,

* Here by the assumption (d) in § 0, the series $\sum_{l=1}^{\infty} l b_l z_s{}^l$ $[\equiv\sum_{l=1}^{\infty} l f(l)]$ is convergent (and then, of course, the series $\sum_{l=1}^{\infty} b_l z_s{}^l$ $[\equiv\sum_{l=1}^{\infty} f(l)]$ is convergent), where $z_s(\equiv b_0^{-1})$ is the first singularity (on the real positive axis of z) of $\sum_{l=1}^{\infty} l b_l z^l$ (and of $\sum_{l=1}^{\infty} b_l z^l$).

there exists, for almost any N, at least one set $\{N_2, \boldsymbol{i}[N-N_2]\}$ for which

$$|(1/N)\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) - (1/N)\max_{N_2, \boldsymbol{i}[N-N_2]}\ln\tau_N(N_2, \boldsymbol{i}[N-N_2])| < \epsilon \quad (5\cdot1)$$

and

$$|N_2/N - (1-\sum_{l=1}^{\infty} l v b_l z_s^l)| < \epsilon \quad (5\cdot2)$$

are simultaneously satisfied.

Here (5·1) *may be replaced by*

$$|(1/N)\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) - (1/N)\ln\Omega_N| < \epsilon \quad (5\cdot1')$$

and by

$$|(1/N)\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) - (\sum_{l=1}^{\infty} v b_l z_s^l - \ln z_s)| < \epsilon. \quad (5\cdot1'')$$

In (5·1), (5·1′), *and* (5·1″), $(1/N)\ln\tau_N(N_2, \boldsymbol{i}[N-N_2])$ *may be replaced by* $(1/N)\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2])$.

[*Proof*] Consider $\max'_{N_2, \boldsymbol{i}[N-N_2]}\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2])$, where $\max'_{N_2, \boldsymbol{i}[N-N_2]}$ means that the maximum is taken with N_2, m_1, m_2, $\cdots$, $m_{l^*(N)}$ assumed to be continuous variables which take positive real values and are subject to the condition

$$\sum_{l=1}^{l^*(N)} l m_l + N_2 = N. \quad (5\cdot3)$$

Comparing this max′ with the (actual) max, we have

$$\max'_{N_2, \boldsymbol{i}[N-N_2]}\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) \geqq \max_{N_2, \boldsymbol{i}[N-N_2]}\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2]). \quad (5\cdot4)$$

With the use of the undetermined multiplier $\ln z$, the set (of real numbers) $\{N_2, m_1, m_2, \cdots, m_{l^*(N)}\}$ which gives the max′ must satisfy

$$(\partial/\partial m_l)\{\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) + (\sum_{l=1}^{l^*(N)} l m_l + N_2)\ln z\} = 0 \quad (l=1, 2, \cdots, l^*(N)) \quad (5\cdot5a)$$

and

$$(\partial/\partial N_2)\{\ln\tau_N{}^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) + (\sum_{l=1}^{l^*(N)} l m_l + N_2)\ln z\} = 0. \quad (5\cdot5b)$$

By the definition (4·8), we have from (5·5a)

$$m_l = N v b_l z^l \quad (l=1, 2, \cdots, l^*(N)), \quad (5\cdot6a)$$

and we have from (5·5b)

$$z = b_0^{-1} (\equiv z_s \text{ say}). \quad (5\cdot6b)$$

By (5·6a, b) and (5·3), the set giving the max′ must satisfy

$$m_l = \overline{\overline{m}}_l \equiv N v b_l z_s^l \quad (l=1, 2, \cdots, l^*(N)) \quad (5\cdot7a)$$

$$N_2 = \overline{\overline{N}}_2 \equiv N - \sum_{l=1}^{l^*(N)} l \overline{\overline{m}}_l \equiv N - N\sum_{l=1}^{l^*(N)} l v b_l z_s^l. \quad (5\cdot7b)$$

By the assumption that $b_l > 0$, we have

$$[1/v_s] \sum_{l=1}^{\infty} l b_l z_s^l > \sum_{l=1}^{l^*(N)} l b_l z_s^l . \tag{5·8}$$

Hence by (5·7b) we have

$$\overline{\overline{N}}_2 > N(1-v/v_s) > 0, \tag{5·9}$$

(since $v < v_s$) ; and by (5·7a) we have

$$\overline{\overline{m}}_l > 0 \quad (l=1, 2, \cdots, l^*(N)). \tag{5·10}$$

These are consistent with the assumption that N_2, m_1, m_2, $\cdots$ are positive real numbers. Therefore we have

$$\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) = N \sum_{l=1}^{l^*(N)} v b_l z_s^l - \overline{\overline{N}}_1 \ln z_s + \overline{\overline{N}}_2 \ln b_0, \tag{5·11}$$

where

$$\overline{\overline{N}}_1 = \sum_{l=1}^{l^*(N)} l \overline{\overline{m}}_l = N \sum_{l=1}^{l^*(N)} l v b_l z_s^l, \tag{5·12}$$

that is,

$$\overline{\overline{N}}_1 + \overline{\overline{N}}_2 = N. \tag{5·13}$$

By $z_s = b_0^{-1}$ and by (5·13), (5·11) becomes

$$\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) = N \sum_{l=1}^{l^*(N)} v b_l z_s^l - N \ln z_s. \tag{5·14}$$

Now, for each N, we take the set (of integers) $\{N_2, \hat{m}_1, \hat{m}_2, \cdots \hat{m}_{l^*(N)}\}$:

$$\hat{m}_l = [\overline{\overline{m}}_l] = [N v b_l z_s^l] \quad (l=1, 2, \cdots, l^*(N)) \tag{5·15a}$$

$$N_2 = N - \sum_{l=1}^{l^*(N)} l \hat{m}_l = N - \sum_{l=1}^{l^*(N)} l [N v b_l z_s^l], \tag{5·15b}$$

where, for real x, $[x]$ means† the largest integer not exceeding x. Then we have

$$\sum_{l=1}^{l^*(N)} l \hat{m}_l + N_2 = N, \tag{5·16}$$

and, by (5·7b), (5·9), and (2·3), we have for almost every N

$$\hat{N}_2 \geqq \overline{\overline{N}}_2 > l^*(N) \tag{5·17}$$

and, by (5·10), we have

$$\hat{m}_l \geqq 0 \quad (l=1, 2, \cdots, l^*(N)). \tag{5·18}$$

By (5·16), (5·17), and (5·18), for almost every N this set $\{N_2, \hat{m}_l\}$ satisfies the conditions of $\{N_2, \boldsymbol{i}[N-N_2]\}$ (that is, this set is one of the sets $\{N_2, \boldsymbol{i}[N-N_2]\}$). Hence for almost every N we shall denote this set by $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$.

† Here this square bracket should not be confused with a square bracket following a bold-faced letter, e. g. $\boldsymbol{i}[N-N_2]$.

Then, introducing this set into (4·8) for almost every N, and using (5·14) and $z_s \equiv b_0^{-1}$, we have

$$(1/N)\ln \tau_N^{\Delta}(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - (1/N)\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) = A(N)+B(N), \tag{5·19}$$

where

$$A(N) = (1/N)\sum_{l=1}^{l^*(N)}{}' \hat{m}_l\{\ln(Nvb_l) - \ln \hat{m}_l\} + \{(N-N_2)/N\}\ln z_s$$

$$= -(1/N)\sum_{l=1}^{l^*(N)}{}' [Nvb_l z_s^l] \ln([Nvb_l z_s^l]/Nvb_l z_s^l) \tag{5·20}$$

$$B(N) = (1/N)\sum_{l=1}^{l^*(N)} \hat{m}_l - \sum_{l=1}^{l^*(N)} vb_l z_s^l \tag{5·21}$$

(where the prime of $\sum'$ means that in the summation the terms for which $[Nvb_l z_s^l] = 0$, i.e. $Nvb_l z_s^l < 1$, are excluded). Hence we have [by the lemma in the footnote†]

$$|A(N)| \leqq (1/N)\sum_{l=1}^{l^*(N)}{}' K_0 \leqq K_0 \cdot l^*(N)/N \tag{5·22}$$

(where K_0 is a positive absolute constant), and we have

$$|B(N)| = (1/N)\sum_{l=1}^{l^*(N)} (Nvb_l z_s^l - [Nvb_l z_s^l]) < l^*(N)/N. \tag{5·23}$$

Now, take any $\epsilon > 0$. Then by (2·3) we have (4·17) for almost every N. Using (5·22), (4·17) [in which ϵ is replaced by $\epsilon/(6K_0)$], (5·23), and (4·17) [in which ϵ is replaced by $\epsilon/6$], we have from (5·19)

$$|(1/N)\ln \tau_N^{\Delta}(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - (1/N)\max'_{N_2\, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2])|$$

$$[\leqq |A(N)| + |B(N)|] < \epsilon/3 \tag{5·24}$$

for almost every N.

It is obvious that, for almost every N,

$$\max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2]) \geqq \ln \tau_N^{\Delta}(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]). \tag{5·25}$$

From this and (5·4) and (5·24), we have for any given $\epsilon > 0$

$$|(1/N)\ln \tau_N^{\Delta}(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - (1/N)\max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Delta}(N_2, \boldsymbol{i}[N-N_2])| < \epsilon/3 \tag{5·26}$$

for almost every N.

Now, applying (4·7) [in which ϵ is replaced by $\epsilon/3$] to the set $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, we have for any given $\epsilon > 0$

$$|(1/N)\ln \tau_N(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - (1/N)\ln \tau_N^{\Delta}(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2])| < \epsilon/3 \tag{5·27}$$

† **Lemma.** *The function* $[x]\ln([x]/x)$ *is bounded in the half-open real interval* $1 \leqq x < +\infty$.
[*Proof*] We have

$$x\ln(1-1/x) < [x]\ln([x]/x) \leqq 0 \quad (\text{for } x > 1).$$

And the function $x\ln(1-1/x)$ is bounded for $x \geqq 2$; [for it is continuous for $x \geqq 2$ and its limit for $x \to +\infty$ is -1]. And obviously the function $\ln(1/x)$ is bounded for $1 \leqq x < 2$.

for almost every N. From (4·9) [in which ϵ is replaced by $\epsilon/3$] and (5·26) and (5·27), we have for any given $\epsilon>0$

$$|(1/N)\ln\tau_N(\hat{N}_2,\ \hat{\boldsymbol{i}}[N-\hat{N}_2])-(1/N)\max_{N_2,\boldsymbol{i}[N-N_2]}\ln\tau_N(N_2,\ \boldsymbol{i}[N-N_2])|<\epsilon \quad (5\cdot28)$$

for almost every N.

Now by (5·15b) we have

$$\hat{N}_2/N-(1-\sum_{l=1}^{\infty}lvb_lz_s^l)=(1/N)\sum_{l=1}^{l^*(N)}l(Nvb_lz_s^l-[Nvb_lz_s^l])+\sum_{l=l^*(N)+1}^{\infty}lvb_lz_s^l. \quad (5\cdot29)$$

Hence

$$0<\hat{N}_2/N-(1-\sum_{l=1}^{\infty}lvb_lz_s^l)<(1/N)\sum_{l=1}^{l^*(N)}l+\sum_{l=l^*(N)+1}^{\infty}lvb_lz_s^l. \quad (5\cdot30)$$

By (2·3) we have for any given $\epsilon>0$

$$[0<(1/N)\sum_{l=1}^{l^*(N)}l=](1/2N)l^*(N)\,(l^*(N)+1)<(2/3)\epsilon \quad (5\cdot31)$$

for almost every N. For we have

$$l^*(N)\,(l^*(N)+1)/2N\leqq\{l^*(N)/N^{1/2}\}^2 \quad (5\cdot32)$$

for every N.

Since the series $\sum_{l=1}^{\infty}lb_lz_s^l$ is assumed to be convergent, there exists an integer $M(\epsilon)>0$, corresponding to any given $\epsilon>0$, such that

$$[0<\sum_{l=l^*+1}^{\infty}vb_lz_s^l<]\sum_{l=l^*+1}^{\infty}lvb_lz_s^l<\epsilon/3 \quad \text{(whenever } l^*\geqq M(\epsilon)). \quad (5\cdot33)$$

By (2·4) we have (3·5); hence

$$l^*(N)\geqq M(\epsilon) \quad (5\cdot34)$$

for almost every N. Hence by (5·33) we have

$$[0<\sum_{l=l^*(N)+1}^{\infty}vb_lz_s^l<]\sum_{l=l^*(N)+1}^{\infty}lvb_lz_s^l<\epsilon/3 \quad (5\cdot35)$$

for almost every N.

By (5·30), (5·31), and (5·35), we have for any given $\epsilon>0$

$$|\hat{N}_2/N-(1-\sum_{l=1}^{\infty}lvb_lz_s^l)|<\epsilon \quad (5\cdot36)$$

for almost every N.

From (5·28) and (5·36) we see that, for any given $\epsilon>0$, the set $\{\hat{N}_2,\ \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost any N, satisfies both (5·1) and (5·2). Thus the main part of the theorem is proved.

Now, from (5·28) [in which ϵ is replaced by $\epsilon/2$] and (4·1) [in which ϵ is replaced by $\epsilon/2$], we see that the set $\{\hat{N}_2,\ \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost every N, satisfies (5·1′). In the theorem, therefore, (5·1) may be replaced by (5·1′).

From (5·14) and (5·35), we have for any given $\epsilon>0$

$$|(1/N)\max'_{N_2,\,\boldsymbol{i}[N-N_2]}\ln\tau_N^{\Delta}(N_2,\,\boldsymbol{i}[N-N_2])-(\sum_{l=1}^{\infty}vb_lz_s^l-\ln z_s)|[=\sum_{l=l^{\#}(N)+1}^{\infty}vb_lz_s^l]<\epsilon/3 \tag{5·37}$$

for almost every N. From (5·24), (5·27), and (5·37), we see that the set $\{\hat{N}_2,\ \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost every N, satisfies (5·1″). In the theorem, therefore, (5·1) may be replaced by (5·1″).

By using Lemma 3 (i), it is proved that, in (5·1), (5·1′), and (5·1″), $(1/N)\cdot\ln\tau_N(N_2,\,\boldsymbol{i}[N-N_2])$ may be replaced by $(1/N)\ln\tau_N^{\Delta}(N_2,\,\boldsymbol{i}[N-N_2])$.

Thus the proof is completed.

Corollary 1. *If $v<v_s$, then $\lim_{N\to\infty}(1/N)\ln\Omega_N$ exists and*

$$\lim_{N\to\infty}(1/N)\ln\Omega_N=\sum_{l=1}^{\infty}vb_lz_s^l-\ln z_s. \tag{5·38}$$

[Proof] This is proved by using (5·1′) and (5·1″) of the theorem.

Corollary 2. *If $v\leqq v_s$, then, for any given $\epsilon>0$, there exists, for almost any N, at least one N_2 for which*

$$|(1/N)\ln\sum_{\boldsymbol{i}[N-N_2]}\tau_N(N_2,\,\boldsymbol{i}[N-N_2])-(1/N)\ln\Omega_N|<\epsilon \tag{5·39}$$

and (5·2) are simultaneously satisfied.

[*Proof*] By the theorem, there exists, for almost any N, at least one set $\{N_2,\ \boldsymbol{i}[N-N_2]\}$ for which (5·1′) and (5·2) are simultaneously satisfied. Since this N_2 satisfies (5·1′), it satisfies (5·39). For we have

$$\tau_N(N_2,\,\boldsymbol{i}[N-N_2])\leqq\sum_{\boldsymbol{i}[N-N_2]}\tau_N(N_2,\,\boldsymbol{i}[N-N_2])\leqq\sum_{N_2,\,\boldsymbol{i}[N-N_2]}\tau_N(N_2,\,\boldsymbol{i}[N-N_2])=\Omega_N. \tag{5·40}$$

Thus the proof is completed.

Corollary 3. *If $v<v_s$, we have for any given $\epsilon>0$*

$$|(1/N)\ln\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])-(1/N)\ln\Omega_N|<\epsilon \tag{5·41}$$

for almost every N.

[*Proof*] By the theorem, there exists, for almost any N, at least one set $\{N_2,\ \boldsymbol{i}[N-N_2]\}$ for which (5·1′) and (5·2) are simultaneously satisfied. For almost every N we have $N_2>0$ (if $v<v_s$) by (5·2). Hence $\tau_N(N_2,\,\boldsymbol{i}[N-N_2])$ given by this set $\{N_2,\ \boldsymbol{i}[N-N_2]\}$ is a part of the sum $\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])$. Hence

$$\tau_N(N_2,\,\boldsymbol{i}[N-N_2])\leqq\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])\leqq\Omega_N. \tag{5·42}$$

By this and (5·1′), we have (5·41). Thus the proof is completed.

[*The physical meaning of Theorem II and its corollaries*]

Theorem II shows what happens in the range $v<v_s$, that is, beyond the gaseous range ($v\geqq v_s$). The theorem states that, for any given positive number ϵ (however small it may be), if N is large enough, there exists at least one set $\{N_2,\ \boldsymbol{i}[N-N_2]\}$

satisfying both (5·1) and (5·2). Now take such a set $\{N_2, \boldsymbol{i}[N-N_2]\}$. Then, since this set $\{N_2, \boldsymbol{i}[N-N_2]\}$ satisfies (5·1) and $\epsilon(>0)$ may be as small as we please, the term τ_N given by this set $\{N_2, \boldsymbol{i}[N-N_2]\}$ is practically maximum. This means that the whole configuration integral Ω_N may practically be represented by this term τ_N only [see (5·1′)]. That is, this term is predominant in the system;—physically speaking, this set $\{N_2, \boldsymbol{i}[N-N_2]\}$ gives the equilibrium state. This N_2 (that is, N_2 in the equilibrium state) satisfies (5·2) at the same time; hence this N_2 is of the order of magnitude of N if $v<v_s$ [since $(1-\sum_{l=1}^{\infty} lvb_l z_s^l)$ is finite and positive (if $v<v_s$) and $\epsilon(>0)$ may be as small as we please]. Consequently, for our equilibrium-state term τ_N we have by (4·2)

$$\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) = \ln\varphi_N(\boldsymbol{i}[N-N_2]) + \ln\sum_{\boldsymbol{k}[N_2]}\Psi_N(\boldsymbol{k}[N_2]) \qquad (5\cdot43)$$

where N_2 is larger than $l^*(N)$ and is of the order of magnitude of N (if $v<v_s$). And in (5·43), by Theorem I (Corol. 5), $\ln\sum\Psi_N(\boldsymbol{k}[N_2])$ [a number of "large" clusters] may (if N is very large) be replaced by $\ln b_{N_2}$ [one "large" cluster]. Therefore, if $v<v_s$, in thermodynamic equilibrium there exists one "large" cluster ($\ln b_{N_2}$) which is of a macroscopic size; we shall call such a cluster a "huge" cluster [see *Remark* 1 on the next page].

If $v\geqq v_s$, there is no "large" cluster in thermodynamic equilibrium (by Theorem III). The $p-v$ isotherm is not horizontal for $v\geqq v_s$ (by Corol. 1 of Theorem III), but is horizontal for $v<v_s$ (by Corol. 1 of Theorem II). Our "huge" (i. e. macroscopic) cluster $\ln b_{N_2}$ existing for $v<v_s$ should (according to the physical interpretation of a cluster) be regarded as representing the liquid phase which has appeared by the two-phase separation in the condensation process and which coexists with the saturated vapour [represented by $\ln\varphi_N(\boldsymbol{i}[N-N_2])$ in (5·43)] in the condensation range $(v<v_s)$. [We thus see that $1/v_s$ represents the density of the saturated vapour, and also we see from (5·2) that the "huge" cluster $\ln b_{N_2}$ grows large steadily as v decreases.] Thus we have proved the condensation phenomenon and the appearance of liquid phase, by means of a mathematically rigorous method.

This result is obtained also in § A3 of Appendix A (though not so rigorously). The equations (5·7a) and (5·7b)—or the equations (5·15a) and (5·15b)—correspond to the equations (A3·14) and (A3·16) in Appendix A, where, however, the difference between max and max′ and the difference between $\bar{\bar{m}}_l$ and $\hat{m}_l$ ($\overline{\overline{N_2}}$ and $\hat{N}_2$) are left out of account from the beginning, and the notations Max, $\bar{m}_l$, $\overline{N_2}$, are used.

It is now suggested that we may discuss the detailed properties of the liquid phase by analyzing $\ln b_{N_2}$ (the "huge" cluster) itself. [About this point, see the discussions in §§ A5 and A6 of Appendix A.]

Corollary 1 teaches us all thermodynamic properties for $v<v_s$, and especially it shows that the isotherm is horizontal (i. e., the pressure is independent of the specific volume) in the condensation range $v<v_s$).†

† The argument (about the uniform convergence of sequences) given in the footnote on page 382 of the previous paper[8c] is also true in this paper. Hence we have for the pressure p_∞,

(*Continued on the following page.*)

Corollary 2 shows that τ_N in Theorem II may be replaced by the sum of τ_N over $\boldsymbol{i}[N-N_2]$. That is, here we have given the equilibrium state by determining only the number N_2 of particles composing the liquid phase, with no specification of the way $\boldsymbol{i}[N-N_2]$ of grouping $N-N_2$ particles into "small" clusters (i. e., of the detailed state of the vapour phase).

Corollary 3 shows that, when Ω_N is expressed as (2·5), the whole configuration integral Ω_N may practically be represented by the second part only (if $v<v_s$).

[*Remark* 1] We define the word "huge" as "of the order of magnitude of N" when N is large enough. Hence "huge" means "macroscopic" in a physical word. We should distinguish "huge" from "large"; the latter has been defined as "larger than $l^*(N)$" (§ 2). [As is seen from (2·3) and (5·2), the extent of the concept "large" includes the extent of the concept "huge"; that is, every "huge" quantity is "large", but the converse is not true.]

In Appendix A, we give a theory which is not so rigorous; so we do not distinguish between "large" and "huge", but adopt only the word "large", which means (though not so rigorously) "of the order of magnitude of N". [Also "small" in Appendix A is different (in definition) from "small" in the text.]

[*Remark* 2] In Theorem I, N_2 has been mathematically given as any integer such that $l^*(N)+1 \leqq N_2 \leqq N$, while, in Theorem II, for $v<v_s$ the physical character of N_2 in thermodynamic equilibrium has been discussed and the "hugeness" of N_2 in thermodynamic equilibrium has been proved.

§ 6. Gaseous state

Theorem III. *If $v \geqq v_s$, then, for any given $\epsilon>0$, there exists, for almost any N, at least one set $\boldsymbol{i}[N]$ for which*

$$|(1/N)\ln\varphi_N(\boldsymbol{i}[N]) - (1/N)\max_{N_2,\,\boldsymbol{i}[N-N_2]}\ln\tau_N(N_2,\ \boldsymbol{i}[N-N_2])| < \epsilon \qquad (6\cdot1)$$

is satisfied.

Here (6·1) *may be replaced by*

$$|(1/N)\ln\varphi_N(\boldsymbol{i}[N]) - (1/N)\ln\Omega_N| < \epsilon \qquad (6\cdot1')$$

and by

$$|(1/N)\ln\varphi_N(\boldsymbol{i}[N]) - (\sum_{l=1}^{\infty} v b_l z^l - \ln z)| < \epsilon, \qquad (6\cdot1'')$$

where z is the least positive real root of the equation $\sum_{l=1}^{\infty} l v b_l z^l = 1$.

$$p_\infty/kT = \lim \partial \ln \Omega_N/\partial V = \partial/\partial v \lim (1/N)\ln\Omega_N \qquad (5\cdot44)$$

[where lim is taken for $N\to\infty$ and $V\to\infty$ with $v=V/N$ fixed]. Thus by Corollary 1 we have (for $v<v_s$)

$$p_\infty = kT\sum_{l=1}^{\infty} b_l z_s^{\,l} \ (\equiv p_s \text{ say}), \qquad (5\cdot45)$$

which is independent of v. And we have for the Gibbs free energy G

$$\lim (G/N) = \lim\{(A+pV)/N\} = kT\ln\lambda^3 - kT\lim(1/N)\ln\Omega_N + p_\infty v = kT\ln(\lambda^3 z_s) \qquad (5\cdot46)$$

(for $v<v_s$). Hence z_s is the fugacity (or activity) of the system, which is independent of v, (for $v<v_s$). [Note that $G=NkT\ln a$ where a is the absolute activity and a/λ^3 is the activity or fugacity.]

In (6·1), (6·1′), *and* (6·1″), $\varphi_N(\boldsymbol{i}[N])$ *may be replaced by* $\varphi_N^{\Delta}(\boldsymbol{i}[N])$ [$\equiv \tau_N^{\Delta}(0, \boldsymbol{i}[N])$].

Corollary 1. *If* $v \geqq v_s$, *we have*

$$\lim_{N\to\infty} (1/N) \ln \Omega_N = \sum_{l=1}^{\infty} v b_l z^l - \ln z. \tag{6·2}^{\dagger}$$

Corollary 2. *If* $v \geqq v_s$, *we have for any given* $\epsilon > 0$

$$|(1/N) \ln \sum_{\boldsymbol{i}[N]} \varphi_N(\boldsymbol{i}[N]) - (1/N) \ln \Omega_N| < \epsilon \tag{6·3}$$

for almost every N.

[*Remark*] In Theorem III and its corollaries, a fact which has been rigorously proved in the previous paper[8] (see Lemma 6 of ref. 8)—namely, the fact that for the gaseous state ($v \geqq v_s$) the whole configuration integral may practically be represented by the "small" cluster part $\sum' \varphi_N(\boldsymbol{i}[N])$ only, that is, that (for the gaseous state) there is no "large" cluster in the predominant term φ_N (i. e. in thermodynamic equilibrium)—is stated in a modified form which corresponds to the form of Theorem II (with its corollaries). Therefore we need not give the detailed proof here. We have only to note that a condition for $l^*(N)$—namely, (2·4) which means that $l^*(N)$ increases rapidly enough with increasing N—is utilized for this proof. [Cf. note added in proof.]

§ 7. Concluding remarks

We shall now make some remarks on the essential features of our theory. (In this section we shall use concise notations and expressions.)

The old theory (based on, for instance, Mayer's method of maximum term or the method of saddle-point) gives [cf. §§ A1 & A4 of Appendix A and § B2 of Appendix B]

$$N = \sum_l lVb_l z^l \quad \text{or} \quad 1/v = \sum_l l b_l z^l \tag{7·1}$$

$$\ln \Omega \simeq V \sum_l b_l z^l - N \ln z \tag{7·2}$$

for the condensation range ($v < v_s$) [for which, however, the theory itself is invalid] as well as for the gaseous state ($v \geqq v_s$). On the other hand, our theory gives for the gaseous state ($v \geqq v_s$) [cf. § 6]

$$N = \sum_l lVb_l z^l \quad \text{or} \quad 1/v = \sum_l l b_l z^l \tag{7·3}$$

$$\ln \Omega \simeq V \sum_l b_l z^l - N \ln z \tag{7·4}$$

and gives for the condensation range ($v < v_s$) [cf. § 5]

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† With the use of Corollary 1 (of Theorem III) and the thermodynamic relations and with the help of the argument (about the uniform convergence of sequences) given in the footnote on page 382 of the previous paper,[8c] it is proved (for $v \geqq v_s$) that the pressure p_∞ of the gas is given by

$$p_\infty = kT \sum_{l=1}^{\infty} b_l z^l \tag{6·4}$$

and that z is the fugacity (or activity) of the gas.

$$N=\bar{N}_1+\bar{N}_2=\sum_l lVb_l z_s^l+\bar{N}_2 \quad \text{or} \quad 1/v=\sum_l lb_l z_s^l+1/v_2 \quad (v_2=V/\bar{N}_2) \qquad (7\cdot5)^*$$

$$\ln \Omega \simeq \max \ln \tau \simeq \ln \bar{\tau}=\ln \bar{\varphi}+\ln \sum \bar{\Phi} \qquad (7\cdot6)^{**}$$

where

$$\ln \bar{\varphi} \simeq V\sum_l b_l z_s^l-\bar{N}_1 \ln z_s \quad \text{(saturated vapour)} \qquad (7\cdot6\text{a})^{\dagger}$$

$$\ln \sum \bar{\Phi} \simeq \ln b_{\bar{N}_2} \simeq \bar{N}_2 \ln b_0 \quad \text{(liquid drop)}. \qquad (7\cdot6\text{b})^{\dagger\dagger}$$

The last terms $\bar{N}_2$ and $\ln \sum \bar{\Phi}$ (or $\ln b_{\bar{N}_2}$) obtained by our theory are results of the detailed analysis of the sizes of clusters, have some analogy to F. London's[9] first term [see § B1 of Appendix B], and are essential to the existence of a horizontal part of the isotherm and to the appearance of a liquid drop cluster. If one keeps compressing the gas slowly at constant temperature, the series $\sum l b_l z^l$ (and $\sum b_l z^l$ at the same time) becomes singular at the point $v=v_s$ $(z=z_s)$, i. e., the starting point of condensation, and then instantly the last term $\bar{N}_2$ (and $\ln b_{\bar{N}_2}$ at the same time) appears—that is to say, one "huge" cluster $\ln b_{\bar{N}_2}$ consisting of $\bar{N}_2$ molecules, which represents the liquid phase coexisting with the saturated vapour, appears and is described by the following expression [cf. § A5 of Appendix A]:

$$\ln b_{\bar{N}_2} \simeq \bar{N}_2 \sum_k \beta_k \gamma_0^k-\bar{N}_2 \ln \gamma_0 \qquad (7\cdot7)$$

or

$$\ln b_{\bar{N}_2} \simeq \bar{N}_2 \sum_k \beta_k \gamma_s^k-\bar{N}_2' \ln \gamma_s+\ln \beta_{\bar{N}_2}. \qquad (7\cdot8)$$

Here it should be emphasized that, when the series for the gaseous state becomes singular, our theory of a condensing system does *not* fall into the difficulty due to the divergence of the series, but a new term or series appears and plays the leading role, in place of the old series which has already become singular. Further researches for the new term or series itself may lead to more knowledge of the liquid phase.

In conclusion, let us briefly state in what way we wish to go further. In this paper the volume dependence of the cluster integrals have been left out of account. If we wish to introduce their volume dependence into the problem of this paper, we must propose a theory in which the theory in this paper and that in the previous paper[8] are unified in some way. If this is done, we may have some method for evaluating the volume dependence of the "huge" cluster $\ln b_{\bar{N}_2}$ (i. e. the liquid phase) when one compresses the system further after the liquid phase occupies the whole volume of the system. Thus we may have some possibility that we can derive the properties of the pure liquid phase and the singularity corresponding to solidification automatically from the partition function expanded by the Ursell method.

* $\bar{N}_2$ given by (7·5) is practically equal to N_2 in (5·2), $\bar{\bar{N}}_2$ in (5·7b), or $\hat{N}_2$ in (5·15b).

** See (4·1), (5·1), (5·1′), and (5·43).

† See (5·11).

†† See (3·23), (3·12), and (3.13).

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Appendix A

Approximate theory of condensing systems

Introduction and summary

This appendix may be regarded as an independent paper, which criticizes Mayer's theory of condensation (§§ A1 and A4) and gives an improved theory (§§ A2, A3, A5, and A6).

The reader may read this appendix without reading the text; from this appendix (§§ A2 and A3) he can know in a simple way the substantial physical contents of the rigorous and relatively complicated theory given in the text, because the theory given in §§ A2 and A3 (though it is not rigorous from the pure mathematical point of view*) is rather concise and is substantially similar to the rigorous theory in the text.

The relations between the arguments in this appendix (§§ A2 and A3) and those in the text have been stated in paragraphs printed in small type in the text (§§ 1–6). [The rigorous theory corresponding to § A5 has been omitted in the text.]

The words "small" and "large" used in this appendix are different in definition from those used in the text; "small" and "large" in this appendix are not so rigorously defined as in the text and mean "comparable with 1" and "comparable with N" respectively. [Hence "large" in this appendix is roughly equivalent to "huge" in the text; the latter has, however, been defined rigorously in the pure mathematical sense.]

First we show that Mayer's maximum-term method for treating the Ursell expansion of the partition function contains certain approximations which are good for comparatively small clusters but are not good for large clusters; as a result, this method is invalid for that range in which the condensation may occur. Then we propose an improved method, in which we classify all clusters into large clusters and other clusters and give a new suitable treatment to the large clusters. Our method is valid even for the condensation range. Also we apply our method to the expression of large cluster integrals in terms of irreducible cluster integrals. Then we give physical interpretations to the results obtained by our method; especially we deduce the appearance of a liquid drop (in the condensation range) and describe its properties.

§ A1. Approximations in Mayer's method of maximum term

Mayer[5] applied the Stirling approximate formula

$$\ln m_l! = m_l \ln m_l - m_l \tag{A1·1}$$

to $m_l (l=1, 2, \cdots, N)$ in the Ursell expansion [eq. (0·7) in the text], and obtained

$$\Omega_N = \sum_{m_l} \left(\sum_l l m_l = N\right) T(m_l) \tag{A1·2}$$

where

* See § A7.

$$\ln T(m_l) = \sum_l m_l \{\ln(Nvb_l) - \ln m_l + 1\}, \quad (Nv = V), \tag{A1·3}$$

where the cluster integrals b_l are assumed to be volume-independent and positive. Then he found the maximum term $T(\bar{m}_l)$ on the condition $\sum_l l m_l = N$, by using Lagrange's undetermined multiplier $\ln z$. From

$$(\partial/\partial m_l)\{\ln T(m_l) + (\sum_l l m_l)\ln z\} = 0 \quad (l = 1, 2, \cdots, N) \tag{A1·4}$$

one obtains for the maximum term

$$\bar{m}_l = Nvb_l z^l \quad (l = 1, 2, \cdots, N), \tag{A1·5}$$

where z is determined by

$$N\sum_{l=1}^{N} l v b_l z^l = N. \tag{A1·6}$$

Hence, for very large N, or for almost infinitely large N, we have the following [since the logarithm of the total number of terms in $\sum_{m_l}$ in (A1·2) is very small compared with N]:

$$\ln \Omega_N \simeq \ln T(\bar{m}_l) = N\sum_{l=1}^{N} v b_l z^l - N \ln z. \tag{A1·7}$$

Assuming these results (A1·6) and (A1·7) to be valid for the whole range of v, Mayer deduced the condensation phenomenon of a gas.

However, when we deduce conclusions from the equations (A1·6) and (A1·7) obtained by the above method of asymptotic evaluation, we must be careful, lest we should step beyond the range of validity of the above method. If one deduces a conclusion in a range, in which the method is invalid and so the equations have no meaning, then the conclusion will be unreliable. Therefore we wish to discuss in detail the validity of the above method.

The above method has the following two defects (A) and (B):

(A) The Stirling approximate formula, which we shall call the approximation (α), has been applied to *all* m_l. This approximation is good for large m_l but is not good for small m_l.

(B) To find the maximum term, differentiation with respect to m_l has been performed by assuming the integral variables (or discrete variables) m_l as continuous variables. We shall call this procedure the approximation (β). It has been applied to *all* m_l. For large integers m_l, it may be a good approximation, since, for large integers m_l, we can take a difference Δm_l such that $\Delta m_l \ll m_l$, and we can then take such a Δm_l as a differential (dm_l). For small integers m_l (comparable with 1), however, this approximation is apparently invalid, since, for small integers $m_l (\sim 1)$, we cannot take a difference Δm_l such that $\Delta m_l \ll m_l$.

In the next section we shall give a preliminary argument, in order to analyze the approximations (α) and (β).

§ A2. Classification of orders of magnitude

Now N is a very large (in fact, almost infinite) integral number (say $N\sim 10^{23}$), and we have

$$\sum_{l=1}^{N} l m_l = N. \qquad (A2\cdot 1)$$

Here the l's are 1, 2, 3, ⋯, N. And we have $0 \leqq m_l \leqq N$. We shall now classify the l's into the following three kinds in a suitable way. Some of the l's are comparable with N, that is, they are of the order of magnitude of N; we shall call such l's "large". (That is, $l\sim N$ in notation.) Some of the other l's are comparable with 1, that is, they are of the order of magnitude of 1; we shall call such l's "small". (That is, $l\sim 1$ in notation.) All l's which belong neither to the set of "large" l's nor to the set of "small" l's will be called "of intermediate magnitude" and we shall denote such l's by $N \gg l \gg 1$. To any other quantities q, we shall apply this classification into the three kinds of orders of magnitude, "large" ($q\sim N$), "small" ($q\sim 1$), and "intermediate" ($N \gg q \gg 1$), excepting the quantities of higher order of magnitude than N, ($q \gg N$). So we have ("large" q) = ("large" l) × (a quantity comparable with 1), ("small" q) = ("small" l) × (a quantity comparable with 1), and (q "of intermediate magnitude") = (l "of intermediate magnitude") × (a quantity comparable with 1).

Because of (A2·1), the m_l's for "large" l's, i. e. the numbers of "large" clusters, must be "small" (if not zero); and the m_l's for "small" l's, i. e. the numbers of "small" clusters, may be "large" or "of intermediate magnitude" or "small" (if not zero); and the m_l's for l's "of intermediate magnitude", i. e. the numbers of clusters "of intermediate size", may be "of intermediate magnitude" or "small" (if not zero).

We may safely consider that the approximations (α) and (β) [stated in § A1] are valid for "large" m_l and for m_l "of intermediate magnitude", but are not valid for "small" m_l. Hence, in applying the approximations (α) and (β) to m_l's, the existence of "small" clusters and of clusters "of intermediate size" will cause no serious error. For, as stated in the last paragraph, m_l's for such clusters are "large" or "of intermediate magnitude" except for a "small" number of such clusters. This exception is unimportant, since the existence of a "small" number of such clusters (which are not "large") is negligible compared with the existence of a "small" number of "large" clusters.

In the next section, in order to avoid the defects (A) and (B) [stated in § A1] and obtain better results, we shall propose a method of treatment in which the "large" clusters and the other clusters are treated in different ways; the latter are treated by applying the approximations (α) and (β) but the former in a new way.

§ A3. Improved approximate theory

As stated in § A2, we classify the sizes of the clusters into three kinds, viz. "large" ($l\sim N$) and "small" ($l\sim 1$) and "intermediate" ($1 \ll l \ll N$), when N is large ($N\sim 10^{23}$). Hereafter, for clearness' sake we write L for "large" l, though we write simply l for other l.

Let N_1 be the total number of those molecules which compose "small" clusters and clusters "of intermediate size", and let N_2 be the total number of those molecules which compose "large" clusters. Then we have

$$\sum_{l=1}^{l\ll N} l m_l = N_1, \quad \sum_{L\sim N}^{L=N} L m_L = N_2, \quad N_1 + N_2 = N. \tag{A3·1}$$*

Here we may note that N_2 is "large" or else zero, while N_1 may have any kind of magnitude.

Consequently we may write eq. (0·7) [with $V=Nv$] of the text as

$$\Omega_N = \sum_{m_l, m_L} \Big(\sum_{l=1}^{l\ll N} l m_l + \sum_{L\sim N}^{L=N} L m_L = N\Big) \prod_{l=1}^{l\ll N} \{(N v b_l)^{m_l}/m_l!\} \prod_{L\sim N}^{L=N} \{(N v b_L)^{m_L}/m_L!\}$$

$$= \sum_{N_2, m_l} \Big(\sum_{l=1}^{l\ll N} l m_l + N_2 = N\Big) \prod_{l=1}^{l\ll N} \{(N v b_l)^{m_l}/m_l!\} \sum_{m_L} \Big(\sum_{L\sim N}^{L\sim N} L m_L = N_2\Big) \prod_{L\sim N}^{L=N} \{(N v b_L)^{m_L}/m_L!\}. \tag{A3·2}$$

We should treat the former part $\prod_{l=1}^{l\ll N}\{\cdots\}$ and the latter part $\prod_{L\sim N}^{L=N}\{\cdots\}$ in different ways because of the difference of their properties, as discussed in § A2.

Here we assume** that b_l and b_L are independent of the volume V of the vessel, and that $b_l > 0$, $b_L > 0$ for all l, L.

Then the logarithm of any term in $\sum_{m_L}$ in (A3·2) becomes

$$\ln \prod_{L\sim N}^{L=N} \{(N v b_L)^{m_L}/m_L!\} \fallingdotseq \sum_{L\sim N}^{L=N} L m_L \ln b_0 \fallingdotseq N_2 \ln b_0, \tag{A3·3}$$

and in particular the logarithm of the term for which $m_{N_2}=1$ and $m_L=0$ $(L\neq N_2)$ becomes

$$\ln (N v b_{N_2}) \fallingdotseq \ln b_{N_2} \fallingdotseq N_2 \ln b_0, \tag{A3·4}$$

where the equality symbol $X \fallingdotseq Y$ means that on both sides of the equation all "small" quantities and all quantities "of intermediate magnitude" have been neglected, and that both sides X, Y themselves are "large" quantities, and that, in the limiting case when $N\to\infty$ with $v=V/N$ fixed, the relation, $\lim (X/N) = \lim (Y/N) =$ finite and definite and positive, would hold. In deriving (A3·3) and (A3·4) we have used the fact that

$$\lim_{l\to\infty} b_l^{1/l} = b_0 = \text{definite and finite and positive}, \tag{A3·5}***$$

that is, $b_l = f(l) b_0^{\,l}$ [where $\lim_{l\to\infty} (1/l) \ln f(l) = 0$] and hence

$$b_L = f(L) b_0^{\,L} \tag{A3·6}$$

where

$$\ln f(L) \ll L \sim N, \tag{A3·6a}$$

---

* In the second equation, the upper limit $L=N$ is in fact $L=N_2 (\leqq N)$. The same applies hereinafter.

** Assumptions (a) and (b) in § 0 of the text.

*** This is the assumption (c) [for b_l] in § 0 of the text.

and we have also used the fact that $m_L \sim 1$ and $\sum_{L\sim N}^{L=N} 1 \sim 1$ and $\ln(Nv) \ll N$.

Then we have by (A3·3) and (A3·4)

$$\ln \sum_{m_L} \Big(\sum_{L\sim N}^{L=N} L m_L = N_2\Big) \prod_{L\sim N}^{L=N} \{(N v \cdot b_L)^{m_L}/m_L!\} \fallingdotseq \ln(N v b_{N_2}) \fallingdotseq \ln b_{N_2} \fallingdotseq N_2 \ln b_0, \tag{A3·7}$$

since the logarithm of the total number of terms in $\sum_{m_L}$ is $\ll N$.

Here we may remark that the defect (A) in § A1 is now avoided since we do not apply the Stirling approximation to m_L's (which are "small").

By eqs. (A3·3) and (A3·4) we see that the situation where the N_2 molecules compose an arbitrary number of "large" clusters is equivalent to the situation where the N_2 molecules together compose only one "large" cluster. That is to say, we cannot decide how many "large" clusters are present; thus for convenience we may consider that only one "large" cluster (b_{N_2}) is present.

Next we have, by the Stirling approximation (A1·1),

$$\ln \prod_{l=1}^{l\ll N} \{(N v b_l)^{m_l}/m_l!\} \fallingdotseq \sum_{l=1}^{l\ll N} m_l \{\ln(N v b_l) - \ln m_l + 1\}. \tag{A3·8}$$

For, as was shown at the end of § A2, we can safely apply the Stirling approximation to m_l's when there are only "small" clusters and clusters "of intermediate size" (that is, $l \ll N$).

Thus we have

$$\Omega_N = \sum_{N_2, m_l} \Big(\sum_{l=1}^{l\ll N} l m_l + N_2 = N\Big) T(N_2, m_l), \tag{A3·9}$$

where

$$\begin{aligned} \ln T(N_2, m_l) &\fallingdotseq \sum_{l=1}^{l\ll N} m_l \{\ln(N v b_l) - \ln m_l + 1\} + \ln b_{N_2} \\ &\fallingdotseq \sum_{l=1}^{l\ll N} m_l \{\ln(N v b_l) - \ln m_l + 1\} + N_2 \ln b_0. \end{aligned} \tag{A3·10}$$

Since the logarithm of the total number of terms in $\sum_{N_2, m_l}$ in (A3·9) is $\ll N$, we have

$$\ln \Omega_N \fallingdotseq \operatorname{Max} \ln T(N_2, m_l) = \ln T(\bar{N}_2, \bar{m}_l). \tag{A3·11}$$

Here the maximum is taken under the condition

$$\sum_{l=1}^{l\ll N} l m_l + N_2 = N, \tag{A3·12}$$

and so, using the undetermined multiplier $\ln z$, we obtain

$$z = b_0^{-1} (\equiv z_s \text{ say}) \tag{A3·13}$$

from $(\partial/\partial N_2)\{\ln T(N_2, m_l) + (\sum_{l=1}^{l\ll N} l m_l + N_2) \ln z\} = 0$, and obtain

$$\bar{m}_l = N v b_l z^l = N v b_l z_s^l \quad (l=1, 2, 3, \cdots \ll N) \tag{A3·14}$$

from $(\partial/\partial m_l)\{\ ,,\ \}=0$ $(l=1, 2, 3, \cdots \ll N)$. [Hence we know that $\bar{m}_l$ is "large" for every "small" l.] Thus the values of N_1 and N_2 for the maximum term are

$$\bar{N}_1 = \sum_{l=1}^{l\ll N} l\bar{m}_l = N\sum_{l=1}^{l\ll N} l v b_l z_s^l \tag{A3·15}$$

and

$$\bar{N}_2 = N - N\sum_{l=1}^{l\ll N} l v b_l z_s^l, \tag{A3·16}$$

respectively.

Here we may remark that the defect (B) in § A1 is now avoided since we perform differentiation with respect to N_2 and m_l $(l\ll N)$ [where N_2 is "large" and $m_l \gg 1$*] but not with respect to m_L [where m_L's are "small", i. e. $m_L \sim 1$].

Then in order that the solution just now obtained may have a physical meaning, it is necessary that $\sum_{l=1}^{l\ll N} l v b_l z_s^l$ should be definite and finite and $\sum_{l=1}^{l\ll N} l v b_l z^l \leq 1$, (since N_2 should not be negative). In order that this may be so, firstly, the series $\sum_{l=1}^{\infty} l b_l z_s^l$ [$= \sum_{l=1}^{\infty} l f(l)$ where $b_l = f(l) b_0^l$] should be convergent** (since some of the l's "of intermediate magnitude" are very large compared with 1), unless v is infinitesimal. In this case, we have $\sum_{l=1}^{l\ll N} l b_l z_s^l = \sum_{l=1}^{\infty} l b_l z_s^l$ (with a negligible error), and $\sum_{l=1}^{\infty} b_l z_s^l$ [$= \sum_{l=1}^{\infty} f(l)$] is convergent and is equal to $\sum_{l=1}^{l\ll N} b_l z^l$ (with a negligible error). Secondly, the following should hold:

$$1/v \geqq \sum_{l=1}^{\infty} l b_l z_s^l \ (\equiv 1/v_s, \text{ say}). \tag{A3·17}$$

Under the above conditions we obtain [from (A3·11), (A3·10), and (A3·14)]

$$\ln \Omega_N \backsimeq N\sum_{l=1}^{l\ll N} v b_l z_s^l - \bar{N}_1 \ln z_s + \ln b_{\bar{N}_2} \tag{A3·18}$$

with (A3·15) and (A3·16). Here we may (with a negligible error) replace the upper limit $l\ll N$ of the sum by $l=\infty$, since some of the l's "of intermediate magnitude" are very large compared with 1. Thus (A3·18) becomes, with the use of $z_s \equiv b_0^{-1}$ and (A3·4),

$$\ln \Omega_N \backsimeq N\sum_{l=1}^{\infty} v b_l z_s^l - N \ln z_s. \tag{A3·19}$$

Hence the pressure of the system is

$$p = kT(\partial/\partial V)\ln \Omega_N = kT\sum_{l=1}^{\infty} b_l z_s^l \ (\equiv p_s, \text{ say}) \tag{A3·20}$$

---

* Except for a "small" number of clusters (which are not "large"). This exception is unimportant, as was stated before.

** This convergence [which is the assumption (d) in § 0 of the text] is necessary in order that the condensation may occur at a finite density.

which is independent of v. That is, the isotherm ($p-v$ curve) is horizontal in this range ($v \leqq v_s$). The Gibbs free energy G is

$$G=A+pV \backsimeq NkT \ln(\lambda^3 z_s). \tag{A3·21}$$

Hence z_s is the fugacity (or activity) of the system, which is independent of v. This corresponds to the condensation range.

On the other hand, if v is so given that

$$1/v < \sum_{l=1}^{\infty} l b_l z_s^l \ (\equiv 1/v_s), \tag{A3·22}$$

then we have $\bar{N}_2 < 0$ in (A3·16), and so we should have $\bar{N}_2=0$ for the largest term that has a physical meaning. In this case we have

$$\ln \Omega_N \backsimeq \text{Max} \ln T(0, m_l) = \ln T(0, \bar{m}_l) \tag{A3·23}$$

with the condition

$$\sum_{l=1}^{l \ll N} l m_l = N. \tag{A3·24}$$

Using the undetermined multiplier $\ln z$, we obtain* from $(\partial/\partial m_l)\{\ln T(0, m_l) + (\sum_{l=1}^{l \ll N} l m_l) \cdot \ln z\}$

$$\bar{m}_l = N v b_l z^l \quad (l=1, 2, 3, \cdots \ll N), \tag{A3·25}$$

where z is determined by

$$N \sum_{l=1}^{l \ll N} l v b_l z^l = N. \tag{A3·26}$$

Hence, unless $v \to 0$ we have

$$[1/v=] \sum_{l=1}^{l \ll N} l b_l z^l = \sum_{l=1}^{\infty} l b_l z^l = \text{convergent} \tag{A3·26'}$$

and

$$\sum_{l=1}^{l \ll N} b_l z^l = \sum_{l=1}^{\infty} b_l z^l = \text{convergent}.$$

Thus from (A3·23) and (A3·25) we obtain

$$\ln \Omega_N \backsimeq N \sum_{l=1}^{\infty} v b_l z^l - N \ln z \tag{A3·27}$$

and

$$p = kT \sum_{l=1}^{\infty} b_l z^l \tag{A3·28}$$

and

$$G \backsimeq NkT \ln(\lambda^3 z), \tag{A3·29}$$

---

* Here the defect (B) in § A1 is avoided, since there is no $m_L \sim 1$.

which shows that z is the fugacity (or activity) of the system. This corresponds to the gaseous state.

Thus, v_s [(A3·17), (A3·22)] and z_s [(A3·13)] are respectively the specific volume and the fugacity at the starting-point of condensation. Using the Cauchy-Hadamard theorem, we see from (A3·5) that z_s $(\equiv b_0^{-1})$ is the first singularity (on the real positive axis of z) of the power series p [(A3·28)] and $1/v$ [(A3·26′)].

We shall now show a more intelligible method for calculating the maximum term (A3·11). Here for brevity we shall use the usual equality symbol $=$ even when we should use $\doteqdot$; and we shall use the concise symbol $\sum_l$ for $\sum_{l=1}^{l\leqq N}$.

Now we have, by (A3·9) and (A3·10),

$$\Omega_N = \sum_{N_2,\, m_l} \Big(\sum_l l m_l + N_2 = N\Big)\, T(N_2,\, m_l), \tag{A3·30}$$

where

$$\ln T(N_2,\, m_l) = \sum_l m_l \{\ln(N v b_l) - \ln m_l + 1\} + N_2 \ln b_0. \tag{A3·31}$$

Hence

$$\ln \Omega_N = \operatorname{Max} \ln T(N_2,\, m_l) = \ln T(\bar{N}_2,\, \bar{m}_l), \tag{A3·32}$$

subject to the condition

$$\sum_l l m_l + N_2 = N. \tag{A3·33}$$

Then

$$\delta \ln T(N_2,\, m_l) = \sum_l \{\ln(N v b_l) - \ln m_l\}\, \delta m_l + \ln b_0 \cdot \delta N_2, \tag{A3·34}$$

$$\sum_l l\, \delta m_l + \delta N_2 = 0. \tag{A3·35}$$

Eliminating δN_2 from (A3·34) and (A3·35), we have

$$\delta \ln T(N_2,\, m_l) = \sum_l \{\ln(N v b_l) - \ln m_l - l \ln b_0\}\, \delta m_l. \tag{A3·36}$$

Every δm_l is now independent. For the maximum, $\delta \ln T(N_2,\, m_l) = 0$ for every arbitrary δm_l; hence

$$\bar{m}_l = N v b_l b_0^{\,l}. \tag{A3·37}$$

Putting

$$b_0^{-1} \equiv z_s, \tag{A3·38}$$

we have

$$\bar{m}_l = N v b_l z_s^{\,l}. \tag{A3·39}$$

Hence by (A3·33),

$$\bar{N}_2 = N - \sum_l l \bar{m}_l = N - N \sum_l l v b_l z_s^{\,l}. \tag{A3·40}$$

Now we put

$$v_s \equiv \Big(\sum_l l b_l z_s^{\,l}\Big)^{-1}. \tag{A3·41}$$

If $v \leqq v_s$, then $\bar{N}_2 \geqq 0$ by (A3·40), so the solution (A3·39) has a meaning. If $v > v_s$, then $\bar{N}_2 < 0$ by (A3·40), so the solution (A3·39) has no meaning. We should have calculated the maximum under the conditions that $N_2 \geqq 0$ and $m_l \geqq 0$ as well as that $\sum_l l m_l + N_2 = N$.

We shall then calculate the maximum (denoted by Max$\dagger$) of $\ln T(N_2,\, m_l)$ when N_2 is fixed at an arbitrary value. Thus

$$\operatorname{Max}\dagger \ln T(N_2,\, m_l) = \ln T(N_2,\, m_l\dagger(N_2)), \tag{A3·42}$$

subject to the condition

$$\sum_l l m_l = N - N_2. \tag{A3·43}$$

[We see that the $m_l\dagger$ (that is, the m_l for this maximum) are functions of N_2.] Then, using the undetermined multiplier $\ln \zeta$, we obtain from $(\partial/\partial m_l)\{\ln T(N_2,\, m_l) + (\sum_l l m_l) \ln \zeta\} = 0$

$$m_l\dagger(N_2)=Nvb_l\zeta^l, \tag{A3·44}$$

where $\zeta=\zeta(N_2)$ is determined by

$$\sum_l lNvb_l\zeta^l=N-N_2. \tag{A3·45}$$

Therefore by (A3·31),

$$\ln T(N_2,\ m_l\dagger(N_2))=N\sum_l vb_l\zeta^l-(N-N_2)\ln\zeta+N_2\ln b_0. \tag{A3·46}$$

Hence, using (A3·45), we have

$$(d/dN_2)\ln T(N_2,\ m_l\dagger(N_2))=\ln(b_0\zeta). \tag{A3·47}$$

From (A3·45) we see that ζ [and therefore $\ln(b_0\zeta)$] is a monotonic decreasing function of N_2; and we see that

$$\ln(b_0\zeta)=0 \quad \text{if } N_2=\bar{N}_2[=N(v_s-v)/v_s],$$
$$\ln(b_0\zeta)<0 \quad \text{if } N\geqq N_2>\bar{N}_2[=N(v_s-v)/v_s]. \tag{A3·48}$$

Thus we have the figures 1, 2, and 3 for the behaviour of (A3·42), [i.e. (A3·46)] as a function of N_2. [Here we may note that, for the region where $b_0\zeta>1$ (that is, $N_2<\bar{N}_2$)—the broken-curve part in the figures—, the series $\sum_l lvb_l\zeta^l$, (A3·45), is divergent and so the solution (A3·44) has no meaning.]

Therefore, if $v<v_s$, the largest term having a meaning is given by $N_2=\bar{N}_2$ in Fig. 1 and by $m_l\dagger(\bar{N}_2)$ in (A3·44) and (A3·45), that is, $\bar{m}_l$ in (A3·39); thus we have

$$\ln\Omega_N=\ln T(\bar{N}_2,\ \bar{m}_l)=N\sum_l vb_l z_s^l - (N-\bar{N}_2)\ln z_s+\bar{N}_2\ln b_0$$
$$=N\sum_l vb_l z_s^l-N\ln z_s. \tag{A3·49}$$

This corresponds to the condensation range. On the other hand, if $v>v_s$, the largest term having a meaning is given by $N_2=0$ in Fig. 3 and by $m_l\dagger(0)$ in (A3·44) and (A3·45), that is, $\bar{m}_l$ such that

$$\bar{m}_l=Nvb_l z^l \tag{A3·50}$$

where z is determined by

$$\sum_l lNvb_l z^l=N; \tag{A3·51}$$

thus we have

$$\ln\Omega_N=\ln\ T(0,\ m_l\dagger(0))=N\sum_l vb_l z^l-N\ln z. \tag{A3·52}$$

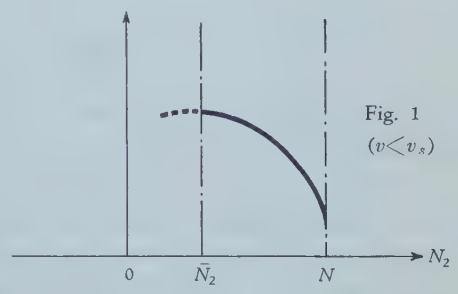

Fig. 1 ($v<v_s$)

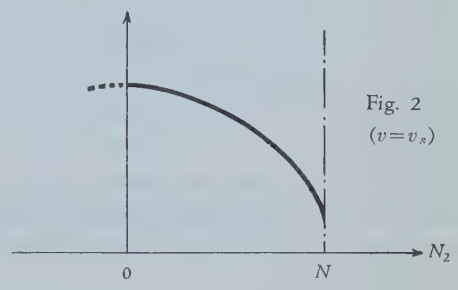

Fig. 2 ($v=v_s$)

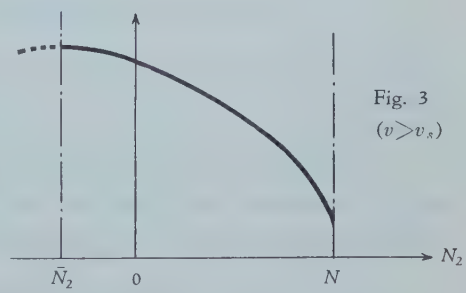

Fig. 3 ($v>v_s$)

Figs. 1, 2, 3: The ordinate is $\ln T(N_2, m_l\dagger(N_2))$.

This corresponds to the gaseous state. Finally, if $v=v_s$ (the starting point of condensation), the above two arguments are both applicable.

§ A4. Criticism on Mayer's theory

Now, making a comparison with the new method given in § A3, we shall again criticize Mayer's method discussed in § A1.

In § A3, in the case that $v\geqq v_s$ (that is, for the gaseous state), we have no "large"

cluster in the maximum term or the predominant term ; so the approximations (α) and (β) [stated in § A1] may safely be applied. Hence Mayer's method (§ A1) is valid in this case. In fact, from the results (A1·6) and (A1·7) of Mayer's method we have for very large N

$$1/v=\sum_{l=1}^{\infty} l b_l z^l \tag{A4·1}$$

$$\ln \Omega_N \simeq N\sum_{l=1}^{\infty} v b_l z^l - N \ln z \tag{A4·2}$$

which are identical respectively with the equations (A3·26′) and (A3·27) given by the new method.

However, in § A3, in the case that $v<v_s$ (that is, for the condensation range), we have some "large" clusters in the predominant term ; so the full application of the approximations (α) and (β) is not justified. Hence Mayer's method (§ A1) is invalid in this case. In fact, by Mayer's method (§ A1) we obtain eq. (A4·1) for $v<v_s$ as well as for $v\geqq v_s$, but eq. (A4·1) has no meaning for $v<v_s$, since $1/v_s$ is the largest value which the right-hand side of eq. (A4·1) can reach (in the reasonable mathematical sense). Trying to deduce a condensation from (A4·1), one might be tempted to consider that $1/v$ increases without bound for z just beyond the singularity z_s, where, however, eq. (A4·1) itself ceases to have a meaning. This does not imply any mathematical proof of the horizontality (for $v<v_s$) of the isotherm. [Even if one takes a large finite N as the upper limit of the sum in (A4·1) and makes z go slightly beyond the singularity z_s, the mathematical unreasonableness cannot be avoided.]

§ A5. Expression of b_L in terms of β_k

In this section and the latter half of § A6, we shall assume (0·4′) and (0·10) of the text.

We shall now attempt to give a more detailed expression to the last term ($\ln b_{\bar{N}_2}$) of the expression (A3·18) for $\ln \Omega_N$ in the condensation range. Of course, $\bar{N}_2$ is "large". Hereafter, for convenience we shall write L for $\bar{N}_2$.

The cluster integrals b_l are expressed as[5),7)]

$$b_l=(1/l!)\sum_{n_k\geqq 0}\Big(\sum_{k=1}^{l-1} k n_k=l-1\Big)\prod_{k=1}^{l-1}\{(l\beta_k)^{n_k}/n_k!\}, \tag{A5·1}$$

where

$$\beta_k=(1/k!\,V)\int_V\cdots\int_V \sum_{k+1\geqq i>j\geqq 1}^{\text{frame}} \prod f_{ij}\, d\{k+1\} \tag{A5·2}$$

are called the "irreducible cluster integrals" or the "frame integrals".* Now, taking L for l, we have

* For the definition of the concept of a "frame", i. e., an "irreducible cluster", see the literatures.[5),7)]

$$\ln b_L \backsimeq \ln \tilde{b}_L \tag{A5·3}$$

where

$$\tilde{b}_L = \sum_{n_k \geq 0} \Big(\sum_{k=1}^{L} k n_k = L\Big) \prod_{k=1}^{L} \{(L\beta_k)^{n_k}/n_k!\}. \tag{A5·4}$$

For the asymptotic evaluation of the expression (A5·4) when L is large, we may apply the same method as in § A3.

In the same way as before, we classify k (that is, the sizes of the frames) into three kinds, viz. "large" ($k \sim L \sim N$) and "small" ($k \sim 1$) and "intermediate" ($1 \ll k \ll L \sim N$), when $L(\sim N)$ is large. Hereafter, for clearness' sake we write K for "large" k, though we write simply k for other k. Now n_k is the number of frames of $k+1$ molecules. Then $n_K \sim 1$ (if not zero) for $K \sim L$, and $n_k \sim 1$ or $n_k \gg 1$ (if not zero) for $k \ll L$.

Let N_2' be the total number of those molecules which compose "small" frames and frames "of intermediate size", and let N_3 be the total number of those molecules which compose "large" frames. Then we have

$$\sum_{k=1}^{k \ll L} k n_k = N_2', \quad \sum_{K \sim L}^{K=L} K n_K = N_3, \quad N_2' + N_3 = L. \tag{A5·5}*$$

Here we may note that N_3 is "large" or else zero, while N_2' may have any kind of magnitude.

Consequently we may write (A5·4) as

$$\begin{aligned}\tilde{b}_L &= \sum_{n_k, n_K} \Big(\sum_{k=1}^{k \ll L} k n_k + \sum_{K \sim L}^{K=L} K n_K = L\Big) \prod_{k=1}^{k \ll L} \{(L\beta_k)^{n_k}/n_k!\} \prod_{K \sim L}^{K=L} \{(L\beta_K)^{n_K}/n_K!\} \\ &= \sum_{N_3, n_k} \Big(\sum_{k=1}^{k \ll L} k n_k + N_3 = L\Big) \prod_{k=1}^{k \ll L} \{(L\beta_k)^{n_k}/n_k!\} \sum_{n_K} \Big(\sum_{K \sim L}^{K=L} K n_K = N_3\Big) \prod_{K \sim L}^{K=L} \{(L\beta_K)^{n_K}/n_K!\}.\end{aligned} \tag{A5·6}$$

Here we assume that β_k, β_K are independent of the volume of the vessel, and that $\beta_k > 0$, $\beta_K > 0$ for all k, K.

Then the logarithm of any term in $\sum_{n_K}$ in (A5·6) becomes

$$\ln \prod_{K \sim L}^{K=L} \{(L\beta_K)^{n_K}/n_K!\} \backsimeq \sum_{K \sim L}^{K=L} K n_K \ln \beta_0 = N_3 \ln \beta_0, \tag{A5·7}$$

and in particular the logarithm of the term for which $n_{N_3} = 1$ and $n_K = 0$ $(K \neq N_3)$ becomes

$$\ln (L\beta_{N_3}) \backsimeq \ln \beta_{N_3} \backsimeq N_3 \ln \beta_0. \tag{A5·8}$$

In deriving (A5·7) and (A5·8) we have used the fact that

$$\lim_{k \to \infty} \beta_k^{1/k} = \beta_0 = \text{definite and finite and positive.} \tag{A5·9}**$$

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† The difference between $L-1$ and L is negligible.

* In the second equation the upper limit $K=L$ is in fact $K=N_3(\leqq L)$. The same applies hereinafter.

** This is an assumption for β_k.

that is, $\beta_k = g(k)\beta_0^k$ [where $\lim_{k\to\infty}(1/k)\ln g(k) = 0$] and hence

$$\beta_K = g(K)\beta_0^K \tag{A5·10}$$

where

$$\ln g(K) \ll K \sim L \sim N, \tag{A5·10a}$$

and we have also used the fact that $n_K \sim 1$ and $\sum_{K\sim L}^{K=L} 1 \sim 1$ and $\ln L \ll L$.

Then we have by (A5·7) and (A5·8)

$$\ln \sum_{n_K} \Big(\sum_{K\sim L}^{K=L} K n_K = N_3\Big) \prod_{K\sim L}^{K=L} \{(L_l\beta_K)^{n_K}/n_K!\} \backsimeq \ln(L_l\beta_{N_3}) \backsimeq \ln\beta_{N_3} \backsimeq N_3 \ln\beta_0, \tag{A5·11}$$

since the logarithm of the total number of terms in $\sum_{n_K}$ is $\ll N$.

By eqs. (A5·7) and (A5·8) we see that the situation where the N_3 molecules compose an arbitrary number of "large" frames is equivalent to the situation where the N_3 molecules together compose only one "large" frame. That is to say, we cannot decide how many "large" frames are present; thus for convenience we may consider that only one "large" frame (β_{N_3}) is present.

Next we have

$$\ln \prod_{k=1}^{k\ll L} \{(L_l\beta_k)^{n_k}/n_k!\} \backsimeq \sum_{k=1}^{k\ll L} n_k \{\ln(L_l\beta_k) - \ln n_k + 1\} \tag{A5·12}$$

by applying the Stirling approximation to n_k (for $k \ll L$).

Thus we have

$$\tilde{b}_L = \sum_{N_3, n_k} \Big(\sum_{k=1}^{k\ll L} k n_k + N_3 = L\Big) T(N_3,\ n_k), \tag{A5·13}$$

where

$$\begin{aligned} \ln T(N_3,\ n_k) &\backsimeq \sum_{k=1}^{k\ll L} n_k \{\ln(L_l\beta_k) - \ln n_k + 1\} + N_3 \ln\beta_0 \\ &\backsimeq \sum_{k=1}^{k\ll L} n_k \{\ln(L_l\beta_k) - \ln n_k + 1\} + \ln\beta_{N_3}. \end{aligned} \tag{A5·14}$$

Since the logarithm of the total number of terms in $\sum_{N_3, n_k}$ in (A5·13) is $\ll L$, we have

$$\ln b_L \backsimeq \ln\tilde{b}_L \backsimeq \mathrm{Max}\ln T(N_3,\ n_k) = \ln T(\bar{N}_3,\ \bar{n}_k). \tag{A5·15}$$

Here the maximum is taken under the condition

$$\sum_{k=1}^{k\ll L} k n_k + N_3 = L, \tag{A5·16}$$

and so, using the undetermined multiplier $\ln y$, we obtain (in the same way as before)

$$y = \beta_0^{-1} (\equiv y_s,\ \text{say}) \tag{A5·17}$$

and

$$\bar{n}_k = L_l\beta_k y^k = L_l\beta_k y_s^k \quad (k = 1,\ 2,\ 3, \cdots \ll L) \tag{A5·18}$$

from differentiation in N_3 and in n_k $(k=1, 2, 3, \cdots \ll L)$ respectively. Thus the values of N_2' and N_3 for the maximum term are

$$\bar{N}_2' = \sum_{k=1}^{k \ll L} k\bar{n}_k = L \sum_{k=1}^{k \ll L} k\beta_k \gamma_s^k \tag{A5·19}$$

and

$$\bar{N}_3 = L - L \sum_{k=1}^{k \ll L} k\beta_k \gamma_s^k, \tag{A5·20}$$

respectively.

Then in order that the solution just now obtained may have a physical meaning, it is necessary that $\sum_{k=1}^{k \ll L} k\beta_k \gamma_s^k$ should be definite and finite and $\sum_{k=1}^{k \ll L} k\beta_k \gamma_s^k \leqq 1$ (since $\bar{N}_3 \geqq 0$). In order that this may be so, firstly, the series $\sum_{k=1}^{\infty} k\beta_k \gamma_s^k \left[= \sum_{k=1}^{\infty} kg(k) \right.$ where $\left. \beta_k = g(k)\beta_0^k \right]$ should be convergent* (since some of the k's "of intermediate magnitude" are very large compared with 1). In this case we have (with a negligible error) $\sum_{k=1}^{k \ll L} k\beta_k \gamma_s^k = \sum_{k=1}^{\infty} k\beta_k \gamma_s^k$ and $\sum_{k=1}^{k \ll L} \beta_k \gamma_s^k = \sum_{k=1}^{\infty} \beta_k \gamma_s^k \left[= \sum_{k=1}^{\infty} g(k) = \text{convergent} \right]$. Secondly, the relation $\sum_{k=1}^{\infty} k\beta_k \gamma_s^k \leqq 1$ should hold. That is, [since in general $\sum_{k=1}^{\infty} k\beta_k \gamma_s^k$ is assumed to be increasing with the temperature], if there exists a temperature T_m such that

$$\sum_{k=1}^{\infty} k\beta_k(T_m) \gamma_s^k(T_m) = 1, \tag{A5·21}$$

then the temperature T should satisfy

$$T \leqq T_m. \tag{A5·21'}$$

Under the above conditions we obtain [from (A5·15), (A5·14), and (A5·18)]

$$\ln b_L \backsimeq L \sum_{k=1}^{k \ll L} \beta_k \gamma_s^k - \bar{N}_2' \ln \gamma_s + \ln \beta_{\bar{N}_3} \tag{A5·22}$$

with (A5·19) and (A5·20). Here we may (with a negligible error) replace the upper limit $k \ll L$ of the sum by $k = \infty$. Thus (A5·22) becomes, with the use of $\gamma_s \equiv \beta_0^{-1}$ and (A5·8),

$$\ln b_L \backsimeq L \sum_{k=1}^{\infty} \beta_k \gamma_s^k - L \ln \gamma_s. \tag{A5·23}$$

From (A3·6) we have

$$\ln b_L \backsimeq L \ln b_0 \equiv -L \ln z_s. \tag{A5·23a}$$

As is seen from (A5·9), $\gamma_s \left[= \beta_0^{-1} \right]$ is the first singularity (on the real positive axis of γ) of the series $\sum_{k=1}^{\infty} \beta_k \gamma^k$ and $\sum_{k=1}^{\infty} k^\nu \beta_k \gamma^k$ $(\nu = 1, 2, 3, \cdots)$ (by the Cauchy-Hadamard theorem).

Now if

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* If this is divergent, then (A5·25) holds and the argument (A5·26) to (A5·30) is valid.

$$T > T_m, \tag{A5·24}$$

then we have

$$\sum_{k=1}^{\infty} k\beta_k \gamma_s^k > 1 \tag{A5·25}$$

and $\bar{N}_3 < 0$ in (A5·20), and so we should have $N_3 = 0$ for the largest term that has a physical meaning. In this case we have

$$\ln b_L \backsimeq \mathrm{Max} \ln T(0, n_k) = \ln T(0, \bar{n}_k) \tag{A5·26}$$

with the condition

$$\sum_{k=1}^{k \ll L} k n_k = L. \tag{A5·27}$$

Using the undetermined multiplier $\ln \gamma_0$, we obtain from $(\partial/\partial n_k)\{\ln T(0, n_k) + (\sum_{k=1}^{k \ll L} k n_k) \cdot \ln \gamma_0\} = 0$

$$\bar{n}_k = L\beta_k \gamma_0^k \quad (k = 1, 2, 3, \cdots \ll L), \tag{A5·28}$$

where γ_0 is determined by

$$L \sum_{k=1}^{k \ll L} k\beta_k \gamma_0^k = L. \tag{A5·29}$$

Hence we have

$$[1 =] \sum_{k=1}^{k \ll L} k\beta_k \gamma_0^k = \sum_{k=1}^{\infty} k\beta_k \gamma_0^k = \text{convergent},$$

$$\sum_{k=1}^{k \ll L} \beta_k \gamma_0^k = \sum_{k=1}^{\infty} \beta_k \gamma_0^k = \text{convergent}.$$

Thus from (A5·26) and (A5·28) we obtain

$$\ln b_L \backsimeq L \sum_{k=1}^{\infty} \beta_k \gamma_0^k - L \ln \gamma_0. \tag{A5·30}$$

From (A3·6) we have

$$\ln b_L \backsimeq L \ln b_0 \equiv -L \ln z_s. \tag{A5·30a}$$

§ A6. Physical interpretation—especially for the condensation range

((From § A3)) If $v > v_s$ we obtain the fugacity z from (A3·26), and $\ln \Omega_N$ from (A3·27); and the system in thermodynamic equilibrium consists of a "large" number of "small" and "medium"* clusters, whose size distribution is given by (A3·25); and, in determining the pressure, each cluster plays the role played by one molecule of a perfect gas, as pointed out by Mayer[5]; –((the gaseous state)). If $v < v_s$, the fugacity (z_s) is independent of v, and $\ln \Omega_N$ is obtained from (A3·18) or (A3·19); and in

* The adjective "medium" is an abbreviated expression for "of intermediate size (magnitude)".

thermodynamic equilibrium the $\bar{N}_1$ molecules [(A3·15)] compose a "large" number of "small" and "medium" clusters, whose size distribution is given by (A3·14), and the remaining $\bar{N}_2$ molecules [(A3·16)] compose *one** "large" cluster ($\ln b_{\bar{N}_2}$). As v decreases, $\bar{N}_1/V$ is constant and equal to $1/v_s$ and $\bar{N}_2$ (also $\bar{N}_2/V$) increases; ($N_2=0$ when $v=v_s$),

$$N/V=\bar{N}_1/V+\bar{N}_2/V \quad \text{or} \quad 1/v=1/v_1+1/v_2 \ (v_1\equiv v_s). \tag{A6·1}$$

The former part is considered as the saturated vapour. The pressure of the system takes the constant value p_s [(A3·20)] (the saturated vapour pressure) which arises from the saturated set of "small" and "medium" clusters [(A3·14) and (A3·15)], each cluster playing the role played by one molecule of a perfect gas; and the "large" cluster $\ln b_{\bar{N}_2}$ makes no contribution to the pressure. This "large" cluster is considered to be the condensed phase which has appeared by two-phase separation;—((the condensation range)).

((From § A5)) More detailed knowledge of the condensation range:—

If $T>T_m$, eqs. (A5·26) to (A5·30) hold for the last term $\ln b_{\bar{N}_2}$ of eq. (A3·18) and hence we have

$$\ln \Omega_N \backsimeq N\sum_{l=1}^{\infty} v b_l z_s^l - \bar{N}_1 \ln z_s + (N-\bar{N}_1)\sum_{k=1}^{\infty}\beta_k \gamma_0^k - (N-\bar{N}_1)\ln\gamma_0, \tag{A6·2}$$

$$\begin{aligned} N &= N\sum_{l=1}^{\infty} l v b_l z_s^l + (N-\bar{N}_1)\sum_{k=1}^{\infty} k\beta_k\gamma_0^k \\ &= \quad \bar{N}_1 \quad + \quad \bar{N}_2. \end{aligned} \tag{A6·3}$$

From these we see that the $\bar{N}_1$ molecules [(A3·15)] compose the saturated vapour as stated before, and that the $N-\bar{N}_1$ molecules compose a connected set of a large number of "small" and "medium" frames whose size distribution is given by (A5·28) in thermodynamic equilibrium (or, in other words, in maximum probability). Any term of the sum (A5·1) or (A5·4)—and therefore the largest term (which we are now considering)—contains the integrals for all possible manners of connecting together the frames whose size distribution is given. Hence, physically speaking, in any infinitesimal time interval these frames are connected together in *every* possible manner.

If $T<T_m$, eqs. (A5·15) to (A5·23) hold for the last term $\ln b_{\bar{N}_2}$ of eq. (A3·18) and hence we have

$$\ln \Omega_N \backsimeq N\sum_{l=1}^{\infty} v b_l z_s^l - \bar{N}_1 \ln z_s + (N-\bar{N}_1)\sum_{k=1}^{\infty}\beta_k \gamma_s^k - \bar{N}_2' \ln\gamma_s - (N-\bar{N}_1-\bar{N}_2')\ln\gamma_s, \tag{A6·4}$$

$$\begin{aligned} N &= N\sum_{l=1}^{\infty} l v b_l z_s^l + (N-\bar{N}_1)\sum_{k=1}^{\infty} k\beta_k\gamma_s^k + (N-\bar{N}_1-\bar{N}_2') \\ &= \quad \bar{N}_1 \quad + \quad \bar{N}_2' \quad + \quad \bar{N}_3. \end{aligned} \tag{A6·5}$$

From these we see that the $\bar{N}_1$ molecules [(A3·15)] compose the saturated vapour, and that the $N-\bar{N}_1$ molecules composing the "large" cluster can be classified into two

---

* Or *a "small" number of……s.*

groups, one group consisting of $\bar{N}_2'$ molecules [(A5·19)], that is, a connected set of "small" and "medium" frames which is the same as before except that the size distribution is now given by (A5·18) instead of (A5·28), and the other group consisting of $\bar{N}_3$ $(=N-\bar{N}_1-\bar{N}_2')$ molecules [(A5·20)] which compose *one** "large" frame $(\ln \beta_{\bar{N}_3})$.

§ A7. Concluding remarks

In §§ A2 and A3, using an improved method in which the defects (A) and (B) [discussed in § A1] are avoided, we have obtained better results. However, the method of §§ A2 and A3 still has the following defects. (The same is true for § A5.)

(1) The way of classifying the l's (and any other quantities) into "large" and "small" and "of intermediate magnitude" is rather ambiguous. That is, the border-lines between these classes have been not so well defined. We can only say that we have defined the border-lines in some suitable way which is believed to give as good an approximation as possible. [But it is very probable that the well-defined positions of the border-lines are not essential to the theory.] We have yet to make this point clearer.

(2) We have assumed that the application of the approximations (α) and (β) is justified for "large" m_l and for m_l "of intermediate magnitude". We have yet to attempt to exclude every approximate procedure.

The theory given in the text is substantially similar to the theory given in §§ A2 and A3, but is a completely *rigorous* one which does not contain either of the above-mentioned defects (1) and (2). [The rigorous theory corresponding to § A5 can be given in a similar manner but it has been omitted in the text.]

Appendix B

Comparison with other theories of condensation

Introduction and summary

We wish to compare our theory with other theories of condensation, in order to make clear the essential features of our theory. We have already compared our theory with Mayer's theory of condensation [see §§ A1 and A4 of Appendix A]. In this appendix we shall compare our theory with (i) F. London's and Kahn-Uhlenbeck's theories of Bose-Einstein condensation and (ii) the condensation theory based on the saddle-point method (or steepest descents method).

First we show that the advantage of our theory over Mayer's theory is somewhat analogous to that of F. London's theory over Kahn-Uhlenbeck's theory, and second we show (from another point of view) that our theory is, in a sense, equivalent to Schubert-Dingle's theory criticizing the saddle-point method.

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* Or *a "small" number of……s.*

§ B1. F. London's and Kahn-Uhlenbeck's theories of Bose-Einstein condensation

There are two theories attempting to treat the perfect Bose-Einstein gas : one is due to Kahn and Uhlenbeck[6] and the other is due to F. London.[9]

By a method analogous to Mayer's cluster theory, Kahn and Uhlenbeck[6] have derived the following equations for the perfect Bose-Einstein gas (consisting of N molecules each of mass m) in a volume $V(=Nv)$ and at a temperature T :

$$1/v=\sum_{l=1}^{\infty} l b_l z^l, \tag{B1·1}*$$

$$p=kT\sum_{l=1}^{\infty} b_l z^l, \tag{B1·2}$$

where

$$b_l=\lambda^{3(l-1)}/l^{5/2}, \tag{B1·3}$$

where

$$\lambda=h/\sqrt{2\pi mkT}. \tag{B1·4}$$

On the other hand, F. London[9] starts from the well-known formula for the Bose-Einstein distribution

$$N_i=g_i/(e^{\beta\epsilon_i+\alpha}-1) \quad (i=1, 2, 3, \cdots), \tag{B1·5}$$

where $\beta=1/kT$, and $\alpha[=\alpha(T)]$ is determined by eq. (B1·7), and g_i is the degree of degenerary of the i-th energy level ϵ_i,

$$\epsilon_i=\epsilon_{n_x, n_y, n_z}=(n_x^2+n_y^2+n_z^2)(h^2/8mL^2) \tag{B1·6}$$

$$\epsilon_1<\epsilon_2<\epsilon_3<\cdots$$

(n_x, n_y, and n_z being positive integers, and $V=L^3$). Then

$$N=\sum_i N_i=\sum_i g_i/(e^{\beta\epsilon_i+\alpha}-1). \tag{B1·7}$$

If one replaces the sum (B1·7) by the integration over p from 0 to ∞ [where $\epsilon_i\to\epsilon_p=p^2/2m$, $g_i\to g(p)dp=(4\pi V/h^3)p^2dp$], then one reaches the equation (B1·1)** obtained by Kahn and Uhlenbeck. Eq. (B1·1) is valid for $v\geqq v_s$, but it has no meaning for $v<v_s$ [where $v_s=(\sum_{l=1}^{\infty} l b_l z_s^l)^{-1}$, and here $z_s(\equiv\lambda^{-3})$ is the first singularity (on the real positive z-axis) of the series (B1·1)], since $1/v_s$ is the largest value which the right-hand side of eq. (B1·1) can reach. This difficulty is analogous to that of the result (A4·1) obtained by Mayer's method for the imperfect gas. In order to avoid this difficulty and obtain a better result, however, F. London divides the sum (B1·7) into two parts :

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* $z=a/\lambda^3=(1/\lambda^3)\exp(\mu/kT)$; a=absolute activity, μ=chemical potential.

** Here $z=e^{-\alpha}/\lambda^3$.

$$N=\sum_{i\leq i^*} g_i/(e^{\beta\epsilon_i+\alpha}-1)+\sum_{i>i^*} g_i/(e^{\beta\epsilon_i+\alpha}-1) \qquad \text{(B1·8)}$$

[where i^* is a suitable small positive integer† which may make the following approximation as good as possible]; the former part contains the lowest energy levels [for which $i=1$ (lowest) or i is comparable with 1] and the latter contains the other levels. The approximation in which the sum is replaced by the integration is suitable only for the latter part, while the former part must be treated in a different way. Thus London obtains

$$N=g_0/(e^{\beta\epsilon_0+\alpha}-1)+(4\pi V/h^3)\int_{\rho h/2L}^{\infty} p^2\,dp/(e^{\alpha+p^2/2mkT}-1)$$

$$=N_0+V\sum_{l=1}^{\infty} l\,b_l z^l, \qquad \text{(B1·9)}$$

[where $\epsilon_0=(3/40)(\rho^2h^2/mV^{2/3})$, $g_0=(\pi/6)\rho^3$, $z=e^{-\alpha}/\lambda^3$, and ρ is the value of $(n_x^2+n_y^2+n_z^2)^{1/2}$ for $i=i^*$]. By this he has established the well-known theory of Bose-Einstein condensation; that is, he has deduced that, if $v<v_s$, the N_0 particles [given by the first term in (B1·9)] are condensed into the lowest energy level (i. e. the two-phase separation in the momentum space) and that the isothermal $p-v$ curve is horizontal for $v<v_s$.

This theory due to F. London is analogous to our theory of condensing systems, in that the whole sum is divided into two parts, which are then treated in the respective suitable ways. London's first term [in (B1·9)] or N_0 is somewhat analogous to the last term $\ln\sum\Phi$ [in (5·43)], $\ln\sum\bar{\Phi}$ [in (7·6)], $\ln b_{N_2}$ [in (A3·10)], $\ln b_{\bar{N}_2}$ [in (A3·18)], or $\bar{N}_2$ [in (A3·16) or (7·5)] (i. e. the liquid phase) in our theory.

The Kahn-Uhlenbeck theory[6] of condensation (which covers both cases of the perfect Bose-Einstein gas and the imperfect gas) does not deduce the two-phase separation in the momentum space (for the case of the perfect Bose-Einstein gas) or in the usual space (for the case of the imperfect gas), i. e., the appearance of the part condensed into the lowest enery level (i. e. the so-called "superfluid" part) [which is deduced by F. London[9]] or the appearance of the liquid phase [which is deduced by our theory].

§ B2. Condensation theory based on the saddle-point method

Besides the maximum-term method (§ A1), there is the saddle-point method (or the steepest descents method) for treating the Ursell expansion for the imperfect gas and discussing the condensation. The saddle-point method was applied to the case of the imperfect gas by Born and Fuchs.[7] But, as was shown by Schubert[10] and Dingle,[11] the saddle-point method, when it is applied to the perfect Bose-Einstein gas, is invalid for that range in which the Bose-Einstein condensation may occur. It may be shown that this criticism of Schubert and Dingle on the saddle-point method for the Bose-Einstein condensation is also true for the condensation of the imperfect gas (treated in

† It is shown that the final result is practically independent of i^* (if i^* is small), so one may put $i^*=1$.

this paper). Thus the saddle-point method, when it is applied to the imperfect gas, is valid for $v > v_s$, but invalid for $v \leqq v_s$; for, when $v \leqq v_s$, the saddle-point and the singular point coincide with each other in position and the method breaks down. Hence the maximum-term method discussed in § A1 and the saddle-point method just discussed have the same range of validity. This is also consistent with the fact that (as was shown by Born and Fuchs[7]) these two methods give the same result. Thus, in short, we may say that the saddle-point method is essentially equivalent to the maximum-term method discussed in § A1. It should be remarked that our theory of condensing systems is quite free from the above-mentioned faults contained in the older methods, i. e., the maximum-term method and the saddle-point method.

One should not confuse the argument in this section with that in § B1.

In § B1 we assumed the Bose-Einstein distribution function (B1·5) to be valid for the whole range, and showed an analogy between our theory of condensation of the imperfect gas and F. London's theory [which started from (B1·5) and reached the Bose-Einstein condensation].

However, as is stated in this section, Schubert and Dingle showed that the derivation of (B1·5) itself from the partition function of the perfect Bose-Einstein gas by means of the saddle-point method is invalid for the condensation range, and thus criticized London's theory which started from (B1·5). Hence we can also show the mathematical equivalence (in a sense) between this criticism or theory of Schubert-Dingle (for the case of the perfect Bose-Einstein gas) and our criticism or theory (for the case of the imperfect gas).

In short, from the two different standpoints we can compare our theory with both London's and Schubert-Dingle's theories.

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Note added in proof. Lemma 6 on page 360 of ref. 8c is still true if (5) and (6) are replaced by $v \geqq v_s$ and $z \leqq z_s$ respectively, and if (d) in § 0 of the present paper is assumed. [Note that, when $v = v_s$ (i.e., $z = z_s$), $Q_4(N)$ exactly vanishes and (41) – (44) are still true.]

Errata

On the Theory of Condensation [Kazuyosi Ikeda, Proc. Internat. Conf. Theor. Phys. Kyoto and Tokyo (1953), 544]

p. 544, line 11 ;	$f_{ij}=\exp\{-u(r_{ij})/kT\}$	should read	$f_{ij}=\exp\{-u(r_{ij})/kT\}-1$
p. 550, line 24 ;	(I)	,,	(1)

On the Theory of Condensing Systems [Kazuyosi Ikeda, Prog. Theor. Phys. **11** (1954), 336]

p. 336, Eqs. (7), (10) ;	bN_2	should read	b_{N_2}
p. 337, left, lines 10, 11 ;	$(l=1, 2, \cdots \ll l)$	,,	$(l=1, 2, \cdots \ll N)$
p. 337, Eq. (16) ;	$lub_l z_s{}^l$	,,	$lvb_l z_s{}^l$
p. 337, Eq. (18) ;	$bN_2{}^*$	,,	$b_{N_2{}^*}$
p. 337, right, line 3 ;	$\lambda^{-3N} Q_N$	,,	$\lambda^{-3N} \mathcal{Q}_N$
p. 337, Eq. (23) ;	$=\ln T(0, m_l)$	,,	$=\ln T(0, m_l{}^*)$
p. 338, left, lines 28, 29 ;	bN_2	,,	b_{N_2}
p. 338, left, lines 28, 29, 39 ;	$bN_2{}^*$	,,	$b_{N_2{}^*}$
p. 338, Eq. (34) ;	$\beta N_3{}^*$	,,	$\beta_{N_3{}^*}$
p. 338, reference 2) ;	10	,,	101

On the Theory of Condensation [Kazuyosi Ikeda, Prog. Theor. Phys. **16** (1956), 341]

p. 343, line 27 ;	$b_l{}^+(V, T)$	should read	$b_l{}^\dagger(V, T)$
p. 344, line 20 ;	(9)	,,	(9) [§ 0·1]
p. 344, line 20 ;	(10)	,,	(10) [§ 0·2]
p. 348, line 4 ;	$b_l{}^{(3)}(x_l, T)$	,,	$b_l{}^{(1)}(x_l, T)$
p. 352, line 14 ;	$\theta(l', T)=\bar{\theta}(l', T)$	,,	$\theta(l', T)=\bar{\theta}(l, T)$
p. 355, line 25 ;	for every N	,,	for every $N(\geqq \nu_0)$
p. 356, line 20 ;	for every N	,,	for every $N(\geqq \nu_0)$
p. 357, (15a) ;	$\ln \tilde{T}_N$	,,	$\ln \tilde{T}_N{}^\dagger$
p. 357, line 7 ;	for every N	,,	for every $N(\geqq \nu_0)$
p. 363, $(28)^3$;	$<(1/N)\sum' l(\mu_l+1)\lvert \ln z\rvert$	,,	$\leqq(1/N)\sum' l(\mu_l+1)\lvert \ln z\rvert$
p. 363, (29) ;	$<P(N)+Q(N)$	,,	$\leqq P(N)+Q(N)$
p. 363, (30) ;	$Q(N)$	,,	$Q(N)\leqq$
p. 364, (31) ;	$\lvert m_l \ln\{$	,,	$\lvert \bar{m}_l \ln\{$
p. 364, (31) ;	$<X(N)+Y(N)$	,,	$\leqq X(N)+Y(N)$
p. 364, $(33)^6$;	$<(1/N)\sum' l\lvert \ln z\rvert$	,,	$\leqq(1/N)\sum' l\lvert \ln z\rvert$
p. 364. (37) ;	$\lvert D(N)\rvert=$	,,	$\lvert D(N)\rvert\leqq$
p. 366, line 11 ;	for every N	,,	for every $N(\geqq \nu_0)$
p. 371, line 18 ;	for every N	,,	for every $N(\geqq \nu_0)$
p. 379, Fig. 2·3 ; (d_1) ;	the nib of the broken line should touch the solid line at P.		
p. 380, line 13 ;	ont	should read	not
p. 382, footnote (line 1) ;	ω^+ and $\omega^{(0)}$	,,	$-kT\,\omega^\dagger$ and $-kT\omega^{(0)}$
p. 388, line 3 ;	$\lambda(-\infty<\lambda<+\infty)$	,,	$\bar{\lambda}(-\infty<\bar{\lambda}<+\infty)$
p. 388, reference 4) ;	Willey	,,	Wiley
p. 388, reference 6) ;	**4** (1938), 299	,,	**5** (1938), 399

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Quantum-Mechanical Many-Body Problem with Hard-Sphere Interaction

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Theory of the surface charge pseudopotential for the hard-sphere potential is developed. It is shown that the formulas known to be exact are obtained by the present method. The excitation energy spectrum of the Bose system is calculated, and is shown to have the nonmonotonic behaviour similar to the roton spectrum in liquid He^4. The dispersion formula for the sound velocity is calculated and compared with that derived by Landau and Khalatonikov.

§ 1. Introduction

In the previous paper[1] we have suggested that the "surface charge pseudopotential" method for the hard-sphere potential is more suitable for the investigations of the excitations in the short-wave regions than the "point charge pseudopotential" method. The purpose of this paper is to construct the surface charge pseudopotential in a systematic way.

Before entering into the main subject, it may be helpful first to give a simple example as an illustration of our method, the spherically symmetric wave function of a system of two particles with hard-sphere interaction. The wave function in the center-of-mass coordinate system then satisfies the equations

$$(d^2/dr^2+k^2)\psi=0 \quad (r>a) \tag{1·1}$$

$$\psi=0 \quad (r\leq a)$$

where a is the hard-sphere diameter, $r=|\boldsymbol{x}|$, with $\boldsymbol{x}$ the relative position vector, and ψ the wave function multiplied by r. We replace the above equations by

$$(d^2/dr^2+k^2)\psi=A\delta(r-a) \tag{1·2}$$

without any condition in the region $r<a$, and see how the solution of (1·2) reproduces the vanishing of ψ for $r<a$. Here A is an arbitrary constant which is not zero. To settle the problem we impose the boundary conditions

$$\psi=0, \text{ for } r=R \text{ and } r=0, \ (R>a). \tag{1·3}$$

A complete set of eigenfunctions in the region $0\leq r\leq R$ satisfying the condition (1·3) are

$$\psi_n=\sqrt{2/R}\sin\kappa_n r, \ (n=1, 2, \cdots), \tag{1·4}$$

with eigenvalues $\kappa_n=\pi n/R$. Expanding ψ, a solution of (1·2), in terms of the above set, under the assumption that k is equal to none of κ_n, we obtain

$$\psi(r)=\sum_{n=1}^{\infty}\frac{A}{k^2-\kappa_n^2}\int_0^R\delta(r'-a)\psi_n(r')dr'\psi_n(r)$$

$$=\frac{AR}{\pi^2}\sum_n\frac{1}{\alpha^2-n^2}\left\{\cos\frac{n\pi}{R}(r-a)-\cos\frac{n\pi}{R}(r+a)\right\},$$

with $\alpha=kR/\pi$. By using the equations

$$\sum_{n=1}^{\infty}\cos n\theta/(\alpha^2-n^2)=(\pi/2)\cos\alpha(\pi-\theta)/\alpha\sin\alpha\pi-1/2\alpha^2,\quad(0\leq\theta\leq2\pi)$$

$$=(\pi/2)\cos\alpha(\pi+\theta)/\alpha\sin\alpha\pi-1/2\alpha^2,\quad(-2\pi\leq\theta\leq0),$$

we have

$$\psi(r)=-(AR/\pi\alpha\sin\alpha\pi)\sin\{\alpha\pi(1-r/R)\}\sin(\alpha\pi a/R),\quad(r\geq a)\qquad(1\cdot5a)$$

$$=-(AR/\pi\alpha\sin\alpha\pi)\sin\{\alpha\pi(1-a/R)\}\sin(\alpha\pi r/R),\quad(r\leq a).\qquad(1\cdot5b)$$

From (1·5b) it follows that $\psi(r)$ identically vanishes for $r\leq a$ once it vanishes at $r=a$. We have then

$$\alpha\pi\{1-(a/R)\}=n\pi,\ \text{or}\ k^2=\pi^2n^2/(R-a)^2,$$

which are just the exact eigenvalues for the present problem. It is easily verified that the same considerations can also be applied to the higher partial waves.

We conclude from this argument that the wave function vanishes inside the hard-sphere if the additional term $A\delta(r-a)$, which corresponds to the surface charge on the sphere, is added to the Schrödinger wave equation and once the solution vanishes on the surface of the hard-sphere. In other words, the hard-sphere potential can be replaced effectively by the term $A\delta(r-a)$ in this case. On the other hand, in the point charge pseudopotential method the hard-sphere potential is replaced by the differential operator which includes the term $\delta(r)$; in addition, the solution is extended to the region $0<r<a$ and does not vanish inside the hard-sphere though it does on the surface of the hard-sphere. These points are the essential differences between the two methods.

It should be pointed out here that (1·2) includes another type of solution by the requirement $\psi(a)=0$ alone. This is the case when $\alpha\pi a/R=n\pi$, $(n=1, 2, \cdots)$. Then $\psi(r)$ vanishes for $r\geq a$ but does not vanish for $r<a$. Of course, the solution of this type should be excluded from the solutions of (1·2). We shall return to this point later in the next section (see also the footnote in § 3).

§ 2. Generalized two-body system

In the previous section we have demonstrated a simple example as an illustration of our method. In this section we proceed to a more general case in which the inhomogeneous term appears in the two-body Schrödinger equation and find an effective

operator which is equivalent to the hard-sphere potential. Once such an operator is found in this case, generalization to a many-body system is straightforward.

In accordance with the above statements, we consider the equation

$$(\nabla_1^2+\nabla_2^2+2\kappa^2)\Psi(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=f(\boldsymbol{x}_1,\ \boldsymbol{x}_2) \tag{2·1}$$

with the boundary condition

$$\Psi=0 \quad \text{for} \quad |\boldsymbol{x}_1-\boldsymbol{x}_2|\leq a. \tag{2·2}$$

Here $\hbar^2\kappa^2/2m$ corresponds to the energy eigenvalue per particle.

We assume here:

(a) $f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)$ does not include the singularity $\delta(r_{12}-a)$ and

(b) $f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)$ vanishes for $|\boldsymbol{x}_1-\boldsymbol{x}_2|\leq a$, with relevant conditions at infinity.

As we have done in § 1, we replace (2·1) and (2·2) by the equation including the surface charge at $|\boldsymbol{x}_1-\boldsymbol{x}_2|=a$. Thus, we consider the equation

$$(\nabla_1^2+\nabla_2^2+2\kappa^2)\Psi(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)+2\delta(r-a)\sum_{\alpha}\exp(i\boldsymbol{K}\cdot\boldsymbol{X})\,Y_{lm}(\theta,\ \varphi)\,\lambda_\alpha \tag{2·3}$$

where $\boldsymbol{K}$ and $\boldsymbol{X}$ are the wave vector and the coordinate of the center of mass, respectively, Y_{lm} the normalized spherical harmonic, r, θ, φ the polar coordinates of the relative position vector $\boldsymbol{x}$ $(=\boldsymbol{x}_1-\boldsymbol{x}_2)$, and λ_α a constant to be determined later on. α stands for $\boldsymbol{K}$, l and m. In the following we denote the unit vector with z-direction as $\boldsymbol{z}$ and write $Y_{lm}(\theta,\ \varphi)$ as $Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z})$.

Expanding Ψ and f in terms of $\exp(i\boldsymbol{K}\cdot\boldsymbol{X})$ as

$$\Psi(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=\sum_{\boldsymbol{K}}\exp(i\boldsymbol{K}\cdot\boldsymbol{X})\,\varphi_{\boldsymbol{K}}(\boldsymbol{x}), \tag{2·4}$$

$$f(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=2\sum_{\boldsymbol{K}}\exp(i\boldsymbol{K}\cdot\boldsymbol{X})\,f_{\boldsymbol{K}}(\boldsymbol{x}), \tag{2·5}$$

we have

$$(\nabla^2+k^2)\varphi(\boldsymbol{x})=f(\boldsymbol{x})+\sum_{l,m}\delta(r-a)\,Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z})\,\lambda_\alpha \tag{2·6}$$

where k is the magnitude of the wave vector for the relative motion defined by

$$k^2=\kappa^2-(\boldsymbol{K}^2/4), \tag{2·7}$$

and the suffix $\boldsymbol{K}$ is not written down explicitly.

In solving (2·6), we impose on $\varphi(\boldsymbol{x})$ the periodic boundary condition in a cubical box of volume V and expand $\varphi(\boldsymbol{x})$ in terms of $\exp(i\boldsymbol{q}\cdot\boldsymbol{x})$, where $\boldsymbol{q}$ is defined by

$$\boldsymbol{q}=(2\pi/L)\,(n_1,\ n_2,\ n_3), \qquad L=V^{1/3}, \qquad n_1,\ n_2, \qquad n_3=\pm \text{ integers}.$$

Then the solution of (2·6) is given by

$$\varphi(\boldsymbol{x})=\int G(\boldsymbol{x},\ \boldsymbol{x}')\,\{f(\boldsymbol{x}')+\sum_{l,m}\delta(r'-a)\,Y_{lm}(\boldsymbol{x}'\cdot\boldsymbol{z})\,\lambda_\alpha\}\,d\boldsymbol{x}', \tag{2·8}$$

where $G(\boldsymbol{x},\ \boldsymbol{x}')$ is the Green function defined by

$$G(\boldsymbol{x},\ \boldsymbol{x}')=\frac{1}{V}\sum_{q}\frac{1}{k^2-q^2}\exp[i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{x}')]. \tag{2·9}$$

Using the equation

$$\exp(i\boldsymbol{q}\cdot\boldsymbol{x})=\sum_{l=0}^{\infty}(2l+1)\,i^l j_l(qr)\,P_l(\boldsymbol{q}\cdot\boldsymbol{x}) \tag{2·10}$$

and replacing $\sum_q'$ by an integral, we have

$$G(\boldsymbol{x},\ \boldsymbol{x}')=(1/4\pi)\sum_{l=0}^{\infty}(2l+1)\,P_l(\boldsymbol{x}\cdot\boldsymbol{x}')\,G_l(r,\ r') \tag{2·11}$$

where

$$G_l(r,\ r')=\frac{2}{\pi}\int_0^{\infty}\frac{q^2}{k^2-q^2}\,j_l(qr)\,j_l(qr')\,dq. \tag{2·12}$$

This integral is evaluated by means of the theory of residues (taking the principal values) to give

$$\begin{aligned}G_l(r,\ r')&=k\,j_l(kr)\,n_l(kr'),\quad(r\leq r')\\&=k\,n_l(kr)\,j_l(kr'),\quad(r\geq r').\end{aligned} \tag{2·13}$$

Now, if we expand $f(\boldsymbol{x})$ in the spherical harmonics,

$$f(\boldsymbol{x})=\sum_{l,m}f_{lm}(r)\,Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z}), \tag{2·14}$$

we have

$$\varphi(\boldsymbol{x})=\sum_{l,m}\left\{\int_0^{\infty}G_l(r,\ r')[f_{lm}(r')+\lambda_\alpha\delta(r'-a)]\,r'^2dr'\right\}Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z}), \tag{2·15}$$

with the aid of the theorem,

$$P_l(\boldsymbol{x}\cdot\boldsymbol{x}')=(4\pi/2l+1)\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z})\,Y_{lm}{}^*(\boldsymbol{x}'\cdot\boldsymbol{z}).$$

From (2·15) the solution for $r\leq a$ becomes

$$\varphi(\boldsymbol{x})=\sum_{l,m}\left\{k\int_a^{\infty}n_l(kr')f_{lm}(r')\,r'^2dr'+k\lambda_\alpha a^2 n_l(ka)\right\}j_l(kr)\,Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z}), \tag{2·16}$$

if we use the assumption (b) mentioned previously. We see from (2·16) that $\varphi(\boldsymbol{x})=0$ for $r<a$ once it vanishes at $r=a$, that is, if λ_α is determined by

$$\int_a^{\infty}n_l(kr')f_{lm}(r')\,r'^2dr'+\lambda_\alpha a^2 n_l(ka)=0. \tag{2·17}$$

On the other hand, the solution outside the hard-sphere is given by, with the aid of (2·13), (2·15) and (2·17),

$$\varphi(\boldsymbol{x}) = \sum_{l,m} \Big\{ k\lambda_\alpha a^2 [n_l(kr) j_l(ka) - j_l(kr) n_l(ka)] + k \int_a^r [n_l(kr) j_l(kr') - j_l(kr) n_l(kr')] r'^2 f_{lm}(r') dr' \Big\} Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z}), \qquad (2\cdot18)$$

which is shown to be the solution of (2·6) by a direct substitution. It is easily seen that the above solution does not identically vanish, since if $\varphi(\boldsymbol{x}) \equiv 0$ for $r \geq a$, then $f(\boldsymbol{x}) = 0$ for $r > a$, [see (2·6)] or $f_{lm}(r) \propto \delta(r-a)$ [see (2·18)]. The latter case is forbidden by the assumption (a). Hence the appearance of the solution discussed at the end of § 1 is rather exceptional and, in general, such a solution is automatically excluded from the solutions of (2·6). (See also the footnote in § 3).

In order to have an effective Hamiltonian which is equivalent to the boundary condition for $r \leq a$, it is necessary to eliminate the constants λ_α's from the equation by using some operational procedure. This may be performed by a differential procedure in the pseudopotential method, but is not suitable here, for the wave function has the discontinuous derivative at $r=a$. We shall therefore choose another way in which λ_α's are eliminated by an integral procedure. It is not, however, convenient to use (2·17) or (2·18), since these equations do not take account of the periodic boundary condition which is to be imposed on the wave function. Thus we shall make use of the original form for the solution expressed in (2·8),

$$\varphi(\boldsymbol{x}) = V^{-1} \sum_{\boldsymbol{q}}{}' \frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{x})}{k^2 - q^2} \int f(\boldsymbol{x}') \exp(-i\boldsymbol{q}\cdot\boldsymbol{x}') d\boldsymbol{x}' + \sum_{l,m} \lambda_\alpha Y_{lm}(\boldsymbol{x}\cdot\boldsymbol{z}) G_l(r, a) a^2. \qquad (2\cdot19)$$

Multiplying both sides by $Y_{lm}^*(\boldsymbol{x}\cdot\boldsymbol{z})$ and integrating over the solid angles, and by using

$$\int Y_{lm}^*(\boldsymbol{x}\cdot\boldsymbol{z}) \exp(i\boldsymbol{q}\cdot\boldsymbol{x}) d\Omega = 4\pi i^l Y_{lm}^*(\boldsymbol{q}\cdot\boldsymbol{z}) j_l(qr),$$

we have

$$\lambda_\alpha = -\frac{4\pi}{a^2 G_l(a, a)} \sum_{\boldsymbol{q}}{}' \frac{i^l Y_{lm}^*(\boldsymbol{q}\cdot\boldsymbol{z})}{k^2 - q^2} j_l(qa) V^{-1} \int \exp(-i\boldsymbol{q}\cdot\boldsymbol{x}') f(\boldsymbol{x}') d\boldsymbol{x}', \qquad (2\cdot20)$$

from the condition that $\varphi(\boldsymbol{x})$ should vanish at $r=a$. This equation together with (2·3) and (2·5) leads to

$$(\nabla_1^2 + \nabla_2^2 + 2\kappa^2) \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = f(\boldsymbol{x}_1, \boldsymbol{x}_2) + \delta(r_{12} - a) K_{12} f, \qquad (2\cdot21)$$

as an equation which replaces the boundary condition for $r \leq a$. Here K_{12} is an integral operator defined by

$$K_{12} = -\sum_{\boldsymbol{K},l}{}' \frac{(2l+1) i^l}{a^2 G_l(a, a)} \exp(i\boldsymbol{K}\cdot\boldsymbol{X}) \sum_{\boldsymbol{q}}{}' \frac{P_l(\boldsymbol{q}\cdot\boldsymbol{x})}{k^2 - q^2} j_l(qa) V^{-2} \int \exp(-i\boldsymbol{K}\cdot\boldsymbol{X}' - i\boldsymbol{q}\cdot\boldsymbol{x}') d\boldsymbol{X}' d\boldsymbol{x}'. \qquad (2\cdot22)$$

We have so far implicitly assumed that k^2 is equal to none of q^2. Since we have shown in § 1 that exact eigenvalues are obtained under this assumption in the two-body

problem, we may take it for granted that this still remains to hold in the many-body problem. A more complete discussion on this problem will be given in the following paper[2] from a general point of view.

§ 3. Many-body system (Fermi particles)

In this section we will try to find a many-body operator equivalent to the hard-sphere potential, using the results obtained in § 2. For a while, we neglect quantum statistics; this will be taken into account after the hard-sphere potential is replaced by an equivalent operator. In this section we shall discuss the Fermi system and in the next section the Bose system. In our theory, the equivalent operator is independent of quantum statistics, just as the pseudopotential obtained by Huang and Yang[3] is.

It is clear that the product $v\Psi$ (v: hard-sphere potential) is a delta function at $|\boldsymbol{x}_i-\boldsymbol{x}_j|=a$, which we shall write $\delta(r_{ij}-a)\,t_{ij}\Psi$ with t_{ij} some operator to be determined later on. The Schrödinger equation then becomes

$$\sum_i (\nabla_i^2+\kappa^2)\Psi=\sum_{i<j}\delta(r_{ij}-a)\,t_{ij}\Psi. \tag{3·1}$$

Writing this equation as

$$(\nabla_1^2+\nabla_2^2+2\kappa^2)\Psi=-\sum_{n\neq 1,2}(\nabla_n^2+\kappa^2)\Psi+\sum_{k<l}{}'\delta(r_{kl}-a)\,t_{kl}\Psi+\delta(r_{12}-a)\,t_{12}\Psi, \tag{3·1}'$$

we consider the first and second terms on the right-hand side as corresponding to $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in (2·21) ($\boldsymbol{x}_3, \boldsymbol{x}_4, \cdots, \boldsymbol{x}_N$ are considered as parameters). Here the prime on $\sum$ means that the pair (12) is excluded from the summation. Then it is easily seen that the function $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ defined in this way satisfies the assumptions (a) and (b) mentioned at the beginning of § 2, and hence the results obtained in § 2 can be applied.

Substituting $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ defined above in (2·21) and equating $K_{12}f$ to $t_{12}\Psi$ in (3·1)′ (see also the following paper), we have

$$t_{12}\Psi=K_{12}\Big[-\sum_{n\neq 1,2}(\nabla_n^2+\kappa^2)+\sum_{k<l}{}'\delta(r_{kl}-a)\,t_{kl}\Big]\Psi$$

from the condition that the wave function should vanish for $r_{12}\leq a$. If we use (3·1)′ and drop Ψ,* we have

$$t_{12}=K_{12}[(\nabla_1^2+\nabla_2^2+2\kappa^2)-\delta(r_{12}-a)\,t_{12}]. \tag{3·2}$$

Or, if we write

$$u_{12}=K_{12}(\nabla_1^2+\nabla_2^2+2\kappa^2) \tag{3·3}$$

the equation for the t-operator becomes

$$t_{12}=u_{12}-K_{12}\delta(r_{12}-a)\,t_{12}. \tag{3·4}$$

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* As a matter of fact any operator P_{12} defined by $P_{12}\Psi=0$ may be added to t_{12} and thus t_{12} is not uniquely defined. For our purpose, however, it is sufficient to find one operator equivalent to the hard-sphere potential.

In this way, the effective operator which is mathematically equivalent to the boundary conditions imposed on the wave function is found.* To carry out the further calculations, it is convenient to make use of the second-quantization formalism; we consider a system of Fermi particles with spin 1/2, and introduce the field operator $\psi(1)$ which can be expanded in a Fourier series:

$$\psi(1)=V^{-1/2}\sum_{\alpha}a_{\alpha}\exp(i\boldsymbol{k}_{\alpha}\cdot\boldsymbol{x}_1)\,\delta(s_{\alpha},\ \sigma_1) \tag{3·5}$$

where σ is the spin coordinate and a_{α}, a_{α}^*, respectively, denote the annihilation and creation operator for the single particle state of momentum $\boldsymbol{k}_{\alpha}$ and spin s_{α}. The total Hamiltonian can be written as

$$\mathcal{H}=(\hbar^2/2m)\int\Gamma\psi^*\cdot\Gamma\psi\,d\tau+(\hbar^2/4m)\int\psi^*(1)\,\psi^*(2)\,\delta(r_{12}-a)\,t_{12}\,\psi(2)\,\psi(1)\,d\tau_1 d\tau_2, \tag{3·6}$$

where $d\tau$ means the integration over the coordinates together with the summation over the spin coordinates. Introducing (3·5) into (3·6), we have

$$\mathcal{H}=(\hbar^2/2m)\sum_{\alpha}k_{\alpha}^2a_{\alpha}^*a_{\alpha}+(\hbar^2/4mV)\sum_{\alpha\beta\gamma\lambda}a_{\alpha}^*a_{\beta}^*a_{\gamma}a_{\lambda}\,\delta(s_{\alpha},\ s_{\lambda})\,\delta(s_{\beta},\ s_{\gamma})\,t(\alpha\beta,\ \lambda\gamma), \tag{3·7}$$

where $t(\alpha\beta,\ \lambda\gamma)$ is defined by

$$t(\alpha\beta,\ \lambda\gamma)=(1/V)\iint\exp(-i\boldsymbol{k}_{\alpha}\cdot\boldsymbol{x}_1-i\boldsymbol{k}_{\beta}\cdot\boldsymbol{x}_2)\,\delta(r_{12}-a)\,t_{12}\exp(i\boldsymbol{k}_{\lambda}\cdot\boldsymbol{x}_1+i\boldsymbol{k}_{\gamma}\cdot\boldsymbol{x}_2)\,d\boldsymbol{x}_1 d\boldsymbol{x}_2. \tag{3·8}$$

The matrix element $t(\alpha\beta,\ \lambda\gamma)$ should be determined by the linear integral equation (3·4). Let us first calculate the matrix elements of the inhomogeneous part and the kernel of the integral equation. If we define $K(\alpha\beta,\ \lambda\gamma)$ with t replaced by K in (3·8), we have

$$K(\alpha\beta,\ \lambda\gamma)=-4\pi\delta(\alpha\beta,\ \lambda\gamma)\sum_{l=0}^{\infty}\frac{(2l+1)}{G_l(a,\ a)}\,\frac{j_l(k_{\alpha\beta}a)\,j_l(k_{\lambda\gamma}a)}{k^2-k_{\lambda\gamma}^2}\,P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\gamma}), \tag{3·9}$$

where

$$G_l(a,\ a)=kj_l(ka)\,n_l(ka), \tag{3·10}$$

$$\boldsymbol{k}_{\alpha\beta}=(\boldsymbol{k}_{\alpha}-\boldsymbol{k}_{\beta})/2,\ \boldsymbol{k}_{\lambda\gamma}=(\boldsymbol{k}_{\lambda}-\boldsymbol{k}_{\gamma})/2\ \text{and}\ k^2=\kappa^2-(\boldsymbol{k}_{\lambda}+\boldsymbol{k}_{\gamma})^2/4. \tag{3·11}$$

In (3·9) $\delta(\alpha\beta,\ \lambda\gamma)$ expresses the conservation of total momentum, i.e. $\delta(\alpha\beta,\ \lambda\gamma)=\delta(\boldsymbol{k}_{\alpha}+\boldsymbol{k}_{\beta}-\boldsymbol{k}_{\lambda}-\boldsymbol{k}_{\gamma})$.

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* The solution of the eigenvalue problem (3·1) with t_{ij} given by (3·4) may include the wave function of the following type: $\Psi=0$ for $r_{ij}>a$ and $\Psi\neq 0$ for $r_{ij}<a$, since in this case $f(x)=0$ ($r<a$) and the discussion in § 2 can equally be applied; therefore K_{12} does not exclude such a solution. This may be excluded, however, as long as the perturbation procedure starting from the free particles is used, for the energy eigenvalue corresponding to such a solution becomes ∞ as $a\to 0$ in order that Ψ may stay normalized and is not connected to the counterpart for a system of free particles.

If we further define $u(\alpha\beta,\ \lambda\gamma)$ analogously to (3·8), we have from (3·3)

$$u(\alpha\beta,\ \lambda\gamma)=-8\pi\delta(\alpha\beta,\ \lambda\gamma)\sum_{l=0}^{\infty}(2l+1)G_l^{-1}(a,\ a)\,j_l(k_{\alpha\beta}a)\,j_l(k_{\lambda\tau}a)\,P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau}). \tag{3·12}$$

Finally, from (3·4) it follows that the integral equation for the t-operator becomes in the momentum representation,

$$t(\alpha\beta,\ \lambda\gamma)=u(\alpha\beta,\ \lambda\gamma)-(1/V)\sum_{\sigma,\tau}K(\alpha\beta,\ \sigma\tau)\,t(\sigma\tau,\ \lambda\gamma) \tag{3·13}$$

with the aid of the equation

$$\delta(r_{12}-a)\,t_{12}\exp(i\boldsymbol{k}_\lambda\cdot\boldsymbol{x}_1+i\boldsymbol{k}_\tau\cdot\boldsymbol{x}_2)=(1/V)\sum_{\sigma,\tau}\exp(i\boldsymbol{k}_\sigma\cdot\boldsymbol{x}_1+i\boldsymbol{k}_\tau\cdot\boldsymbol{x}_2)\,t(\sigma\tau,\ \lambda\gamma).$$

Before discussing the solution of (3·13), we wish to compare the u-matrix with the corresponding matrix obtained by the point charge pseudopotential method. The latter can be written as

$$u_p(\alpha\beta,\ \lambda\gamma)=-8\pi\delta(\alpha\beta,\ \lambda\gamma)\sum_{l=0}^{\infty}\frac{[(2l+1)!!]^2}{(2l)!}\left\{\frac{\tan\delta_l\,j_l(k_{\alpha\beta}r)}{r^l k^{2l+1}}\,\frac{\partial^{2l+1}}{\partial r^{2l+1}}\left[r^{l+1}j_l(k_{\lambda\tau}r)\right]\right\}_{r=0}$$
$$\times P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau}) \tag{3·14}$$

where $\{\ \}_{r=0}$ means that the substitution $r=0$ is carried out after the operation of a_α's and the summation over α, β, λ, γ are performed.[4] For the wave function which is regular at $r_{ij}=0$, we may put $r=0$ in (3·14) and are led to

$$u_p(\alpha\beta,\ \lambda\gamma)=-8\pi\delta(\alpha\beta,\ \lambda\gamma)\sum_{l}(2l+1)\tan\delta_l k_{\alpha\beta}^l k_{\lambda\tau}^l k^{-2l-1}P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau}). \tag{3·15}$$

If the phase shifts are expanded in powers of ka and the lowest order term is retained, (3·15) is reduced to

$$u_p(\alpha\beta,\ \lambda\gamma)=8\pi a\delta(\alpha\beta,\ \lambda\gamma)\sum_{l}\{(2l-1)!!\}^{-2}a^{2l}k_{\alpha\beta}^l k_{\lambda\tau}^l P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau}). \tag{3·16}$$

On the other hand, under the same approximation, that is, if $G_l(a,\ a)$ is expanded in terms of ka and the lowest order term is retained, (3·12) is reduced to

$$u(\alpha\beta,\ \lambda\gamma)=8\pi a\delta(\alpha\beta,\ \lambda\gamma)\sum_{l=0}^{\infty}(2l+1)^2 j_l(k_{\alpha\beta}a)\,j_l(k_{\lambda\tau}a)\,P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau}). \tag{3·17}$$

If $j_l(k_{\alpha\beta}a)$ and $j_l(k_{\lambda\tau}a)$ in the above expression are expanded in terms of a, the lowest order term turns out to be identical with $u_p(\alpha\beta,\ \lambda\gamma)$ given by (3·16). The essential difference between (3·16) and (3·17) lies in the behaviours of the u-matrix for large momentum transfers, that is, it is oscillatory in (3·17), while monotonic in (3·16). Such behaviours of $u_p(\alpha\beta,\ \lambda\gamma)$ indicate that the point charge pseudopotential method cannot treat correctly the problems in which the large momentum transfers are important, e. g. the roton spectrum in liquid He^4, as we have discussed previously.[1]

Returning now to the integral equation (3·13), we try to solve the equation by an iteration procedure. The first iterated term is

$$t_1(\alpha\beta,\ \lambda\gamma) = -(1/V)\sum_{\sigma,\tau} K(\alpha\beta,\ \sigma\tau)\, u(\sigma\tau,\ \lambda\gamma). \qquad (3\cdot18)$$

By substituting $(3\cdot9)$ and $(3\cdot12)$ in this and replacing the summation over $\boldsymbol{k}_{\sigma\tau}$ by integration, we have

$$t_1(\alpha\beta,\ \lambda\gamma) = u(\alpha\beta,\ \lambda\gamma).$$

If we formally carry out further iterations, $t(\alpha\beta,\ \lambda\gamma)$ becomes

$$t(\alpha\beta,\ \lambda\gamma) = u(\alpha\beta,\ \lambda\gamma) + u(\alpha\beta,\ \lambda\gamma) + u(\alpha\beta,\ \lambda\gamma) + \cdots,$$

which is obviously divergent. This indicates that there is no solution of the integral equation $(3\cdot13)$.

In order to clarify the reason why such situations are present, we shall retain the term of the order a in $K(\alpha\beta,\ \lambda\gamma)$ and in $u(\alpha\beta,\ \lambda\gamma)$, and see how the iterated term contributes to the energy. We have then

$$u(\alpha\beta,\ \lambda\gamma) = 8\pi a\delta(\alpha\beta,\ \lambda\gamma),$$

$$K(\alpha\beta,\ \lambda\gamma) = 4\pi a\delta(\alpha\beta,\ \lambda\gamma)/(k^2 - k_{\lambda\tau}^2).$$

Hence $t(\alpha\beta,\ \lambda\gamma)$ becomes

$$t(\alpha\beta,\ \lambda\gamma) = 8\pi a\delta(\alpha\beta,\ \lambda\gamma) - (32\pi^2 a^2/V)\,\delta(\alpha\beta,\ \lambda\gamma)\sum_q 1/(k^2 - q^2), \qquad (3\cdot19)$$

up to the order of a^2. Now, let us calculate the perturbed energy of the order of a^2. It consists of the second-order energy resulting from the first term in $(3\cdot19)$ and the first-order one from the second term. They are, respectively, given by

$$(16\pi^2 a^2\hbar^2/mV^2)\sum{}'\,\delta(\alpha\beta,\ \lambda\gamma)\, n_\lambda{}^+ n_\tau{}^- (1 - n_\alpha{}^+ - n_\beta{}^-)/(k_{\lambda\tau}^2 - k_{\alpha\beta}^2), \qquad (3\cdot20)$$

$$-(16\pi^2 a^2\hbar^2/mV^2)\sum_{k_\lambda,\,k_\tau} n_\lambda{}^+ n_\tau{}^- \sum_q 1/(k^2 - q^2), \qquad (3\cdot21)$$

where $\sum'$ in $(3\cdot20)$ means that $\boldsymbol{k}_\alpha \neq \boldsymbol{k}_\lambda \neq \boldsymbol{k}_\tau \neq \boldsymbol{k}_\lambda$, n_α's are the occupation numbers and $+$ or $-$ denotes the spin direction. It is to be noted that both expressions are divergent. However, when added together, the result is no longer divergent and is given by

$$E_2 = (16\pi^2 a^2\hbar^2/mV^2)\sum{}'\frac{\delta(\boldsymbol{k}_\alpha + \boldsymbol{k}_\beta - \boldsymbol{k}_\tau - \boldsymbol{k}_\lambda)}{k_{\alpha\beta}^2 - k_{\lambda\tau}^2}\, n_\lambda{}^+ n_\tau{}^- (n_\alpha{}^+ + n_\alpha{}^-). \qquad (3\cdot22)$$

In deriving this equation, we have used

$$\sum_q\left[\frac{1}{k_{\lambda\tau}^2 - q^2} - \frac{1}{k^2 - q^2}\right] = \sum_q{}'\left[\frac{k_{\lambda\tau}^2}{q^2(k_{\lambda\tau}^2 - q^2)} - \frac{k^2}{q^2(k^2 - q^2)}\right] = 0,$$

which can be easily proved by evaluating Cauchy's principal value of the integral.

The energy of the order of a^2 given by $(3\cdot22)$ is just the same as obtained by the pseudopotential method and is exact as long as the order of a^2 is concerned. For the ground state, after rather lengthy calculations, we obtain

$$E_2/N = 6\hbar^2\rho k_F(11 - 2\log 2)\, a^2/35\, m \qquad (3\cdot23)$$

where ρ is the average density N/V (N: particle number) and k_F the Fermi momentum defined by $k_F=(3\pi^2\rho)^{1/3}$. The above expression is identical with that given by Lee and Yang.[5]

Now, it is easy to see how the iterated solutions of (3·13) contribute to the energy. We have noticed above that the terms given by (3·20) and (3·21) are, respectively, divergent. However, the sum of both terms leads to a correct result. In other words, the iterated terms lead to subtractions which yield a correct finite result. Such situations may hold to any higher orders of a, for we have so far made no approximations. In general, all the iterated terms which give rise to the energy of a specified order should be added before integration, and then the integration over the momentum space is performed. It is interesting to note that we have encountered with the similar situation in the pseudopotential method. There the operation $\partial/\partial r$ [see (3·14)] must not be carried out before the summation over the momentum space is done. Contrary to the pseudopotential method, many-body collision terms may be automatically included by the present method, if one performs formally the perturbation calculation of the u-matrix and of the iterated terms, with the remark given above in mind.

§ 4. Many-body system (Bose particles)

It is evident that the Hamiltonian for the Bose system is given by (3·7), with $\delta(s_\alpha, s_\lambda)\delta(s_\beta, s_\tau)$ omitted and a_α, a_α^* the annihilation and creation operators obeying the commutation rules for the Bose statistics. The perturbational procedure discussed in § 3 for the Fermi system is, however, no longer valid for the Bose system, because such a procedure leads to the divergence of energy owing to the contributions arising from particle excitations with small momentum transfers. These points were fully discussed by Brueckner and Sawada[6]; they have proposed another way in which the multiple excitation of one or more pairs to the state with opposite momenta and their interaction with the unexcited particles are taken into account from the beginning, then the particle Hamiltonian is transformed into the phonon Hamiltonian by a transformation similar to that used by Bogolyubov.[7] The result is written, using the notations in the present paper, as

$$\hbar\omega=(\hbar^2/2m)\,(\beta^2-\alpha^2N^2)^{1/2}, \tag{4·1}$$

$$\kappa^2=\frac{\rho}{2}t(00,00)+\frac{1}{4\pi^2\rho}\int_0^\infty[(\beta^2-\alpha^2N^2)^{1/2}-\beta]q^2dq, \tag{4·2}$$

where $\hbar\omega$ is the excitation energy corresponding to a single phonon or roton state, $\hbar^2\kappa^2/2m$ the ground state energy per particle, and αN and β are defined by

$$\alpha N=(\rho/2)\,\{t(00,\ \boldsymbol{q}-\boldsymbol{q})+t(00,\ -\boldsymbol{q}\boldsymbol{q})\} \tag{4·3}$$

$$\beta=q^2+\rho\{t(0\boldsymbol{q},\ 0\boldsymbol{q})+t(0\boldsymbol{q},\ \boldsymbol{q}0)-t(00,\ 00)\}. \tag{4·4}$$

Before calculating (4·1) or (4·2), we wish to make some remarks about the contributions to the energy from phonon-phonon interactions. The lowest order term of

these contributions was given by Bogolyubov and Zubarev,[8] and was shown to be expressed as the multiple integrals over the momentum space, i. e., $\iint d\boldsymbol{q}_1 d\boldsymbol{q}_2(\cdots)$. If we take account of multiple excitations of phonons, there should be similar terms such as $\iiint d\boldsymbol{q}_1 d\boldsymbol{q}_2 d\boldsymbol{q}_3$, $\iiiint d\boldsymbol{q}_1 d\boldsymbol{q}_2 d\boldsymbol{q}_3 d\boldsymbol{q}_4$, etc.

Now we consider to solve the equation (3·13) by an iteration procedure. As we mentioned in the previous section, the iterated terms must not be integrated at once, but should remain in the original form of the integration over the momentum space. Substituting the iterated terms in (4·2) and expanding the integrand of the last term in (4·2) in powers of such terms, we have

$$\kappa^2=\frac{\rho}{2}\left[u(00,\ 00)-\frac{1}{(2\pi)^3}\int K(00,\ \boldsymbol{q}-\boldsymbol{q})\,u(\boldsymbol{q}-\boldsymbol{q},\ 00)\,d\boldsymbol{q}+\int d\boldsymbol{q}_1 d\boldsymbol{q}_2+\cdots\right]$$

$$+\frac{1}{4\pi^2\rho}\int_0^{\infty}[t=u]dq+\int d\boldsymbol{q}_1 d\boldsymbol{q}_2+\cdots,$$

where $[t=u]$ indicates that t is replaced by u in (4·3) and (4·4). As a matter of fact, the contributions from phonon-phonon interactions should be added to the above equation. Those terms are written in a form of the multiple integrals as we have mentioned above, and are combined with the corresponding terms arising from the iteration; thus an exact energy is expanded as

$$\kappa^2=\frac{\rho}{2}u(00,\ 00)+\int d\boldsymbol{q}_1(\cdots)+\int d\boldsymbol{q}_1 d\boldsymbol{q}_2(\cdots)+\int d\boldsymbol{q}_1 d\boldsymbol{q}_2 d\boldsymbol{q}_3(\cdots)+\cdots,$$

which is the well-defined series, though extremely complicated one.

The same consideration can be applied to the excitation energy spectrum. Under the assumption that the phonon-phonon interaction may be neglected, we have

$$\kappa^2=\frac{\rho}{2}\left[u(00,\ 00)-\frac{1}{(2\pi)^3}\int K(00,\ \boldsymbol{q}-\boldsymbol{q})\,u(\boldsymbol{q}-\boldsymbol{q},\ 00)\,d\boldsymbol{q}\right]$$

$$+\frac{1}{4\pi^2\rho}\int_0^{\infty}\left[(\beta^2-\alpha^2N^2)^{1/2}-\beta\right]q^2dq, \tag{4·5}$$

with t's replaced by u's in (4·1), (4·3) and (4·4). From (3·9), (3·12), (4·3) and (4·4), it follows

$$\alpha N=-8\pi\rho j_0(qa)/G_0(a,\ a), \tag{4·6}$$

$$\beta=q^2+\rho\left\{-\sum_{l(\text{even})}16\pi\frac{(2l+1)}{G_l(a,\ a)}j_l^2(qa/2)+8\pi\frac{1}{G_0(a,\ a)}\right\} \tag{4·7}$$

and

$$\kappa^2=-\frac{4\pi\rho}{G_0(a,\ a)}-8\rho\int_0^{\infty}\frac{j_0^2(qa)}{G_0^2(a,\ a)}\frac{q^2}{k^2-q^2}dq+\frac{1}{4\pi^2\rho}\int_0^{\infty}\left[(\beta^2-\alpha^2N^2)^{1/2}-\beta\right]q^2dq. \tag{4·8}$$

It is noted that equations (4·6) and (4·7) are quite similar to those obtained by Brueckner and Sawada,[6] the only difference being that $G_l(a, a)$'s are given by (3·10) in the present method while the ground state energy is proportional to $G_0^{-1}(a, a)$ in Brueckner and Sawada's theory.

To see how the formula (4·8) leads to an exact result obtained by Lee and Yang[5] and how the iterated term contributes to the energy, we shall confine ourselves only to the S-wave scattering, approximating $j_0(qa)$ by 1; then we have

$$\alpha N=-8\pi\rho G_0^{-1}(a, a), \quad \beta=q^2-8\pi\rho G_0^{-1}(a, a).$$

Furthermore, if we expand $G_0^{-1}(a, a)$ in powers of ka and retain the lowest order term, we have

$$\alpha N=k_0^2, \quad \beta=q^2+k_0^2, \tag{4·9}$$

where k_0^2 is given by

$$k_0^2=8\pi a\rho. \tag{4·10}$$

From (4·8) and (4·9), we obtain

$$\begin{aligned}\kappa^2&=4\pi a\rho+\frac{1}{4\pi^2\rho}\int_0^\infty\left\{\sqrt{q^4+2k_0^2q^2}-q^2-k_0^2+\frac{k_0^4}{2q^2}\right\}q^2dq\\&=4\pi a\rho\left\{1+\frac{128}{15\sqrt{\pi}}(\rho a^3)^{1/2}\right\},\end{aligned} \tag{4·11}$$

which is just the exact formula obtained by Lee and Yang[5] or by Lee, Huang and Yang.[9]

In these calculations, the iterated term plays a role such that it leads to subtractions which yield a finite result, in the same manner as in the Fermi system. This makes the energy higher than that calculated without subtractions, since, neglecting the iterated term, κ^2 is given by

$$\kappa^2=-4\pi\rho/G_0(a, a)+\frac{1}{4\pi^2\rho}\int_0^\infty[(\beta^2-\alpha^2N^2)^{1/2}-\beta]q^2dq,$$

which is smaller than $-4\pi\rho/G_0(a, a)$, for the second term is always negative. However, when the iterated term is added, the negative contribution from the second is cancelled by the added term and κ^2 becomes larger than $-4\pi\rho/G_0(a, a)$. The same considerations can also be applied to the Fermi system. In general, one may attribute such a behaviour of the iterated term to the non-hermitic property of the t-operator, since, if it were possible to find an operator of the order of a, which is hermitian and equivalent to the hard-sphere potential, then the second-order perturbation energy for the ground state, E_2, should naturally be negative as usual, which is, however, contradictory to the exact positive value given by (3·23). In order to save this, a positive contribution should be added by the addition of the iterated term.

Formula (4·11) forms an expansion of the energy in powers of $(\rho a^3)^{1/2}$, appropriate for the low density limit. On the other hand, at high densities such an expansion is no longer valid and another procedure should be taken.

As the density approaches the close-packed density, it is expected that the energy becomes very large; this suggests that ka is assumed to be large compared with 1. Then the asymptotic expansions for the spherical Bessel or Neumann function may be applied. It is easily verified in this case that

$$G_l^{-1}(a, a) = (-1)^l G_0^{-1}(a, a). \tag{4·12}$$

By using (4·12), we have from (4·6) and (4·7),

$$\alpha N = -8\pi\rho G_0^{-1}\frac{\sin qa}{qa}, \quad \beta = q^2 - 8\pi\rho G_0^{-1}\frac{\sin qa}{qa}, \tag{4·13}$$

where use is made of the equation

$$\sum_{l(\text{even})}(2l+1)j_l^2(x) = (1/2) + \sin 2x/4x.$$

From (4·1) and (4·13), the excitation energy spectrum becomes

$$\omega(q) = \frac{\hbar q}{2m}\sqrt{q^2 - 16\pi\rho G_0^{-1}\frac{\sin qa}{qa}}, \tag{4·14}$$

or, if we eliminate G_0^{-1} by the relation $\omega(q) = c_0 q$ (at small q, c_0: sound velocity), we have

$$\omega(q) = q\sqrt{c_0^2\frac{\sin qa}{qa} + \left(\frac{\hbar q}{2m}\right)^2}, \tag{4·15}$$

which is of the form suggested in the previous paper.[1] Also, this expression is formally identical with that given by Brueckner and Sawada.[6] The excitation spectrum given by (4·15) has a characteristic nonmonotonic behaviour similar to the roton spectrum in liquid He^4.

It is interesting here to calculate the dispersion formula for the sound velocity from (4·15). A simple computation yields

$$c(q) = c_0(1 - \gamma/2 \cdot q^2 + \cdots) \tag{4·16}$$

where γ is given by

$$\gamma = a^2/6 - (\hbar/2mc_0)^2. \tag{4·17}$$

Substituting $a = 2.2 \times 10^{-8}$ cm and values appropriate for liquid He^4, we have $\gamma = 6.9 \times 10^{-17}$ cm^2, which agrees fairly with the value 6.2×10^{-17} cm^2 deduced by Landau and Khalatonikov.[10]

§ 5. Concluding remarks

A method which makes it possible to treat the hard-sphere potential by the perturbational procedure has been developed. The essential point of the method lies in the

transformation of the original potential function into an equivalent integral operator. Such a method is not only restricted to the hard-sphere potential, but can be generalized to arbitrary singular potentials. In the following paper[2] we shall discuss the generalization of the method, with an application to the Lennard-Jones potential.

We must mention here some ambiguous feature of the t-operator defined in this paper. As shown in § 3, $\delta(r_{12}-a)t_{12}$ is equivalent to the hard-sphere potential v_{12}. However, the defining equation for t_{12}, $(3\cdot2)$, is valid for an arbitrary value of κ^2 not necessarily proportional to the energy eigenvalue per particle. Hence k^2 defined by $(3\cdot11)$ may be assumed to be arbitrary and yet the subsequent discussion is valid. As we have shown in $(3\cdot17)$, when the u-matrix is expanded in powers of k^2, the lowest order term is independent of k^2 and free from such an arbitrariness. Though it is not yet clear how the arbitrariness of k^2 plays a role in the determination of the energy eigenvalue, we hope that the results may be independent of the value of k^2 if the perturbation calculation using the u-matrix is carried out up to infinite order.

Now the question is how to remove the ambiguity for k^2 when the perturbation series is terminated in the finite terms, (this series is dependent on k^2 owing to the dependence of the u-matrix on k^2). We have not yet a definite answer to this question; however, we consider that the following procedure will be useful. We assume that k^2 is given by $(3\cdot11)$ with $\hbar^2\kappa^2/2m$ the energy eigenvalue per particle and consider that κ^2 is an unknown quantity, then the perturbation series is some function of κ^2. This series in turn is equal to κ^2 and we obtain an equation to determine κ^2. The approximate nature of this procedure is not clarified yet, but we expect that it is a good approximation for a system at low density.

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Quantum Mechanics of Strongly Interacting Particles with an Application to Lennard-Jones Potential

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The perturbation theoretic procedure is developed for a system of strongly interacting particles, by replacing the singular potential with an equivalent operator. As an application of the method, an expression for the ground state energy of a Bose system, which corresponds to Lenz's formula for the hard-sphere system, is obtained for the Lennard-Jones potential. The cohesive energy of liquid He^4 is calculated and is shown to be about twice as large as the experimentally observed one. Some difficulties arising from the negative value of the scattering length are discussed, and it is suggested that the assumption, usually taken for Bose system, that nearly all the particles are in the single particle state with momentum 0 should be altered for the actual liquid He^4.

§ 1. Introduction

Recently the quantum-mechanical treatment of many-body systems with strong interactions has made considerable progress in connexion with the problems in the low-temperature properties of liquid He[1-4], the correlation energy of electron gas[5-7], and the nuclear matter[8-10]. As one method for dealing with the hard-sphere system, there is the pseudopotential method developed by Huang and Yang[1]. The great merit of this method consists in being able to replace the hard-sphere potential by the equivalent potential operator to which the usual perturbation calculation may be applied. However, this method is only applicable to the hard-sphere potential and its generalization to more general potential functions seems to be difficult in its original form developed by Huang and Yang. In this paper we will generalize the pseudopotential method from a different point of view which enables us to treat such a singular potential. This is essentially a generalization of the method developed in the previous paper[11] (to be referred to as I).

The main difficulty in quantum-mechanical many-body problem with strong interaction is that the usual method appropriate for the weakly interacting system is not applicable owing to the divergence of the Fourier components of the potential function. For example, if one calculates the energy levels of the hard-sphere system by the usual perturbation method, taking an assembly of free particles as the unperturbed system, then the results diverge to infinity. In order to save such a difficulty Huang and Yang[1] have introduced the concept of pseudopotential, replacing the hard-sphere potential $V(i, j)$ by

$$V(i, j) = (4\pi a\hbar^2/m)\,\delta(\boldsymbol{x}_i - \boldsymbol{x}_j)\,(\partial/\partial r_{ij})\,r_{ij} + O(a^3)$$

where a is the hard-sphere diameter. The perturbation theoretic procedure is not applicable to $V(i, j)$, but may work on the operator on the right-hand side in the above expression.

Now the generalization of the pseudopotential method to arbitrary potentials can be carried out if one finds an appropriate potential operator which is equivalent to the original one in the sense mentioned above. To be more precise, let us consider the Fourier component of the potential, $\nu(\boldsymbol{q})$. This is divergent for the singular potential under consideration, but we assume that $\nu(\boldsymbol{q})$ may be expanded in some way as

$$\nu(\boldsymbol{q}) = a\nu_1(\boldsymbol{q}) + a^2\nu_2(\boldsymbol{q}) + \cdots, \tag{1·1}$$

where a is an expansion parameter which depends on the particular choice of the potential function, for instance, hard-sphere diameter for hard-sphere potential. This power series diverges when the sum is carried out up to infinite orders. However, if one retains the finite terms and applies the usual perturbation method for these terms, an expression for the energy levels will be obtained which is exact up to a specified order. For example, if the first term is retained, the lowest term of the order of a is obtained, just as the corresponding term for hard-sphere system is calculated by making use of the pseudopotential mentioned above. Thus, the problem is reduced to finding an exact expression for $\nu(\boldsymbol{q})$ as in eq. (1·1). In short, the essence of the present method is to replace the original singular potential by an equivalent operator of which the Fourier component does exist, and then to apply the usual methods in quantum mechanics or quantum statistical mechanics appropriate for the system with weak interaction.

In § 2 we show that $\nu(\boldsymbol{q})$ is in fact written in the form of eq. (1·1), in terms of the solution of the two-body problem. It is clarified that the present formulation corresponds to summing up the divergent terms of the conventional perturbation series to give a convergent result. The method is applied to the Bose system with Lennard-Jones potential in § 3, and an expression for the lowest order term of the ground state energy, which corresponds to Lenz's formula[12] for the hard-sphere potential, is obtained. Substituting the values appropriate for liquid He^4 in this expression, the cohesive energy is calculated and is shown to be about twice as large as the experimentally observed one. In § 4 some difficulties which arise from the negative value of the two-body scattering length are discussed, and it is suggested that some of the features of the actual liquid He^4 are likely to be markedly different from those of the systems with purely repulsive forces.

§ 2. General theory

As we have remarked in § 1, the perturbation method may still be applicable for a system interacting through singular potentials if the expression like eq. (1·1) is found for the Fourier components of the potential. It is obvious that the correlations between particles should be taken into account for the system under consideration, since the divergence difficulty associated with the singular potentials is due to the neglect of the

correlations between particles. This consideration suggests that the required expression will be obtained in terms of the solution of the two-body problem. This is the main content of this section. First, we neglect quantum statistics, which will be taken into account after the potential function is replaced by an operator of which the Fourier components exist.

We consider N particles in a cubical box of volume V. Let the potential function between i-th and j-th particles be $V(i, j)$. Then the Schrödinger equation is

$$-(\hbar^2/2m)\sum_i \nabla_i^2\Psi+\sum_{i<j}V(i, j)\Psi=E\Psi. \tag{2·1}$$

If we define $v(i, j)$ and ε by

$$v(i, j)=(2m/\hbar^2)V(i, j), \tag{2·2}$$

$$\varepsilon=2mE/\hbar^2N, \tag{2·3}$$

eq. (2·1) is reduced to

$$\sum_i(\nabla_i^2+\varepsilon)\Psi=\sum_{i<j}v(i, j)\Psi. \tag{2·4}$$

If we consider the coordinates of particles 3, 4,$\cdots$, N as parameters and define the function $f(1, 2)$ by

$$f(1, 2)=-\sum_{n\neq 1,2}(\nabla_n^2+\varepsilon)\Psi+\sum_{i<j}{}'v(i, j)\Psi, \tag{2·5}$$

where $\sum'$ means that the pair (12) is excluded from the summation, then eq. (2·4) takes the form*

$$[\nabla_1^2+\nabla_2^2+2\varepsilon-v(1, 2)]\Psi=f(1, 2). \tag{2·6}$$

This equation may be interpreted as follows: the particles 3, 4,$\cdots$, N are at rest and considered as the medium; the particles 1 and 2 move in this medium, exerting the force each other through the two-body potential $v(1, 2)$ and being influenced by the medium through the function $f(1, 2)$. We shall show below that the potential $v(1, 2)$ can be expressed in terms of an integral operator if we investigate the solution of this equation.

In solving the equation (2·6) we impose on Ψ the periodic boundary condition. Furthermore, Ψ should approach zero as $r_{12}\to 0$, owing to the rapid increase of the potential. Let a complete set of eigenfunctions for the two-body problem satisfying the above conditions be ψ_n and the corresponding eigenvalue be ε_n, i.e.,

$$[\nabla_1^2+\nabla_2^2+2\varepsilon_n-v(1, 2)]\psi_n=0.$$

If we assume that the ψ_n's are orthonormal in the volume V, i.e.,

$$\int\psi_n^*(1, 2)\psi_m(1, 2)\,d\boldsymbol{x}_1d\boldsymbol{x}_2=\delta_{n,m},$$

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* Though the following discussions are valid for an arbitrary value of ε, we define ε by eq. (2.3), i.e. ε is proportional to the energy value per particle. Hence the t-operator with which we shall deal below is dependent on the energy value. See also the notice given in § 5 of I.

then the solution of eq. (2·6) is given by

$$\Psi = \int G_v(12, 1'2') f(1'2')\, d\boldsymbol{x}_1' d\boldsymbol{x}_2', \tag{2·7}$$

where $G_v(12, 1'2')$ is Green's function defined by

$$G_v = \sum_n \psi_n(1, 2)\psi_n^*(1'2')\, 2^{-1}(\mathcal{E}-\mathcal{E}_n)^{-1}.$$

As is well known, if $\mathcal{E}$ is equal to one of the eigenvalues $\mathcal{E}_m$'s, either eq. (2·6) has no solution, or the solution of the corresponding homogeneous equation may be added to eq. (2·7) to get a general solution. However, we may safely exclude such a case without loss of generality.

If we consider Green's function G_v as an integral operator, eq. (2·7) is formally written as

$$\Psi = G_v f. \tag{2·8}$$

Multiplying both sides by v, we have

$$v\Psi = vG_v f. \tag{2·9}$$

Substituting eq. (2·6) in the right-hand side of eq. (2·9), we have

$$v\Psi = vG_v[(\nabla_1^2+\nabla_2^2+2\mathcal{E}) - v]\Psi. \tag{2·10}$$

If we define the operator u_{12} by

$$u_{12} = vG_v(\nabla_1^2+\nabla_2^2+2\mathcal{E}), \tag{2·11}$$

and drop Ψ in eq. (2·10) (see the footnote in § 3 of I), we have

$$v = u_{12} - vG_v v. \tag{2·10}'$$

This equation implies that the potential v may be expressed in terms of integral operator, since v is expanded as

$$v = u_{12} - vG_v u_{12} + vG_v vG_v u_{12} - vG_v vG_v vG_v u_{12} + \cdots,$$

where u_{12} and vG_v are integral operators. In order to emphasize that v is equivalent to an integral operator, we shall use the notation t_{12} instead of v. Then we have an equation for t_{12} as follows:

$$t_{12} = u_{12} - vG_v t_{12}. \tag{2·12}$$

If we assume that u_{12} has finite Fourier components (that this is true will be proved later) and neglect the higher order terms, t_{12} is given by $t_{12}=u_{12}$ and the usual perturbation method may be applied. In this way, the concept of pseudopotential is generalized to an arbitrary singular potential.

Before entering into discussion on the u-operator, we want to express eq. (2·12) in the momentum representation. Introducing the matrix elements of t and u as,

$$t(\alpha\beta, \lambda\gamma) = (1/V)\int \exp(-i\boldsymbol{k}_\alpha\cdot\boldsymbol{x}_1 - i\boldsymbol{k}_\beta\cdot\boldsymbol{x}_2)\, t_{12} \exp(i\boldsymbol{k}_\lambda\cdot\boldsymbol{x}_1 + i\boldsymbol{k}_\gamma\cdot\boldsymbol{x}_2)\, d\boldsymbol{x}_1 d\boldsymbol{x}_2, \tag{2·13}$$

$$u(\alpha\beta, \lambda\gamma) = (1/V)\int \exp(-i\boldsymbol{k}_\alpha \cdot \boldsymbol{x}_1 - i\boldsymbol{k}_\beta \cdot \boldsymbol{x}_2)\, u_{12} \exp(i\boldsymbol{k}_\lambda \cdot \boldsymbol{x}_1 + i\boldsymbol{k}_\gamma \cdot \boldsymbol{x}_2)\, d\boldsymbol{x}_1 d\boldsymbol{x}_2, \quad (2\cdot14)$$

we have from eqs. (2·11) and (2·12),

$$t(\alpha\beta, \lambda\gamma) = u(\alpha\beta, \lambda\gamma) - \frac{1}{2V}\sum_{\sigma,\tau} \frac{u(\alpha\beta, \sigma\tau)}{k^2 - k_{\sigma\tau}^2}\, t(\sigma\tau, \lambda\gamma), \quad (2\cdot15)$$

where

$$k^2 = \mathcal{E} - (\boldsymbol{k}_\lambda + \boldsymbol{k}_\gamma)^2/4 \quad \text{and} \quad \boldsymbol{k}_{\sigma\tau} = (\boldsymbol{k}_\sigma - \boldsymbol{k}_\tau)/2. \quad (2\cdot16)$$

This is an equation to determine $t(\alpha\beta, \lambda\gamma)$ and corresponds to eq. (3·13) of I. If eq. (2·15) is solved by an iteration procedure and $u(\alpha\beta, \lambda\gamma)$ is assumed to be of the order of a, the solution takes the form of the power series of eq. (1·1). In the following, we shall show that $u(\alpha\beta, \lambda\gamma)$ does exist even if v is singular and that it is an hermitian matrix.

From eqs. (2·11) and (2·14), it follows that

$$u(\alpha\beta, \lambda\gamma) = 2(k^2 - k_{\lambda\gamma}^2)\,(1/V)\int \exp(-i\boldsymbol{k}_\alpha \cdot \boldsymbol{x}_1 - i\boldsymbol{k}_\beta \cdot \boldsymbol{x}_2)\, v(1, 2)\, \Phi_{\lambda\gamma}(1, 2)\, d\boldsymbol{x}_1 d\boldsymbol{x}_2, \quad (2\cdot17)$$

where

$$\Phi_{\lambda\gamma}(1, 2) = G_v \exp(i\boldsymbol{k}_\lambda \cdot \boldsymbol{x}_1 + i\boldsymbol{k}_\gamma \cdot \boldsymbol{x}_2). \quad (2\cdot18)$$

From eqs. (2·6) and (2·8), $\Phi_{\lambda\gamma}(1, 2)$ satisfies the equation

$$[\nabla_1^2 + \nabla_2^2 + 2\mathcal{E} - v(1, 2)]\Phi_{\lambda\gamma}(1, 2) = \exp(i\boldsymbol{k}_\lambda \cdot \boldsymbol{x}_1 + i\boldsymbol{k}_\gamma \cdot \boldsymbol{x}_2). \quad (2\cdot19)$$

Introducing the center-of-mass coordinate $\boldsymbol{X}$, the relative coordinate $\boldsymbol{x}$, $[\boldsymbol{x}_1 = \boldsymbol{X} + (\boldsymbol{x}/2)$, $\boldsymbol{x}_2 = \boldsymbol{X} - (\boldsymbol{x}/2)]$, and the function $\varphi_{\lambda\gamma}(\boldsymbol{x})$ defined by

$$2(k^2 - k_{\lambda\gamma}^2)\Phi_{\lambda\gamma} = \exp[i(\boldsymbol{k}_\lambda + \boldsymbol{k}_\gamma)\cdot\boldsymbol{X}]\varphi_{\lambda\gamma}(\boldsymbol{x}), \quad (2\cdot20)$$

we have

$$u(\alpha\beta, \lambda\gamma) = \delta(\alpha\beta, \lambda\gamma)\int \exp(-i\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{x})\, v(\boldsymbol{x})\, \varphi_{\lambda\gamma}(\boldsymbol{x})\, d\boldsymbol{x} \quad (2\cdot21)$$

and

$$\left[\nabla^2 + k^2 - \frac{v(\boldsymbol{x})}{2}\right]\varphi_{\lambda\gamma}(\boldsymbol{x}) = (k^2 - k_{\lambda\gamma}^2)\exp(i\boldsymbol{k}_{\lambda\gamma}\cdot\boldsymbol{x}), \quad (2\cdot22)$$

where $\delta(\alpha\beta, \lambda\gamma)$ represents the conservation of total momentum.

When $v(\boldsymbol{x})$ is singular at $r \equiv |\boldsymbol{x}| = 0$, $\varphi_{\lambda\gamma}(\boldsymbol{x})$ approaches zero rapidly as $r \to 0$, and $v(\boldsymbol{x})\varphi_{\lambda\gamma}(\boldsymbol{x})$ remains finite at $r = 0$. Therefore, as is seen from eq. (2·21), there is no divergence difficulty in calculating $u(\alpha\beta, \lambda\gamma)$. On the other hand, if $\varphi_{\lambda\gamma}(\boldsymbol{x})$ is replaced by $\exp(i\boldsymbol{k}_{\lambda\gamma}\cdot\boldsymbol{x})$, $u(\alpha\beta, \lambda\gamma)$ is reduced to the Fourier component of $v(\boldsymbol{x})$ and diverges to infinity, owing to the rapid increase of $v(\boldsymbol{x})$ as $r \to 0$.

It can be proved that the relation

$$u^*(\alpha\beta, \lambda\gamma) = u(\lambda\gamma, \alpha\beta) \quad (2\cdot23)$$

holds for the u-matrix. This implies that $u(\alpha\beta, \lambda\gamma)$ is an Hermitian matrix. The proof is as follows: from the defining equations for $\varphi_{\lambda\tau}$ and $\varphi^*_{\alpha\beta}$, we have

$$(k^2-k^2_{\lambda\tau})\int \exp(i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x})\,\varphi^*_{\alpha\beta}(\boldsymbol{x})\,d\boldsymbol{x}=(k^2-k^2_{\alpha\beta})\int \exp(-i\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{x})\,\varphi_{\lambda\tau}(\boldsymbol{x})\,d\boldsymbol{x}, \quad (2\cdot24a)$$

$$\int \exp(-i\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{x})\left[\nabla^2+k^2-\frac{v(x)}{2}\right]\varphi_{\lambda\tau}(\boldsymbol{x})\,d\boldsymbol{x}=(k^2-k^2_{\lambda\tau})V\delta(\boldsymbol{k}_{\alpha\beta},\ \boldsymbol{k}_{\lambda\tau}) \quad (2\cdot24b)$$

and

$$\int \exp(i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x})\left[\nabla^2+k^2-\frac{v(\boldsymbol{x})}{2}\right]\varphi^*_{\alpha\beta}(\boldsymbol{x})\,d\boldsymbol{x}=(k^2-k^2_{\alpha\beta})V\delta(\boldsymbol{k}_{\alpha\beta},\ \boldsymbol{k}_{\lambda\tau}). \quad (2\cdot24c)$$

Subtracting eq. (2·24c) from eq. (2·24b) and using Green's theorem and eq. (2·24a), we immediately have eq. (2·23).

We have shown above that the u-matrix is finite even if v is singular at $r=0$. In order to see the structure of the u-matrix, it may be helpful to consider the system with weak interaction for which the Fourier components of v exist. Then eq. (2·22) can be solved by the usual perturbation method. The substitution of the solution in (2·21) leads to

$$u(\alpha\beta, \lambda\gamma)=\delta(\alpha\beta, \lambda\gamma)\left[\nu(\boldsymbol{k}_{\alpha\beta}-\boldsymbol{k}_{\lambda\tau})+\frac{1}{2V}\sum_{\boldsymbol{q}'}\frac{\nu(\boldsymbol{k}_{\alpha\beta}-\boldsymbol{q}')\nu(\boldsymbol{q}'-\boldsymbol{k}_{\lambda\tau})}{k^2-q'^2}\right.$$
$$\left.+\frac{1}{2^2V^2}\sum_{\boldsymbol{q}',\boldsymbol{q}''}\frac{\nu(\boldsymbol{k}_{\alpha\beta}-\boldsymbol{q}')\nu(\boldsymbol{q}'-\boldsymbol{q}'')\nu(\boldsymbol{q}''-\boldsymbol{k}_{\lambda\tau})}{(k^2-q'^2)(k^2-q''^2)}+\cdots\right] \quad (2\cdot25)$$

where $\nu(\boldsymbol{q})$ is the Fourier component of $v(\boldsymbol{x})$ defined by

$$\nu(\boldsymbol{q})=\int \exp(-i\boldsymbol{q}\cdot\boldsymbol{x})\,v(\boldsymbol{x})\,d\boldsymbol{x}.$$

From eq. (2·25) it follows that the solution of eq. (2·15) is given by

$$t(\alpha\beta, \lambda\gamma)=\delta(\alpha\beta, \lambda\gamma)\nu(\boldsymbol{k}_{\alpha\beta}-\boldsymbol{k}_{\lambda\tau}).$$

This equation is expected to hold from the defining equation for the t-matrix, (2·13), if one notes that this matrix is equivalent to $v(\boldsymbol{x})$ though the letter "t" is used to indicate explictly that it is an integral operator. We see that $t(\alpha\beta, \lambda\gamma)$ is independent of the value of k^2 though $u(\alpha\beta, \lambda\gamma)$ depends on k^2. (See also the note given in § 5 of I).

On the other hand, for the system with strong interaction $\nu(\boldsymbol{q})$ diverges to infinity, and each term of the series of eq. (2·25) diverges. However, when the summation is carried out to infinite orders, $u(\alpha\beta, \lambda\gamma)$ is no longer divergent. Therefore, if one performs the first-order perturbation calculation by using the u-matrix, it is expected that such a calculation picks up all the divergent terms of the conventional perturbation series which lead to the convergent term of the order of a (u is assumed to be of the order of a).

Summarizing the results, the perturbation method for the singular potential will be

as follows. First, solve the differential equation (2·22) under the periodic boundary condition and by the requirement that $\varphi_{\lambda\tau}(\boldsymbol{x})$ is zero at $r=0$. Then, calculate the matrix element $u(\alpha\beta, \lambda\gamma)$ by eq. (2·21). If one confines oneself to the first-order energy, apply the conventional perturbation method, replacing the martrix element of v by $u(\alpha\beta, \lambda\gamma)$. If one wishes to proceed to the second-order energy, replace $t(\sigma\tau, \lambda\gamma)$ in eq. (2·15) by $u(\sigma\tau, \lambda\gamma)$ and apply the conventional method. The similar procedure may be applied to the higher order terms of the energy. One example of this procedure has been discussed in I for the hard-sphere potential. In applying the usual perturbation formulas some care must be taken as we remarked in § 3, § 4 and § 5 of I.

It should be noted here that the u-matrix depends on k^2 which may be assumed to take arbitrary values. One way to remove such an arbitrariness may be as follows (see also § 5 of I). We expand the u-matrix in powers of k^2 as

$$u=u_0+k^2u_1+k^4u_2+\cdots.$$

It is expected that u_1 is of higher order, e.g. $u_1\sim a^3$, while $u_0\sim a$ for the hard-sphere system. Hence the lowest order term of the energy can be calculated without any ambiguities associated with the arbitrariness of k^2. Therefore, if we expand $\mathcal{E}$ in powers of a as

$$\mathcal{E}=\mathcal{E}_0+a\mathcal{E}_1+a^2\mathcal{E}_2+\cdots,$$

$\mathcal{E}_1$ is calculated exactly. Next, the factor $(k^2-k_{\sigma\tau}^2)^{-1}$ in eq. (2·15) is expanded in powers of a with the aid of eq. (2·16). In such an expansion the term of the order of a is known from the argument given above. If we further carry out the similar procedure, the energy eigenvalue may be expanded as a power series of a, which corresponds to the perturbation series for the system with weak interactions.

In an actual calculation of the u-matrix, it is difficult to solve eq. (2·22) under the periodic boundary condition, so we will follow the method of partial waves, which is apparently valid for the limit $V\to\infty$. If we expand $\varphi_{\lambda\tau}(\boldsymbol{x})$ as

$$\varphi_{\lambda\tau}(\boldsymbol{x})=\sum_{l=0}^{\infty}(2l+1)\,i^lR_l(r)\,P_l(\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x}), \tag{2·26}$$

$R_l(r)$ satisfies the equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{r^2}+k^2-\frac{v(r)}{2}\right]R_l(r)=(k^2-k_{\lambda\tau}^2)\,j_l(k_{\lambda\tau}r), \tag{2·27}$$

and the u-matrix is given by

$$u(\alpha\beta, \lambda\gamma)=4\pi\delta(\alpha\beta, \lambda\gamma)\sum_{l=0}^{\infty}(2l+1)\,P_l(\boldsymbol{k}_{\alpha\beta}\cdot\boldsymbol{k}_{\lambda\tau})\int_0^{\infty}j_l(k_{\alpha\beta}r)\,v(r)\,R_l(r)\,r^2dr. \tag{2·28}$$

In solving eq. (2·27), $R_l(r)$ should be zero at $r=0$ as a boundary condition. On the other hand, the asymptotic form of $R_l(r)$ for $r\to\infty$ becomes

$$R_l(r)\to j_l(k_{\lambda\tau}r)+A_lkn_l(kr), \quad (r\to\infty), \tag{2·29}$$

corresponding to the periodic boundary condition imposed on $\varphi_{\lambda\tau}(\boldsymbol{x})$. Here A_l is some

function of k and $k_{\lambda\tau}$, independent of r. Proof of eq. (2·29) is as follows: we write eq. (2·22) in the following form,

$$(\nabla^2+k^2)\varphi_{\lambda\tau}(\boldsymbol{x})=(k^2-k_{\lambda\tau}^2)\exp(i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x})+v(\boldsymbol{x})\varphi_{\lambda\tau}(\boldsymbol{x})/2.$$

The solution of this equation subject to the periodic boundary condition is given by

$$\varphi_{\lambda\tau}(\boldsymbol{x})=\int G(\boldsymbol{x},\boldsymbol{x}')[(k^2-k_{\lambda\tau}^2)\exp(i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x}')+v(\boldsymbol{x}')\varphi_{\lambda\tau}(\boldsymbol{x}')/2]d\boldsymbol{x}', \tag{2·30}$$

where $G(\boldsymbol{x},\boldsymbol{x}')$ is defined by eq. (2·9) of I. From eq. (2·9) of I and eq. (2·30), we have

$$\varphi_{\lambda\tau}(\boldsymbol{x})=\exp(i\boldsymbol{k}_{\lambda\tau}\cdot\boldsymbol{x})+(1/2)\int G(\boldsymbol{x},\boldsymbol{x}')v(\boldsymbol{x}')\varphi_{\lambda\tau}(\boldsymbol{x}')d\boldsymbol{x}'.$$

Substituting eq. (2·11) of I and eq. (2·26) in the second term, and using eq. (2·10) of I, we have

$$R_l(r)=j_l(k_{\lambda\tau}r)+(1/2)\int_0^\infty G_l(r,r')v(r')R_l(r')r'^2dr'.$$

With the aid of eq. (2·13) of I, $R_l(r)$ becomes, for large r,

$$R_l(r)=j_l(k_{\lambda\tau}r)+kn_l(kr)\,(1/2)\int_0^\infty j_l(kr')v(r')R_l(r')r'^2dr',$$

which is the asymptotic form expressed in eq. (2·29).*

§ 3. Application to Lennard-Jones potential

As an application of the method developed in § 2, we will calculate in this section the lowest order term of the ground state energy for a Bose system with the Lennard-Jones potential. This term is given by

$$E/N=(\hbar^2\rho/4m)u(00,\,00), \tag{3·1}$$

where ρ is the average density N/V. The energy given by eq. (3·1) corresponds to the first order energy arising from the u-matrix.

In calculating $u(00,\,00)$, k^2 is assumed to be zero, since ε is zero in the unperturbed system and hence $k^2=0$, as is seen from eq. (2·16). Furthermore, from eq. (2·28), it follows that the contribution from the terms $l>0$ vanishes, since $k_{\alpha\beta}=0$ and hence $j_l(k_{\alpha\beta}r)=0$ for $l>0$; only the contribution from the S-state does not vanish. From these considerations, it follows that

$$u(00,\,00)=4\pi\int_0^\infty v(r)R(r)r^2dr, \tag{3·2}$$

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{v(r)}{2}\right]R(r)=0, \tag{3·3a}$$

* It should be noted that for $k^2<0$ another form of the asymptotic behaviour of $R(r)$ should be employed. This is the case when the system has the positive binding ($\varepsilon<0$).

and

$$R(r)\to 1, \quad (r\to\infty). \tag{3·3b}$$

The asymptotic form for $R(r)$, eq. (3·3b), is derived from eq. (2·29) putting $k=0$. Combining eqs. (3·1), (3·2) and (3·3a), we have

$$E/N=2\pi\hbar^2\rho/m\cdot(r^2\cdot dR/dr)_{r=\infty}. \tag{3·4}$$

This is an exact expression for the lowest order term of the ground state energy. The function $R(r)$ is the solution of differential equation (3·3a), subject to the condition that it is zero at $r=0$ in addition to eq. (3·3b). For the hard-sphere potential, $R(r)$ is given by $R(r)=1-(a/r)$, $(r\geq a)$, and we have

$$E/N=2\pi\hbar^2\rho a/m,$$

which is just Lenz's[12] formula.

In applying eq. (3·4) to the Lennard-Jones potential, it is convenient to employ the form

$$V(r)=V_0[(\sigma/r)^{10}-(\sigma/r)^6], \tag{3·5}$$

which was used by Green[13] to calculate the second virial coefficient of liquid helium at low temperatures. For this type of potential, it is easily verified that $R(r)$ is given by

$$R(r)=\frac{\Gamma(5/8-\eta)}{\Gamma(1/4)}\left(\frac{\sqrt{\lambda}}{2}\right)^{-3/8}r^{3/2}W_{\eta,1/8}\left(\frac{\sqrt{\lambda}}{2}r^{-4}\right), \tag{3·6}$$

where

$$\lambda=mV_0\sigma^{10}/\hbar^2, \tag{3·7}$$

$$\eta=\sqrt{mV_0}\sigma/8\hbar, \tag{3·8}$$

Γ and W are the gamma function and Whittaker's function,[14] respectively. In these expressions, η is proportional to the reciprocal of de Boer's[15] quantum mechanical parameter and is the non-dimensional number characteristic of the Lennard-Jones potential.

By using the properties of Whittaker's function, we have for $r\to\infty$

$$R(r)=1+\frac{\Gamma(-1/4)\Gamma(5/8-\eta)}{\Gamma(1/4)\Gamma(3/8-\eta)}\left(\frac{\sqrt{\lambda}}{2}\right)^{1/4}\frac{1}{r}+O\left(\frac{1}{r^4}\right). \tag{3·9}$$

Making use of the relation $\Gamma(-1/4)=-4\Gamma(3/4)$, we have from eqs. (3·4) and (3·7)

$$\frac{E}{N}=4\pi\frac{2^{3/4}\Gamma(3/4)\Gamma(5/8-\eta)}{\Gamma(1/4)\Gamma(3/8-\eta)}m^{-7/8}V_0^{1/8}\sigma^{5/4}\hbar^{7/4}\rho, \tag{3·10}$$

which corresponds to Lenz's formula for the hard-sphere potential. The appearance of $V_0^{1/8}$ is interesting, since it clearly shows the non-validity of the conventional perturbation method in which E/N is expanded in powers of V_0.

In order to clarify the meaning of eq. (3·10), we will discuss the meaning of Lenz's formula. As we have noted in I, an exact expression of E/N for the hard-sphere system in the low density limit, is given by

$$\frac{E}{N}=\frac{2\pi\hbar^2\rho a}{m}\left\{1+\frac{128}{15\sqrt{\pi}}(\rho a^3)^{1/2}\right\}.$$

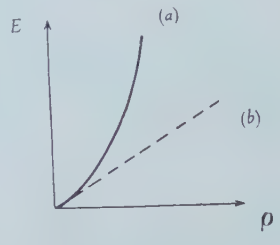
From this, it follows that Lenz's formula gives the value of $\partial(E/N)/\partial\rho$ at $\rho=0$ exactly. In other words, Lenz's formula corresponds to approximating the energy versus density curve by a straight line which is tangent to the exact curve at $\rho=0$, (Fig. 1).

Correspondingly, it is expected that eq. (3·10) gives an exact value of $\partial(E/N)/\partial\rho$ at $\rho=0$ for the Bose system with the Lennard-Jones potential (3·5). From the properties of the gamma function, it follows that

$$(\partial E/\partial\rho)_{\rho=0}>0 \quad \text{for} \quad 0\leq\eta<3/8,$$

while

$$(\partial E/\partial\rho)_{\rho=0}<0 \quad \text{for} \quad 3/8<\eta<5/8, \quad \text{(see Fig. 2)}.$$



(*a*): exact curve.
(*b*): Lenz's formula.
Fig. 1.

The physical interpretation of these results may be given in the following manner. If one neglects the attractive force in eq. (3·5), one can easily verify that E/N is given by eq. (3·10) with $\eta=0$. In other words, $\eta=0$ corresponds to the purely repulsive force. In this case, it is physically obvious that the energy is an increasing function of density, and hence $(\partial E/\partial\rho)_{\rho=0}>0$. As η becomes larger than 0, the attractive force comes into play and $(\partial E/\partial\rho)_{\rho=0}$ decreases until it becomes negative for $\eta>3/8$.

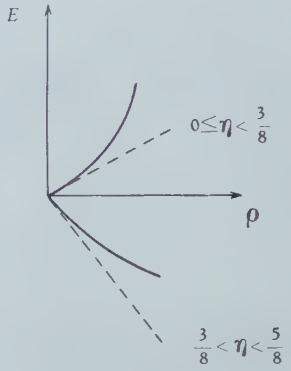


Fig. 2.

To apply the above results to liquid He^4, we take the values (in C.G.S. units),

$$V_0=4.25\times10^{-15}, \quad \hbar=1.05\times10^{-27},$$

$$\sigma=2.72\times10^{-8}, \quad m=6.64\times10^{-24},$$

and

$$\rho=2.18\times10^{22},$$

where the values of V_0 and σ are taken from Green's text book[16]. From these values, we have $\eta=0.543$, which is intermediate between 3/8 and 5/8. Hence, the energy is a decreasing function of density at low densities. At sufficiently high densities, however, the effect of attractive force will be small compared with that of repulsive one and the energy will increase as the density becomes larger (Fig. 3). Therefore, there should occur the energy minimum at an appropriate density. It is obvious that the minimum energy is negative, which accounts for the positive binding of liquid He^4. More quantitatively, substituting the numerical values mentioned above in eq. (3·10), we have

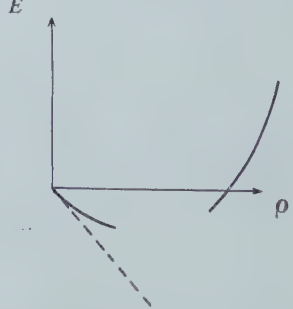


Fig. 3.

$$E/N=-17.6\times10^{-16}\ \text{erg/atom},$$

which is about twice as large as the experimentally observed cohesive energy $E/N = -9.7 \times 10^{-16}$ erg/atom.

In closing this section, we wish to mention some remarks about the more conventional form of the Lennard-Jones potential,

$$V(r) = V_0[(\sigma/r)^{12} - 2(\sigma/r)^6].$$

Analytical treatment of the differential equation (3·3a) is difficult in this case, but by a simple dimensional argument, the dependence of E/N on the physical quantities is shown to be

$$E/N \sim m^{-9/10} V_0^{1/10} \sigma^{6/5} \hbar^{9/5} \rho,$$

corresponding to eq. (3·10).

§ 4. Discussion

Although the formula, eq. (3·10), accounts for the cohesion of liquid He^4, it should be noted that in this case there are some difficulties which cannot be overcome by the methods appropriate for the purely repulsive interactions, as Brueckner and Sawada[3] have pointed out. Such a difficulty arises if one applies eq. (4·1) of I to this case, in order to calculate the excitation energy spectrum; for $u(00, 00)$ is negative and hence the sound velocity becomes imaginary, corresponding to the fact that the two-body scattering length is negative. This feature, however, is to be expected physically, since at low densities the system can find states of much lower energy by contracting to a smaller volume in the configuration space and hence the sound cannot propagate through the medium.

Now, the contraction of system in the configuration space in turn implies the extension in the momentum space. It seems, therefore, that the assumption, usually taken for the Bose system in transforming the particle Hamiltonian to the phonon Hamiltonian[3)17)], that nearly all the particles are in the single particle state with momentum 0 cannot be applied to the actual liquid He^4. It seems in this case that the above mentioned assumption should be altered so as to take into account the spreading out in the momentum space in some way.

The situation is quite different for the Fermi system. In this case the difficulty mentioned above does not occur. The essential difference coming from quantum statistics may be seen from the fact that the expansion parameter for the Bose system is $(\rho a^3)^{1/2}$ while $(\rho a^3)^{1/3}$ for the Fermi system, where a is the scattering length. If one changes the sign of a, $(\rho a^3)^{1/2}$ becomes imaginary while $(\rho a^3)^{1/3}$ merely changes its sign. Hence the difficulty associated with the negative scattering length does not occur for the Fermi system.

In conclusion the author wishes to express his sincere thanks to Professor. N. Fukuda for valuable discussions.

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General Solution of the Bethe-Salpeter Equation in Instantaneous Interaction Approximation*

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The Bethe-Salpeter equation is solved for general values of the angular momentum, using only the ladder approximation and the instantaneous interaction approximation. It is first reduced to a set of coupled equations for the singlet and triplet amplitudes, and these are further reduced to the corresponding Schrödinger equations. The non-relativistic potentials are deduced and are found to contain automatically a "core" singularity and spin orbit coupling terms, which are necessary to account for experimental data. The coupling constant, which is the only adjustable parameter, is fitted to the deuteron binding energy, and the potentials are compared with phenomenological potentials. Satisfactory agreement is obtained.

§ 1. Introduction

In an earlier paper[1] a fully covariant solution of the Bethe-Salpeter (B-S) two-nucleon problem was obtained. The "ladder" approximation was adopted, but pair creation and nucleonic recoil were exactly accounted for. The B-S equation was reduced to a pair of coupled differential equations in which the Dirac matrices appear only in the coupling. Solutions were obtained with varying degrees of explicitness, for instantaneous and delayed interaction.

A feature of the solutions obtained there is that consideration was restricted to solutions corresponding to states of angular momentum $J=0$ but we indicated a possible method of obtaining solutions corresponding to higher angular momentum states from those with $J=0$, making use of the Lorentz transformation.

In this paper we investigate the solutions of the B-S equation corresponding to states of higher angular momentum than $J=0$. However, the instantaneous interaction approximation has been adopted. As explained in the earlier paper[1], previous work[2)5)] has been based on three distinct approximations, as well as the "ladder" approximation, of which it is now proposed to retain only one. That is the instantaneous interaction approximation.

It is found that the B-S equation, with this approximation can be reduced to an equivalent Schrödinger equation even to the extent of predicting a nuclear potential with a repulsive core similar to that of Levy[2] and Jastrow.[3] A singlet even parity potential

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for the proton-proton system has been separated from the triplet even parity potential for the neutron-proton system, and these are compared with the phenomenological potential function of Gammel.[4] Another interesting fact, that the nuclear potential contains a spin-orbit coupling term, has also been confirmed.

Numerical calculations with the instantaneous interaction potential obtained here, have been performed to fit the binding energy of the deuteron and other n-p scattering data.

§ 2. Reduction of the B-S equation

We recall that the B-S equation (See eq. (2·1) of reference 1), hereafter referred to as GBI) can be reduced to a system of differential equations given below, in which Dirac matrices appear only in the coupling:

$$\begin{aligned}
&\{\Box+M^2-p^2+f(x)\}\,\omega'=2p\cdot\nabla\omega\\
&\{\Box+M^2-p^2+f(x)\}\,\varphi=i\{(\nabla f)\,(\omega+i\omega')-(\omega-i\omega')\,(\nabla f)\}\\
&\{\Box+M^2-p^2-f(x)\}\,\omega=-2p\cdot\nabla\omega'+\tfrac{1}{2}\{p,\varphi\}+\tfrac{1}{2}i[\nabla,\varphi]
\end{aligned}\tag{2·1}$$

where the meaning of the variables are the same as in GBI. As an ansatz we table the following form of ω, ω' and φ occurring in (2·1):

$$\omega=\omega_0\Psi_{0,j}+\omega_1\Psi_{1,j}+\omega_2\Psi_{1,j-1}+\omega_3\Psi_{1,j+1}\tag{2·2}$$

$$\omega'=\omega_0'\Psi_{0,j}+\omega_1'\Psi_{1,j}+\omega_2\Psi_{1,j-1}+\omega_3'\Psi_{1,j+1}\tag{2·3}$$

$$\varphi=\varphi_0\Psi_{0,j}\gamma_4+\varphi_1\Psi_{1,j}\gamma_4+\varphi_2\Psi_{1,j-1}\gamma_4+\varphi_3\Psi_{1,j+1}\gamma_4\tag{2·4}$$

where ω_0, ω_1, ω_2, ω_3, ω_0', ω_1', ω_2', ω_3', φ_0, φ_1, φ_2 and φ_3 are all functions of r and x_4, and

$$\Psi_{0,j}=P_j\left(\frac{x_3}{r}\right)\gamma_4\tag{2·5}$$

$$\Psi_{1,j}=\left(\frac{x_1}{r}\gamma_1+\frac{x_2}{r}\gamma_2\right)P_j'\left(\frac{x_3}{r}\right)\gamma_3\gamma_4\tag{2·6}$$

$$\Psi_{1,j-1}=j\gamma_3P_{j-1}\left(\frac{x_3}{r}\right)-\left(\frac{x_1}{r}\gamma_1+\frac{x_2}{r}\gamma_2\right)P_{j-1}'\left(\frac{x_3}{r}\right)\tag{2·7}$$

$$\Psi_{1,j+1}=(j+1)\gamma_3P_{j+1}\left(\frac{x_3}{r}\right)+\left(\frac{x_1}{r}\gamma_1+\frac{x_2}{r}\gamma_2\right)P_{j+1}'\left(\frac{x_3}{r}\right)\tag{2·8}$$

where the P_j's are the Legendre polynomials of order j and argument (x_3/r). Let $\boldsymbol{M}$ represent the vector formed from the components M_{23}, M_{31} and M_{12} of the angular momentum operator, $M_{\lambda\mu}$ given by

$$M_{\lambda\mu}=-i\left(x_\lambda\frac{\partial}{\partial x_\mu}-x_\mu\frac{\partial}{\partial x_\lambda}\right);\ \lambda,\mu=1,\cdots4.$$

Also, let $\boldsymbol{S}$ be total spin operator for the two particles, given by

$$\boldsymbol{S}=\tfrac{1}{2}(\sigma-\sigma')$$

when σ and $-\sigma'$ are the usual spin matrices for the two particles respectively. Then the functions $\Psi_{0,j}$, $\Psi_{1,j-1}$, $\Psi_{1,j+1}$ are the eigenfunctions of the operator $\boldsymbol{M}\cdot\boldsymbol{S}$ We have from $(\boldsymbol{M}+\boldsymbol{S})^2=\boldsymbol{M}^2+\boldsymbol{S}^2+2\boldsymbol{M}\cdot\boldsymbol{S}$ the relation $2\boldsymbol{M}\cdot\boldsymbol{S}=j(j+1)-l(l+1)-s(s+1)$ where j, l, s are the eigenvalues of the total angular momentum, orbital angular momentum and the spin momentum respectively. It is easily seen that $\Psi_{0,j}$ is the eigenfunction of $\boldsymbol{M}\cdot\boldsymbol{S}$ for the singlet state with eigenvalue 0, corresponding to $\boldsymbol{S}^2=0$ and $\boldsymbol{M}^2=j(j+1)$; $\Psi_{1,j}$ is for $\boldsymbol{M}\cdot\boldsymbol{S}=-1$, $\boldsymbol{S}^2=2$, $l=j$, $\boldsymbol{M}\cdot\boldsymbol{S}=-1$, $\boldsymbol{M}^2=j(j+1)$; $\Psi_{1,j-1}$ is for $\boldsymbol{S}^2=2$, $l=j-l$, $\boldsymbol{M}\cdot\boldsymbol{S}=l$, $\boldsymbol{M}^2=j(j-1)$ and $\Psi_{1,j+1}$ for $\boldsymbol{S}^2=2$, $l=j+1$, $\boldsymbol{M}\cdot\boldsymbol{S}=-(l+1)$, $\boldsymbol{M}^2=(j+1)(j+2)$. M_3+S_3 is a constant of the motion. It should be pointed out that the $\boldsymbol{M}\cdot\boldsymbol{S}$ operator does not commute with the B-S operator $\mathfrak{S}$ as mentioned in § 8 of GBI. Thus Ψ's are not proper solutions of the B-S equation, and the general solution of the B-S equation must be taken as linear combination of them with coefficients which are as yet unknown functions of r and time variable. One obtains in this way only solutions for magnetic quantum number $m=0$. To obtain solutions corresponding to the magnetic quantum number $m=+1$ or $m=-1$, we multiply the above solution by (M_-+S_-) and (M_-+S_-) where $M_\pm=M_{23}\pm iM_{31}$ and $S_\pm=(S_1\pm iS_2)$ and for higher values of $m=\pm n$ we multiply the solutions by $(M_\pm+S_\pm)^n$.

Another solution of the B-S equation can be obtained by changing the parity of the functions ω, ω', and φ. This is achieved by multiplying γ_5 on the right, in the right-hand side of the expressions for ω, ω' and φ given in (2·2), (2·3) and (2·4).

Using the recurrence relation for the Legendre functions, we get

$$\frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{r}\Psi_{0,j}\gamma_4=\{\Psi_{1,j+1}+\Psi_{2,j-1}\}\quad(2j+1)$$

and similar results with $\Psi_{1,j}$, etc., can also be easily deduced. Again, noticing that for two matrices f and g the Dirac conjugate satisfies :

$$(fg)^+=g^+f^+,\ \gamma_\lambda^+=\gamma_\lambda=(\gamma_4\gamma_\lambda{}^*\gamma_4)$$

we have

$$\Psi_{0,j}^+=\Psi_{0,j},\qquad \Psi_{1,j}^+=-\Psi_{1,j};\qquad \Psi_{1,j\pm1}^+=\Psi_{1,j\pm1}.$$

Also the following relation holds,

$$\Psi_{0,j}\frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{r}=-\frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{r}\Psi_{0,j}$$

and similar results hold for other Ψ's.

Using these results and introducing the expressions given in (2·2), (2·3) and (2·4) in (2·1) we have the following 12 differential equations after equating the coefficients of the $\Psi_{0,j}$'s, etc., from both sides of (2·1). This system of equations completely determines the unknown functions ω_0, etc.. Thus we write :

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2+f\right\}\omega_0'=2E\frac{\partial\omega_0}{\partial x_4} \tag{2·9}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2+f\right\}\omega_1'=2E\frac{\partial\omega_1}{\partial x_4} \tag{2·10}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j-1)}{r^2}-E^2+M^2+f\right\}\omega_2'=2E\frac{\partial\omega_2}{\partial x_4} \tag{2·11}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{(j+1)(j+2)}{r^2}-E^2+M^2+f\right\}\omega_3'=2E\frac{\partial\omega_3}{\partial x_4} \tag{2·12}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2+f\right\}\varphi_0=2i\dot{f}(i\omega_0')+2ir\tilde{f}\cdot$$
$$\{-ij\omega_2'-i(j+1)\omega_3'\} \tag{2·13}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2+f\right\}\varphi_1=2i\dot{f}(i\omega_1')+2ir\tilde{f}(\omega_2-\omega_3) \tag{2·14}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j-1)}{r^2}-E^2+M^2+f\right\}\varphi_2=-2i\dot{f}\omega_2+\frac{2ir\tilde{f}}{2j+1}\cdot$$
$$\{\omega_0-i(j+1)\omega_1'\} \tag{2·15}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{(j+1)(j+2)}{r^2}-E^2+M^2+f\right\}\varphi_3=-2i\dot{f}\omega_3+\frac{2ir\tilde{f}}{2j+1}\cdot$$
$$\{\omega_0+ij\omega_1'\} \tag{2·16}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2-f\right\}\omega_0=-2E\frac{\partial\omega_0'}{\partial x_4}+E\varphi_0+ij\cdot$$
$$\left\{\frac{j-1}{r}\varphi_2-r\tilde{\varphi}_2\right\}-i(j+1)\left\{\frac{(j+2)}{r}\varphi_3+r\tilde{\varphi}_3\right\} \tag{2·17}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j+1)}{r^2}-E^2+M^2-f\right\}\omega_1=-2E\frac{\partial\omega_1'}{\partial x_4}+E\varphi_1 \tag{2·18}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{j(j-1)}{r^2}-E^2+M^2-f\right\}\omega_2=-2E\frac{\partial\omega_2'}{\partial x_4}-i\frac{\partial\varphi_2}{\partial x_4}$$
$$-i\frac{j+1}{2j+1}\left\{\frac{j+1}{r}\varphi_1+r\tilde{\varphi}_1\right\} \tag{2·19}$$

$$\left\{\frac{\partial^2}{\partial x_4^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{(j+1)(j+2)}{r^2}-E^2+M^2-f\right\}\omega_3=-2E\frac{\partial\omega_3'}{\partial x_4}-i\frac{\partial\varphi_3}{\partial x_4}-i\frac{j}{2j+1}\cdot$$
$$\left\{\frac{j}{r}\varphi_1-r\tilde{\varphi}_1\right\} \tag{2·20}$$

where we have made use of the fact that $p=(\mathbf{0}, E)$; $\dot{f}$ denotes $\frac{d}{dx_4}f$ and $\tilde{f}$ denotes $\frac{\partial}{r\partial r}f$ and similarly $\tilde{\varphi}=\frac{\partial}{r\partial r}\varphi$.

§ 3. Solutions of the equations

Noticing that $$\frac{\partial^2}{\partial r^2}-\frac{j(j+1)}{r^2}=\left(\frac{\partial}{\partial r}-\frac{j}{r}\right)\left(\frac{\partial}{\partial r}+\frac{j}{r}\right)$$
$$=\left(\frac{\partial}{\partial r}+\frac{j+1}{r}\right)\left(\frac{\partial}{\partial r}-\frac{j+1}{r}\right)$$

we may write $D_j^- D_j^+ = D_{j+1}^+$,

where $D_j^- = \left(\frac{\partial}{\partial r}+\frac{j}{r}\right)$. Again it is interesting to note that if $f(D_j^+ D_j^-)$ is any operator function of the product operators $D_j^+ D_j^-$, we have

$$D_j^+ f(D_j^- D_j^+) = f(D_{j-1}^- D_{j-1}^+) D_j^+.$$

Similarly we can write

$$r\tilde{\varphi}_1+\frac{j+1}{r}\varphi_1=\frac{1}{r}D_j^+ r\varphi_1$$

$$r\tilde{\varphi}_3+\frac{j+2}{r}\varphi_3=\frac{1}{r}D_{j+1}^+ r\varphi_3$$

$$r\tilde{\varphi}_2-\frac{j-1}{r}\varphi_2=\frac{1}{r}D_j^- r\varphi_2$$

$$r\tilde{\varphi}_1-\frac{j}{r}\varphi_1=\frac{1}{r}D_{j+1}^- r\varphi_1.$$

Defining $T^2=(M^2-D_j^- D_j^+)$ and $S^2=(M^2-E^2-D_j^- D_j^+)$, we now proceed to write down the solutions of the differential equations (2·9) — (2·19). In the instantaneous interaction we set $f(x)=2iV(r)\ \delta(p\cdot x/E)$, so that $2V(r)$ is an analogue of Yukawa's classical potential energy. From here on the centre of mass frame is adopted, so that $\boldsymbol{p}=0$, $p_4=E$ and $\delta(p\cdot x/E)=\delta(x_4)$.

Because f vanishes when $x_4 \neq 0$, equations (2·9) — (2·20) can be easily solved, since then most of the inhomogeneous terms drop out. The general formal solutions of (2·13) — (2·16) are

$$r\varphi_0=\exp(iS|x_4|)\,u_0 \tag{3·1}$$

$$r\varphi_1=\exp(iS|x_4|)\,u_1\,Sgn\,x_4 \tag{3·2}$$

$$r\varphi_2=D_j^+\exp(iS|x_4|)\,u_2 \tag{3·3}$$

$$r\varphi_3=D_{j+1}^-\exp(iS|x_4|)\,u_3 \tag{3·4}$$

where u_0, u_1, u_2 and u_3 are functions of r only, and S is the integral operator, $(M^2-E^2-D_j^- D_j^+)^{1/2}$. It should be noticed that terms like $\exp(-iS|x_4|)u_0$, etc., are excluded from the solutions by the boundary conditions φ_0, etc., $\to 0$ as $x_4 \to \pm\infty$, since M is supposed to have a small positive imaginary part. Using (3·1) — (3·4), the equations (2·9) — (2·12) and (2·16) — (2·19) are readily solved for $x_4 \neq 0$. We have

$$r\omega_0 = e^{iT|x_4|}(\alpha_0 \cos Ex_4 - \beta_0 \sin Ex_4\, Sgn\, x_4) \tag{3·5}$$

$$r\omega_0' = e^{iT|x_4|}(\alpha_0 \sin Ex_4 + \beta_0\, Sgn\, x_4 \cos Ex_4) - i(2S)^{-1}e^{iS|x_4|}u_0 Sgn x_4$$
$$- jD_j^- D_j^+ (2ES)^{-1}e^{iS|x_4|}u_0 Sgn x_4 - (j+1)D_j^- D_j^+ (2ES)^{-1}e^{iS|x_4|}u_3 \tag{3·6}$$

$$r\omega_1 = e^{iT|x_4|}(\alpha_1 \cos Ex_4\, Sgn x_4 - \beta_1 \sin Ex_4) \tag{3·7}$$

$$r\omega_1' = e^{iT|x_4|}(\alpha_1 \sin Ex_4 Sgn x_4 + \beta_1 \cos Ex_4) - i(2S)^{-1}e^{iS|x_4|}u_1 \tag{3·8}$$

$$r\omega_2 = D_j^+ e^{iT|x_4|}(\alpha_2 \cos Ex_4\, Sgn x_4 - \beta_2 \sin Ex_4) \tag{3·9}$$

$$r\omega_2' = D_j^+ e^{iT|x_4|}(\alpha_2 \sin Ex_4\, Sgn\, x_4 + \beta_2 \cos Ex_4)$$
$$- i\, D_j^+ (2E)^{-1}e^{iS|x_4|}u_2 - \frac{j+1}{2j+1}D_j^+ (2ES)^{-1}e^{iS|x_4|}u_1 \tag{3·10}$$

$$r\omega_3 = D_{j+1}^- e^{iT|x_4|}(\alpha_3 \cos Ex_4\, Sgn x_4 - \beta_3 \sin Ex_4) \tag{3·11}$$

$$r\omega_3' = D_{j+1}^- e^{iT|x_4|}(\alpha_3 \sin Ex_4\, Sgn\, x_4 + \beta_3 \cos Ex_4)$$
$$- iD_{j+1}^- (2E)^{-1}e^{iS|x_4|}u_3 - \frac{j}{2j+1}D_{j+1}^- (ES)^{-1}e^{iS|x_4|}u_1 \tag{3·12}$$

where T is the integral operator $(M^2 - D_j^- D_j^+)^{1/2}$; and α_0, β_0, α_1, β_1, α_2, β_2, α_3 and β_3 are all functions of r only. We notice from the above solutions that φ_0, φ_2, φ_3, ω_0, ω_1', ω_2' and ω_3' are even functions of x_4 while φ_1, ω_0, ω_1, ω_2, ω_3 are odd functions of x_4 which vanish on the space-like surface $x_4 = 0$.

To continue the solutions $(3·1) - (3·12)$ at $x_4 = 0$, we substitute them back into the equation $(2·9) - (2·19)$ and then integrate with respect to x_4 over a small range containing $x_4 = 0$. The validity of the solutions $(3·1) - (3·12)$ at $x_4 = 0$ requires

$$2\beta_0 = 2i(2S)^{-1}u_0 + jD_j^- D_j^+ (2ES)^{-1}2u_2 + (j+1)D_j^- D_j^+ 2(2ES)^{-1}u_3$$

$$2iT\,\beta_1 + 2S(2S)^{-1}u_1 + 2iV(r)\{\beta_1 - i(2S)^{-1}u_1\} = 0$$

$$D_j^+\left\{2iT\,\beta_2 + 2E\alpha_1 + S(2E)^{-1}u_2 - i\frac{j+1}{2j+1}(2S)(2ES)^{-1}u_1\right\}$$
$$+ 2iV(r)\left\{D_j^+\beta_2 - iD_j^+ (2E)^{-1}u_2 - \frac{j+1}{2j+1}D_j^+ (2ES)^{-1}u_1\right\} = 0$$

$$D_{j+1}^-\left\{2iT\beta_3 + 2S(2E)^{-1}u_3 + i\frac{j+1}{2j+1}(2S)(2ES)^{-1}u_1\right\}$$
$$+ 2iV(r)\left\{D_{j+1}^-\beta_3 - iD_{j+1}^-(2E)^{-1}u_3 + \frac{j}{2j+1}D_{j+1}^-(2ES)^{-1}u_1\right\} = 0$$

$$2i(S+V)u_0 = 4iV(r)[iT\beta_0 + E\alpha_0 + S(2S)^{-1}u_0 - jD_j^- D_j^+ (iS)(2ES)^{-1}u_2$$
$$- (j+1)iD_j^- D_j^+ S(2ES)^{-1}u_3] + 4ir\tilde{V}(r)[jD_j^+\beta_2$$
$$- ijD_j^+ (2E)^{-1}u_2 - \frac{j(j+1)}{2j+1}D_j^+ (2ES)^{-1}u_1 + (j+1)D_{j+1}^-(2E)^{-1}u_3$$

$$+\frac{j(j+1)}{2j+1}(2ES)^{-1}v_1\Big]$$

$$-u_1=2iV(r)\{\beta_1-i(2S)^{-1}u_1\}$$

$$D_j^+(2iSu_2)+2iV(r)u_2=+4V(r)D_j^+E\beta_2-\frac{4r\tilde{V}(r)}{2j+1}\{\alpha_0-i(j+1)(\beta_1-i(2S)^{-1}u_1)\}$$

$$D_{j+1}^-(2iSv_3)+2iV(r)v_3=4V(r)D_{j+1}^-E\beta_3-\frac{4r\tilde{V}(r)}{2j+1}\{\alpha_0+ij(\beta_1-i(2S)^{-1}u_1\}$$

$$2i(T-V)\alpha_0=2E\beta_0\,;\ \alpha_1=\alpha_2=\alpha_3=0.$$

From the above set of conditions we at once conclude without loss of generality that $\beta_1=u_1=0$,

Upon eliminating u_0, u_2, u_3, we get the following equations

$$i(T-V)\alpha_0=E\beta_0 \tag{3·13}$$

$$ED_j^+T\beta_2=-\frac{r}{2j+1}\tilde{V}\alpha_0 \tag{3·14}$$

$$ED_{j+1}^-T\beta_3=\frac{r}{2j+1}\tilde{V}\alpha_0 \tag{3·15}$$

$$i(S^2+VT)\beta_0+VE\alpha_0=j(T+V)D_j^-D_j^+\beta_2+(j+1)(T+V)D_j^-D_j^+\beta_3. \tag{3·16}$$

The terms in u_0, u_1, u_2, u_3 contribute nothing to the solution for φ and hence for Ψ when $x_4\neq 0$, but are nevertheless required to neutralise the discontinuity in ω' at $x_4=0$. Further elimination of β_0, β_2, β_3 gives the single equation for the determination of α_0. We get from (3·13) — (3·16).

$$(S^2+VT)(T-V)\alpha_0-E^2V\alpha_0=(T+V)T^{-1}(jD_j^-+(j+1)D_{j+1})\frac{r\tilde{V}(r)}{2j+1}\alpha_0.$$

Putting back the values of D_j^- and D_{j+1}^+, we can write the above equation as

$$(S^2T-VTV+VT^2-T^2V)\alpha_0=(1+VT^{-1})\Big(\frac{1}{r}\frac{\partial}{\partial r}(r^2\tilde{V}\alpha_0)\Big). \tag{3·17}$$

This result is exact, within the limits of the ladder-approximation and the instantaneous interaction approximation. It will be shown subsequently that the equation (3·17) corresponds to the spin singlet system of the nucleon-nucleon system; the corresponding solutions for the spin-triplet states of the nucleon-nucleon system could be easily determined from the solution of the B-S equation with the form of ω, ω', φ given in (2·2) — (2·4) but with the right-hand sides multiplied by γ_5.

Proceeding as before we can obtain* a different set of differential equations to determine the functions ω_0, ω_1, ω_2, etc.. These differential equations are then solved for

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* See Ref. 8.

$x_4 \neq 0$; the continuity conditions for the validity of these solutions near $x_4=0$ will lead to the following simultaneous differential equations,

$$-S^{\circ}_{j-1}(T_{j-1}-V)\alpha_+ = -E^2V\alpha_+ + VT_{j-1}(T_{j-1}-V)\alpha_+ + \frac{rV}{2j+1}T^{-1}\{r\tilde{V}(j\alpha_+ + \overline{j+1}\alpha_-) - (j+1)(T-V)(D_j\alpha_- - D^-_{j-1}\alpha_-)\} - \frac{D^+_j}{2j+1}\{(VT^{-1}+1)r\tilde{V}(j\alpha_+ + \overline{j+1}\alpha_-)\} \qquad (3\cdot18)$$

$$-S^{\circ}_{j+1}(T_{j+1}-V)\alpha_- = -E^2V\alpha_- + VT_{j+1}(T_{j+1}-V)\alpha_- + \frac{rV}{2j+1}T^{-1}\{r\tilde{V}(j\alpha_+ + \overline{j+1}\alpha_-) + j(T-V)(Dj\alpha_+ - D^+_{j+1}\alpha_-)\} - \frac{D^-_{j+1}}{2j+1}\{(VT^{-1}+1)r\tilde{V}(j\alpha_+ + (j+1)\alpha_-)\}. \qquad (3\cdot19)$$

The equations (3·18) and (3·19) are coupled, denoting the fact that for $j=1$, α_+ and α_- stand for the S-state and D-state radial wave functions corresponding to the deuteron problem.

Further $S_{j\pm1}$ and $T_{j\pm1}$ stand for the same operators S and T, but with the modification that the operator $D^-_jD^+_j$ in S and T should be changed to $D^-_{j\pm1}D^+_{j\pm1}$.

The emergence of the simultaneous equations (3·18) and (3·19) representing the triplet state of the nucleon-nucleon system is quite important and implies that the resulting Schrödinger equation contains a tensor-force term in the potential energy. This is essential in explaining the quadrupole moment of deuteron. Also the occurrence of J-dependent terms implies the existence of spin-orbit energy in the potential. This has been found necessary to explain the polarization effect in proton-proton scattering. It is not possible to obtain a single equation explaining all the facts directly.

It now remains to show that the solutions of equations (2·2)—(2·4) and the other set of solutions obtained from (2·2)—(2·4) but differing by a factor of γ_5 do correspond to the singlet and triplet states of the nucleon-nucleon system respectively.

To verify this we notice that since parity is conserved in strong interactions, we need only to analyse the B-S wave function operator which may project the definite parity-state from the incident isotropic mixture of both spin-states. Now the incident wave for nucleon-nucleon interactions is represented by[6]

$$\Psi_0 = (E\gamma_4 - k\gamma_3 + M) \otimes (E\gamma_4 + k\gamma_3 + M)\, e^{ik\cdot x};$$

where $\otimes$ means the direct product and $k=(0, 0, k)$. For Ψ_0 satisfies the B-S wave equation for non-interacting particles, and it contains both spin-states. We now need to know the projection operators Λ for which the expression $(E\gamma_4 - k\gamma_3 + M)\Lambda(E\gamma_4 + k\gamma_3 + M)$ represents either singlet or triplet states of the system. The possible values[5] of Λ are easily found by substituting in turn each of the 16 Dirac γ-matrices. It is found that the simplest form of Λ is I for the singlet state with $S_3=0$; $\Lambda=\gamma_5$ for the triplet state

with $S_3=0$; and $\Lambda=\gamma_\pm$ for the triplet states with $S_3=\pm 1$. $[\gamma_\pm=\gamma_1\pm i\gamma_2]$. The bispinor parts of the corresponding wave functions have the following values of $\omega=2EM\gamma_4$ for the singlet and $-2kM\gamma_3\gamma_5$, $2k(E\gamma_\pm\gamma_3\gamma_4-k\gamma_\pm)$ for the triplet states respectively.

The fact that they do represent singlet or triplet states may be easily verified by applying the spin operator, $\mathbf{S}^2$ to the respective functions. Noting $S\Psi=\frac{1}{2}(\sigma\psi-\psi\sigma)$ where σ is the spin vector, it is easily seen that

$$S_1\gamma_4=S_2\gamma_4=S_3\gamma_4=0$$

whence $S\gamma_4=0$. This shows that $\mathbf{S}^2$ has the eigenvalue 0 on the wave function $2MEY_4$; it is therefore a singlet state with $S_3=0$.

Again applying the operator $\mathbf{S}^2=S_1^2+S_2^2+S_3^2$ (where $S_1=i\frac{1}{2}(\gamma_2\gamma_3-\gamma_3\gamma_2)$, etc.) to $2kM\gamma_3\gamma_5$ we see that $\mathbf{S}\gamma_3\gamma_5=2\gamma_3\gamma_5$ thus $\mathbf{S}^2$ has the eigenvalue 2 showing that $2kM\gamma_3\gamma_5$ is the triplet state wave function. Also $S_3\gamma_3\gamma_5=0$, ensuring that the wave function $2kM\gamma_3\gamma_5$ is for the triplet state with $S_3=0$. Similarly the other two states of the spin triplet are obtained from the projection operator, $\Lambda=\gamma_\pm$ with $S_3\pm 1$. One concludes that ω, ω', and φ given in $(2\cdot2)-(2\cdot4)$ stand for the singlet state while the other set of ω, ω' and φ obtained from $(2\cdot2)-(2\cdot4)$ by multiplying the operator γ_5 from the left corresponding to the triplet state of the system.

A further examination of the functions ω_0, ω_0', φ_0, etc., shows that to characterize the nucleonic-nucleonic interactions with $\Lambda=1$, the part of the solution Ψ is $2ME\Psi_4$; this suggests that for the singlet state of the nucleonic-nucleonic interaction, ω_0 of equation $(2\cdot2)$ should be different from zero, and be an even function of x_4. Again for $\Lambda=\gamma_5$ corresponding to the triplet state, the part of Ψ has the non-vanishing value $2kM\gamma_3\gamma_5$; hence ω_2 and ω_3 of $(2\cdot2)$ should be different from zero, and be even functions of x_4. The odd or even character in x_4 of the rest of the functions can be determined from the consistency of the differential equations for ω_0, etc., as given in $(2\cdot9)-(2\cdot19)$.

In the next section we will discuss the effective nuclear potential predicted by the equations $(3\cdot17)$ and $(3\cdot19)$.

§ 4. Determination of the effective nuclear potential

In this Section we will predict the effective nuclear potential from equations $(3\cdot17)$, $(3\cdot18)$ and $(3\cdot19)$ which are obtained from the original B-S equation in the instantaneous interaction approximation.

We now eliminate velocity-dependent terms exactly from the equation $(3\cdot17)$ and then obtain the effective potential of the nucleon system in the spin singlet state.

Replacing α_0 by $\alpha_0=r\alpha_0'(r)$ and multiplying the whole equation by a factor $1/r$ with this modification the equation $(3\cdot17)$ now reads

$$(S'^2T'-VT'V=VT'^2-T'^2V)\,\alpha'=(1+VT'^{-1})\,(\Delta V\Delta+\Delta V)\,\alpha_0' \qquad (4\cdot1)$$

the S' and T' operators being $S'^2=M^2-E^2-\Delta$, and $T'^2=M^2-\Delta$, where Δ is the usual Laplacian operator. At non-relativistic energies we have $T'\approx M-(\Delta/2M)$, $E\approx M+\frac{1}{2}W$, where W is the negative binding energy. S' has no good non-relativistic approximation,

since M^2 and E^2 are of the same order and their difference is of the same order as the square of the relative momentum $(-\Delta)$. But fortunately, at the end of our calculation S' appears only as the square and does not need to be approximated. Neglecting squares and products of Δ and W, we have from (4·1)

$$\left\{\left(1-\frac{V^2}{2M^2}\right)\Delta+WM+V^2\right\}\alpha_0'(r)=\frac{1}{M}\nabla V\cdot\nabla\alpha_0'(r) \tag{4·2}$$

To eliminate the velocity-dependent term from the righthand side of (4·2) we set $\alpha_0'=f(V)\Psi(r)$; with $f(V)$ satisfying

$$2\left(1-\frac{V^2}{2M^2}\right)\frac{\nabla f}{f}=\frac{\nabla V}{M}.$$

This leads to the following Schrödinger equation:

$$\left\{\left(1-\frac{V^2}{2M^2}\right)\Delta+WM+\frac{\mu V^2}{2M^2}+V^2\right\}\Psi=0 \tag{4·3}$$

where we have dropped terms of order μ^2V^2/M^2 compared to V^2. This is the equation for the singlet nucleon-nucleon system showing the equivalent classical potential

$$-(1+W/M)\left(V^2/M+\frac{\mu^2V}{2M}\right)\Big/(1-V^2/2M^2). \tag{4·4}$$

At large separation of the particles, where the V^2 terms are small compared to V notwithstanding the coefficients μ^2/M^2, it reduces to the usual Yukawa potential with the right coefficient showing that the recoil effect is negligible. In fact our second-order potential $\mu^2V/2M^2$, corresponds exactly with what Levy has found (apart from the tensor-term). However, the V^2 term is important at short distances and one can see that the nucleonic-recoil has considerable effect on pair creations. For the equation differs radically from equations derived by Levy[2] and Klein[5], treating pair effects by a perturbation procedure, and also differs somewhat from that deduced by Arnowitt and Gasiorowicz[7] who neglected nucleon recoil. But so far as the V^2 term is concerned we are in agreement with Arnowitt and Gasiorowicz in predicting a singularity of the potential at $V(r)=\sqrt{2}M$. At distances shorter than these the forces are repulsive, simulating the "core" postulated by Jastrow[3] and Levy.[2] If one tries to improve the non-relativistic approximation by taking terms of order μ^2/M^2, compared with those retained, into account, a difficulty arises; one finds among other terms, the additional term

$$-\frac{(2V-M)V^2}{2M^2-V^2}\left\{\mu^2+\frac{2\mu}{r}+\frac{1}{r^2}\right\}$$

in the effective potential. This extra potential has a peculiar feature. The core, determined at $V(r)=\sqrt{2}M$, comes out to be strongly attractive.

But it is easy to see that this difficulty is due to a break-down of the method of approximation rather than a defect of equation (4·3). From the physical point of view, a particle moving into an attractive potential with a singularity like $(r-a)^{-2}$ attains

momenta inconsistent with the approximation, $T' \approx M - A/2M$. If one wishes to investigate relativistic corrections one ought not therefore to try to amend the Schrödinger equation (4·3) : it is essential to return to the exact equation (3·17) which can be written as an integral equation in momentum space thus,

$$\{E-(\boldsymbol{p}^2+M^2)^{1/2}\}\Psi(\boldsymbol{p})+\int U(\boldsymbol{p}-\boldsymbol{p}'',\ \boldsymbol{p}'')\,\Psi(\boldsymbol{p}'')\,d^3\boldsymbol{p}''=0$$

where

$$\Psi(\boldsymbol{p})=(\boldsymbol{p}^2+M^2)^{1/2}\{E+(\boldsymbol{p}^2+M^2)^{1/2}\}\int \alpha_0(r)\,e^{i\boldsymbol{p}\cdot\boldsymbol{r}}d^3\boldsymbol{r} \qquad (4\cdot5)$$

and $U(\boldsymbol{p}-\boldsymbol{p}'', \boldsymbol{p}')$ is given by

$$U(\boldsymbol{p}-\boldsymbol{p}'', \boldsymbol{p}')=(\boldsymbol{p}''^2+M^2)^{-1/2}\{E+(\boldsymbol{p}''^2+M^2)^{1/2}\}^{-1}[(\boldsymbol{p}-\boldsymbol{p}'')\cdot\boldsymbol{p}''V(\boldsymbol{p}-\boldsymbol{p}'')$$
$$+\int V(\boldsymbol{p}-\boldsymbol{p}')\frac{M^2+\boldsymbol{p}'\cdot\boldsymbol{p}''}{(\boldsymbol{p}'^2+M^2)^{1/2}}V(\boldsymbol{p}'-\boldsymbol{p}'')\,d\boldsymbol{p}'] \qquad (4\cdot6)$$

where $V(\boldsymbol{p}-\boldsymbol{p}'')=\int V(\boldsymbol{r})\,e^{i\boldsymbol{p}\cdot\boldsymbol{r}}d^3\boldsymbol{r}$.

The equation (4·5) is the equivalent relativistic Schrödinger equation in momentum space with $U(\boldsymbol{p}-\boldsymbol{p}'',\ \boldsymbol{p}'')$ as the resulting potential including the velocity-dependent term in momentum space.

It will now be verified that the kernel of the integral equation given in equation (4·6) is not too badly singular to make the eigenvalue problem indeterminate. For the integral over p' in the square bracket of (4·6) is absolutely convergent as $V(\boldsymbol{p}-\boldsymbol{p}'')$ behaves $\approx\mu\{(\boldsymbol{p}-\boldsymbol{p}')^2+\mu^2\}^{-1}$. We now investigate the behaviour of the function $U(\boldsymbol{p}-\boldsymbol{p}'',\ \boldsymbol{p}'')$ near the larger values of p'' which is the region corresponding to small values of r in the coordinate space. We see that the second term in the potential given by

$$(\boldsymbol{p}''^2+M^2)^{-1/2}\{E+(\boldsymbol{p}''^2+M^2)^{1/2}\}\int V(\boldsymbol{p}-\boldsymbol{p}')\frac{M^2+\boldsymbol{p}'\cdot\boldsymbol{p}''}{(\boldsymbol{p}'^2+M^2)^{1/2}}V(\boldsymbol{p}'-\boldsymbol{p}'')\,d\boldsymbol{p}'$$

tends to zero for large momenta, p''. Moreover the term

$$(\boldsymbol{p}''^2+M^2)^{-1/2}\{E+(\boldsymbol{p}''^2+M^2)^{2/1}\}^{-1}(\boldsymbol{p}-\boldsymbol{p}'')\cdot\boldsymbol{p}''\ V(\boldsymbol{p}-\boldsymbol{p}'')$$

does not tend to zero as strongly as the other term for large p''.

Writting this expression as

$$\left\{1-\frac{\boldsymbol{p}\cdot\boldsymbol{p}''+M^2+E(\boldsymbol{p}''^2+M^2)^{1/2}}{\boldsymbol{p}''^2+M^2+E(\boldsymbol{p}''^2+M^2)^{1/2}}\right\}V(\boldsymbol{p}-\boldsymbol{p}'')$$

we notice that the first term is just the Yukawa type potential in the coordinate space. The term $\boldsymbol{p}\cdot\boldsymbol{p}''V(\boldsymbol{p}-\boldsymbol{p}'')/\{\boldsymbol{p}''^2+M^2+E(\boldsymbol{p}''^2+M^2)^{1/2}\}$ in the curly bracket is less singular than that of the Yukawa term. The remainder is non-singular. Thus the singularity in the resulting potential cannot be worse than a $1/r$ singularity.

We now consider the coupled equations (3·18) and (3·19) denoting the S and D state wave functions. Since it is well known that the percentage of D state contribu-

tion in the deuteron is very small (only 4%) we may for our present purposes neglect the α_--component. The equation (3·18), with the re-definition $\alpha_+ = r\alpha_1(r)$ reduces for the state $j=1$, to the following

$$(S'^2T' - VT'V + VT'^2 - T'^2V)\alpha_1 = \left(1 - \frac{1}{3}\frac{V}{M}\right)V\cdot\alpha_1 + \left\{\frac{1}{3}V + \frac{\mu^2}{3M}V^2\right\}\alpha_1$$

where now

$$S'^2 = M^2 - E^2 - \Delta,$$

$$T'^2 = M^2 - \Delta.$$

At non-relativistic energies we assume the same approximation for T and E as is made in equation (4·2). Eliminating the velocity-dependent terms by introducing the transformation

$$\alpha_1 = f(V)\,\Psi(r) \quad \text{where}$$

$$2\left(1 - \frac{V^2}{2M^2}\right)\frac{\Gamma f}{f} = \left(1 - \frac{4V}{3M}\right)\frac{\Gamma V}{M},$$

we have the following equivalent Schrödinger equation

$$\left\{\left(1 - \frac{V^2}{2M^2}\right)\Delta + WM - \frac{1}{3}\frac{\mu^2 V}{2M} + V^2\right\}\Psi = 0 \qquad (4\cdot7)$$

dropping terms of order μ^2V^2/M^2 compared with V^2. This equation is exact within our limits of approximation and denotes the spin triplet states of the deuteron. At large distance of separation when V^2 can be neglected compared to V, the equation predicts the usual Yukawa potential $\mu^2V/6M^2$. This is in exact agreement with the result of Levy[2]: apart from the tensor term, in the charge singlet and spin-triplet state. Again the equation (4·7) predicts the existance of the singularity of the potential at $V(r) = \sqrt{2}M$ simulating the "core" as mentioned earlier. At distances of r shorter than this the forces change sign and become completely repulsive. However, if one takes into consideration higher order terms in V^2 one faces a similar situation as in the case of equation (4·3) and one must go back to the integral equation formalism as discussed before.

It should be mentioned, however, that if one can obtain a single Schrödinger equation from (3·18) and (3·19), the resulting nuclear potential would then predict the existence of a tensor force and spin orbit coupling force along with the central force as determined above.

§ 5. Numerical calculations and discussion

In this Section we shall discuss some low-energy properties of the neutron-proton system together with some high energy nucleon-nucleon scattering data from the nuclear potential determined in the earlier Section. The main feature of our potential derived above is that it is singular and simulates a "repulsive core" of radius a when a is given by $\lambda^2 = 2M^2a^2\exp(2\mu a)$, λ is the coupling constant, M=nucleon mass and μ=meson mass.

A similar potential with same qualitative features as our potential was used by Levy[2] (Levy's potential is infinite for values $r<r_0$, i.e. it has a hard core, where as in our case the potential changes sign for such values of r and is infinite at $r=r_0$, i.e. it has a hard shell rather than a core) to discuss the n-p system at low-energy limit. However, Gammel[4] et al., have attempted to find a phenomenological nucleon-nucleon potential with central and tensor parts of Yukawa shape having a hard core. The depths, ranges and core-radii of such potentials with general spin and parity dependence are adjusted to fit the experimental data.

Here we will first determine the deuteron binding energy with the aid of the triplet even-parity potential of equation (4·7). The core-radius, $a=0.319\times10^{-13}$cm and the coupling constant $\lambda=2.89$ satisfy

$$\lambda^2=2M^2a^2e^{(2\mu a)}.$$

These values will give us the right binding energy of deuteron observed in experiment. For the comparison of our results with some other experimental findings we, at present, compare our potentials with those of Gammel[4]. It is found that Gammel's phenomenological triplet even-parity central potential can be almost predicted by our theory. However, the singlet even parity central potential for the p-p system does not correspond exactly with ours, because near the core our potential is very sharp due to the presence of the factor $(1-V^2/2M^2)^{-1}$. It may be pointed out this potential shape may be obtained as well as Gammel's by fitting ${}^1S_0$ and ${}^1D_2$ $p-p$ phase-shifts for different values of the core-radius and range of Gammel's phenomenological potential.

The analytical calculations for the determination of the binding energy of the deuteron can be performed by means of the usual variational techniques. For this purpose we treat the V^2-term exactly while the V-term can be treated as perturbation. Assuming a trial wave-function of the form $\Psi_t=(r-a)e^{-\alpha r}$ for $r>a$, and $=0$ for $r<a$, where α is fixed from the fact,

$$I(\alpha)=\left[\int r\Psi_t^*\left\{-\frac{d^2}{dr^2}-\frac{V^2}{(1-V^2/2M^2)}\right\}r\Psi_t dr\right]\Big/\int r^2\Psi_t^*\Psi_t dr$$

to be minimum. In this calculation we have not used the D-state contribution simply because we have not the computational facilities for treating the simultaneous equations (3·8) and (3·9) exactly. The value $\alpha=5.93/\mu$ gives I a minimum value. With this choice of α we have calculated the contribution from the potential $-\frac{1}{6}\mu^2\frac{V}{M}\left(1-\frac{V^2}{2M^2}\right)$ and also assuming the percentage of D-state in deuteron as 4% only a value $-\langle L^2/r^2\rangle$. The binding energy thus obtained fits the observed value, 2.228 Mev of deuteron.

With these values of the coupling constant and the core-radius, the other low-energy data, namely the electric quadrupole moment of deuteron and zero energy n-p triplet scattering length should be predicted with these nuclear potential. However, it must be pointed out that for an accurate determination of these data one should solve the simultaneous equations (3·8) and (3·9), which is not feasible at the moment. Then

we propose to compare our potentials with those of Gammel *et al.*, in which all the experimental data have been fairly well described.

The potential suggested by Gammel is of the form

$$V(r) = \infty \text{ for } r < r_0, \quad (r_0 = \text{core-radius})$$

$$= V_c(r) + V_t(r) S_{12} \text{ for } r > r_0$$

where S_{12} is the tensor-force ; $V_c(r) = -V_c \exp(-r/r_c)/(r/r_c)$ and $V_t(r) = -V_t \exp(-r/r_t)/(r/r_t)$. The parameters V_c, V_t, r_c, r_t depend on spin and parity while r_0 is independent of parity and isotopic-spin. Using $r_0 = 0.3 \times 10^{-13}$cm, $V_c = 595.7$Mev $r_c = .5714 \times 10^{-13}$cm. $V_t = 26$Mev, $r_t = 1.633 \times 10^{-13}$cm and taking a D-state admixture of 3.45%, Gammel *et al.* obtained a good fit for the triplet even parity data. It is clear that the tensor force term is small compared to the central part. In this respect we find an almost exact coincidence of Gammel's potential with ours. The potentials are compared in Fig. 1.

For high energy proton-proton scattering (at 310 Mev), Gammel *et al.* with three parameters, namely $r_0 = 0.1 \times 10^{-13}$cm $\mu_c = r_c^{-1} = .97 \times 10^{13}\text{cm}^{-1}$ and $V_c = 77.24$ Mev (there is no tensor force in the singlet even parity potential) have been able to fit the $^1S_0$ and $^1D_2$ p-p phase shift at 310 Mev. The singlet even-parity charge-triplet potential responsible for the proton-proton interaction can be obtained from the potential contribution of the equation (4·3) with $\lambda = .963$ and the corresponding core-radius $a\mu = .079$. It is evident from the curves drawn in the fig. 2 that our potential is sharper than that of Gammel's.

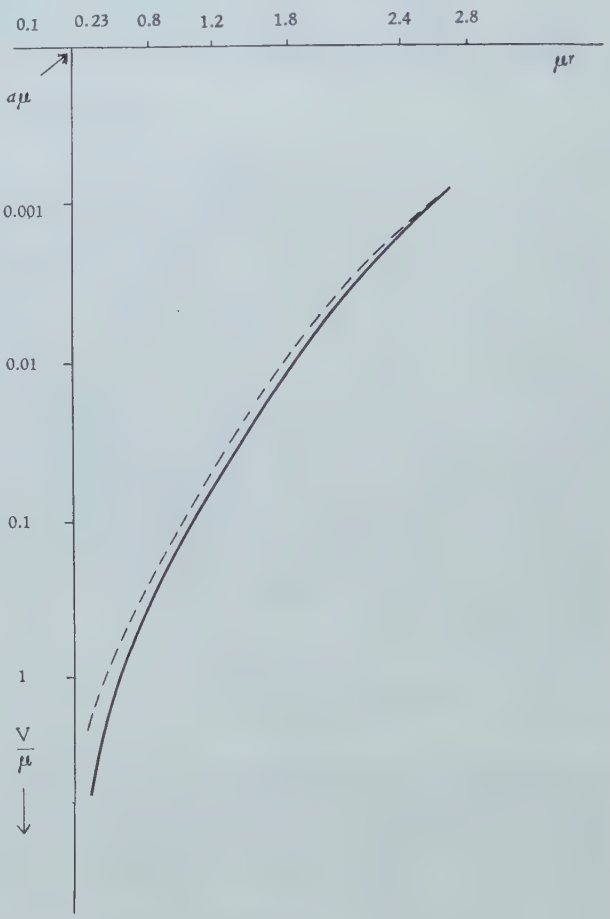

Fig. 1.

Dotted Curve : Gammel's triplet even parity potentials which fit the binding energy of deuteron, the quadrupole of moment of deuteron and the zero-energy n-p triplet scattering length.

Solid Curve : Triplet even-parity-instantaneons interaction potencial predicted by the present calculation eq (4.7). The core-radius $a\mu = .23$ is the same for both the potentials.

However the difference between the two curves is not so great that both might not be consistent with experimental data.

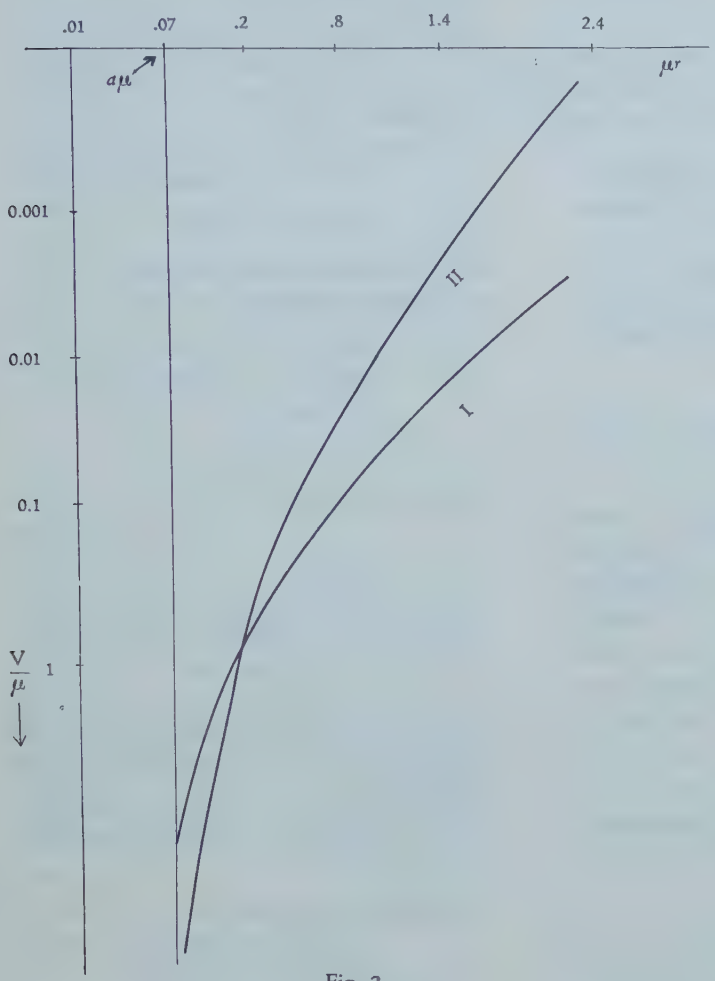

Fig. 2.

Curve 1: Gammel's Singlet even parity potential which fits the zero-energy *n-p* and *p-p* scattering length and effective range as well as high energy proton-proton scattering $^1S_0$ and $^1D_2$ phase shifts.
Curve II: Singlet even-parity potential predicted by the present instantaneous interaction approximation of the *B-S* equation from eq. (4.3.) describing the proton-proton system. The core-radius $a\mu=.07$ is the same for both the cases.

We thus conclude that within the limits of the ladder approximation and the instantaneous interaction approximation, the B-S equation can predict at least qualitatively the nuclear potential which fits the low-energy and high-energy data. Beyond this limit, the non-relativistic approximation is no longer valid, but one might still hope to make useful predictions by a study of the integral equation formalisms of the equations (3·6), (3·7) and (3·8).

The author hopes to present later a detailed calculation of *p-p* scattering data both at low and high energy using the integral equation deduced in this paper.

Acknowledgement

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Letters to the Editor

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A Theory of Lepton which is Invariant under Space Inversion

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October 8, 1957

A theory of lepton which is invariant under space inversion and, at the same time, consistent with the recent asymmetry and polarization experiments is proposed.[1] To account for this experimental evidence, the two component neutrino theory[2] seems to be not necessary and we need not abandon the law of the conservation of parity.

The theory is formulated as follows:

I. Equation of free lepton

$$\left[\rho_1(\boldsymbol{\sigma p})+\rho_3 m\frac{1+\omega_3}{2}\right]\psi=E\psi. \qquad (1)$$

II. Interaction Hamiltonian for β-decay

$$H=(\bar{\psi}_p \Omega\psi_n)\,(\bar{\psi}\Omega(C-C'\rho_1)\omega_+\psi)+\text{h.c.} \qquad (2)$$

where C and C' are real constants.

III. $P^{\pm}\equiv\frac{1}{2}(1\pm\omega_3)$ are the projection operators which pick up the electron and the neutrino state respectively, and $\omega_\pm$ are the operators that give the transition between these two kinds of leptons.

IV. Under space inversion ψ and $\bar{\psi}$ are to be transformed as

$$\psi(x,t)\to\rho_3\psi(-x,t),$$
$$\bar{\psi}(x,t)\to\bar{\psi}(-x,t)\rho_3.$$

V. ρ's and ω's are to be represented by the following matrices:

$$\omega_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \omega_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$\omega_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

$$\rho_1=\begin{pmatrix}\underline{\rho}_1&0\\0&\underline{\rho}_1\end{pmatrix},$$

$$\rho_k=\begin{pmatrix}\underline{\rho}_k & 0\\ 0 & \dfrac{C+C'\underline{\rho}_1}{\sqrt{C^2-C'^2}}\underline{\rho}_k\dfrac{C-C'\underline{\rho}_1}{\sqrt{C^2-C'^2}}\end{pmatrix}$$
$$k=2,\ 3.$$

where 1 is the 2×2 identity and the underlined $\underline{\rho}$'s are the well-known 2×2 Pauli matrices.

From the above, two consequences follow immediately:

A) $$\bar{\psi}\Omega(C-C'\rho_1)\omega_+\psi=\bar{\psi}_e\underline{\Omega}(C-C'\underline{\rho}_1)\psi_\nu.$$

Accordingly, the whole Hamiltonian (2) can be brought into the form

$$H=(\bar{\psi}_p\Omega\psi_n)\,(C\bar{\psi}_e\Omega\psi_\nu+C'\bar{\psi}_e\Omega\gamma_5\psi_\nu)+\text{h.c.}$$

from which, as is well known, we can derive the asymmetry of angular distribution and the polarization of emitted electron.

$$\rho_3(C\mp C'\rho_1)\omega_\pm\rho_2=(C\mp C'\rho)\omega_\pm$$

from which we can conclude that H is

invariant under space inversion.

Similarly, it can be shown that the present theory is invariant under time reversal and therefore also under charge conjugation.

It is easy to verify the fact that the above ρ's and ω's as 4×4 matrices satisfy the Pauli relations:

$$\rho_k^2=1,\ \rho_k\rho_l=-\rho_l\rho_k=i\rho_m,$$
$$\omega_k^2=1,\ \omega_k\omega_l=-\omega_l\omega_k=i\omega_m.$$

And, multiplying (1) by $P^{\pm}$, we can write the free equations of electron and of neutrino separately:

$$[\rho_1(\boldsymbol{\sigma p})+\underline{\rho_3}m]\psi'_e=E_e\psi'_e,$$
$$\underline{\rho_1}(\boldsymbol{\sigma p})\psi'_\nu=E_\nu\psi'_\nu$$

which are exactly the same as the conventional one.

In the present theory both the electron and the neutrino wave functions have, including the spin freedom, four components as usual, neither eight nor two. Free fields, as far as treated separately, are completely the same as in the usual theory. Only the interaction between two fields is changed in a sense, which is characterised by the *non-commutativity* of $\omega_\pm$ with ρ_k $(k=2, 3)$.

The parity operator of electron is ρ_3 and that of neutrino is

$$\frac{C+C'\underline{\rho_1}}{\sqrt{C^2-C'^2}}\underline{\rho_3}\frac{C-C'\underline{\rho_1}}{\sqrt{C^2-C'^2}},$$

which are different from each other as far as the 2×2 matrix representation is concerned, and cannot be diagonalized at the same time. But these two are unified into one ρ_3 in the 4×4 matrix form. The difference between the electron and the neutrino parity operators, or you may say the shift in phase of representation for ρ, is carried away as relative parity by the transition operators $\omega_\pm$ which are now non-commutative with ρ_3.

Although the above discussions are restricted only to the β-interaction, other lepton processes ($\pi-\mu$ decay, $\mu-e$ decay, $K_{\mu2}$ etc.) can be treated in a similar way. Details will be discussed in a later issue of this journal.

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On the Hydrodynamical Theory in Multiple Particle Production

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March 7, 1958

The strong viscosity effects in hydrodynamic description of the multiple production of the particles are discussed. For introducing these effects, we employ the covariant form used in the preceding papers.[1] Consequently, the meson cloud produced by nucleon-nucleon collisions will behave as if the fluid were expanding under the influence of strong viscosity. From the conservation law of total energy-momentum tensor, $\partial_\nu T^{\mu\nu}=0$, we have a system of equations of the one-dimensional motion parallel to x-axis which indicates the direction of the incident nucleon,

$$\left.\begin{aligned}&\frac{4}{3}v^2\frac{\partial\varepsilon}{\partial t}+\frac{8}{3}\varepsilon v\frac{\partial v}{\partial t}+\frac{1}{3}\cdot\frac{\partial\varepsilon}{\partial\xi}\\&\quad=\eta\left(2\frac{\partial^2 v}{\partial t^2}+\frac{9}{2}\cdot\frac{\partial^2 v}{\partial t\partial\xi}+2\frac{\partial^2 v}{\partial\xi^2}\right),\end{aligned}\right\}$$

$$\left.\begin{aligned}&\frac{1}{3}\cdot\frac{\partial\varepsilon}{\partial t}+\frac{1}{3}\cdot\frac{1}{v^2}\cdot\frac{\partial\varepsilon}{\partial\xi}\\&-\frac{2}{3}\cdot\frac{\varepsilon}{v^3}\cdot\frac{\partial v}{\partial\xi}\\&=\eta\Big(\frac{5}{2}\cdot\frac{\partial^2 v}{\partial t^2}+2\frac{\partial^2 r}{\partial t\partial\xi}+\frac{1}{v^3}\cdot\frac{\partial r}{\partial\xi}\\&\times\frac{\partial v}{\partial t}-\frac{1}{2v^2}\cdot\frac{\partial^2 r}{\partial\xi\partial t}\Big),\end{aligned}\right\}\qquad(1)$$

where $\xi=t-x$. In the derivation of the above expression, we have imposed the following conditions upon the four velocity,

$$v_0\sim v^1\equiv v,\quad v^0-v^1=1/(2v),\qquad(2)$$

but these relations may be partially destroyed*[1] because the difference of the above two velocity components becomes large due to the strong viscosity.

We shall now assume $v=F\cdot(t^{k+1}/\xi^k)$ for the velocity in eq. (1) and determine the function F to be independent of t and ξ. Substituting v into eqs. (1) and then eliminating F, we have the differential equation of ε with respect to t and ξ under the condition, $t\gg\xi>\Delta$ (Δ is the longitudinal width of the meson cloud.), $k\neq 0$, -1 and $\eta>\varepsilon\sqrt{\xi^3/t}$,

$$\frac{16}{3}k^3(k+1)^2\varepsilon t+\frac{8}{3}k^2(k+1)^2\Big(\frac{\partial\varepsilon}{\partial\varepsilon}\Big)t\xi$$
$$=-\frac{2}{27\eta^2}\Big\{\Big(\frac{\partial\varepsilon}{\partial t}\Big)\Big(\frac{\partial\varepsilon}{\partial\xi}\Big)^2t\xi^5+\Big(\frac{\partial\varepsilon}{\partial\xi}\Big)^3\xi^6\Big\}.\qquad(3)$$

In the case of a strong interaction, it is very convenient to use the expansion in inverse powers of the viscosity, though the strong viscosity does not always correspond exactly to the strong interaction of the elementary particles.

$$\varepsilon=\varepsilon_0+1/\eta^2\cdot\varepsilon_1+\cdots\cdots.$$

Consequently, we obtain the following solutions using the relation $|\eta|\equiv\eta_0(\varepsilon_0^*+1/\eta^2\cdot\varepsilon_1^*)\sqrt{t_1^3/\Delta}$, where η_0 is a numerical parameter $\eta_0\gg 1$ which signifies the inverse of the Reynolds number,

$$\left.\begin{aligned}&\varepsilon=(\varepsilon_0^*+1/\eta^2\cdot\varepsilon_1^*)\cdot(t/\xi)^2,\\&v=F(t^2/\xi)\approx 1/|\eta|\cdot\varepsilon_0^*+1/\eta^2\cdot\varepsilon_1^*)\\&\qquad\cdot(t^2/\xi).\end{aligned}\right\}\qquad(4)$$

ε_0^* and ε_1^* are the initial values of ε_0 and ε_1 in $t\approx\xi\approx\Delta$, respectively.

In the extreme point of the one-dimensional motion, the energy distribution evaluated by using the quantities (4) shows a great decrease compared with that of the perfect fluid[2] and, on the contrary, the entropy distribution increases excessively. Since the positiveness of the energy distribution is to hold in any stage of the expanding process, we obtain $\eta_0^2\ll\Delta t^4/t_1^3\xi^2$† which, as is easily seen from (2) and (4), means the applicable region of our calculations. When this is not the case, we shall have non-physical results.

From the condition $t\cdot\theta\ll a$ (a : rhe range of nuclear force), the limit of the one-dimensiynal motion and the scattering angle are given by the expressions

$$\left.\begin{aligned}&t_1\sim(a^2/\xi)\cdot(\Delta/\eta_0^2\xi)\cdot\big[1+(\eta_0^6/2)\\&\qquad\cdot(\xi^7/\Delta^3a^4)\big],\\&\theta\sim\eta_0^2(\xi^2/\Delta a)-(\eta_0^8/2)(\xi^9/\Delta^4a^5)\\&\qquad\cdot\exp\big[-\eta_0^6(\xi^7/\Delta^3a^4)\big].\end{aligned}\right\}\qquad(5)$$

* These very considerations should lead some restriction for the applicable region of the solution obtained.

† This is nothing but the purport stressed in reference*.

From the facts that the total sum of the energy and the heat flow in a given cone in the conical scattering process is constantly conserved, that is, the changes of these quantities with the development of time cancel each other, we shall have the relations $\varepsilon \propto 1/t^4$, $v \propto t$ (see eq. (40) in I).

Moreover, considering the maximum point $\xi \approx \Delta$ of the particle distribution, we get the scattering angle of a large number of created particles from the expression (5),

$$\theta \sim \eta_0^2 (M/E') - \eta_0^8/2(M/E')^5,$$

in which E' and M are respectively the energy and the mass of the nucleon in the $c-m$ system.

Finally, the following should be remarked. Although the last expression is quite different from the expression of the angle for a perfect fluid $\theta_p \sim (M/E')^{1/\sqrt{2}}$, the quantitative values of both expressions are of the same order of magnitude in the energy range $E = 10^5 \sim 10^7$ Gev. Neverthless, the limit t_1 is of the order of $(10^{-2} \sim 10^{-3}) \times t_p$ (t_p : the limit of perfect fluid), when η_0^2 has an allowable value ~ 10.

Recently, Blokhintsev[3] has pointed out that the hydrodynamic description at the initial stage of expansion is not valid, but his proposal will be right so far as the meson cloud can be regarded as the "real" fluid in relativistic hydrodynamics. Therefore, the theory constructed in a heuristic meaning that the hydrodynamical quantities are reinterpreted from a certain new point of view should be outside the pale of his discussion. We may rather wish to grasp a highly advanced new point of view transcending the limits of such a hydrodynamics.

I wish to thank Professor T. Inoue for his kind interest to my work. I am also indebted to the "Yukawa Yomiuri Fellowship" for its finantial aid.

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Note on Heavy Ion Induced Nuclear Reactions

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April 3, 1958

Recently many experiments on heavy ion induced nuclear reactions have been reported. These experiments are classified into two categories according to the type of nuclear reaction i. e., the multiple evaporation type of nuclear reaction and the nucleon transfer type of nuclear process. We have analysed in a previous paper[1] the multiple evaporation process, which leads to the residual nuclei a few mass units lighter than the compound nucleus, and concluded that the gross feature of the excitation cross sections is fairly well reproduced by the compound nucleus picture.

In this note, the experimental results[2] of $^{23}\mathrm{Na}(^{14}\mathrm{N}\,;\,\alpha p)^{32}\mathrm{P}$ and $^{23}\mathrm{Na}(^{14}\mathrm{N}\,;\,\alpha 2p)^{31}\mathrm{Si}$ reactions are analysed by means of the compound nucleus model. For the calculation, all decay processes energetically possible

must be taken into account as the competitive processes. The cross section of the nuclear reaction $(l\,;\,i,\,j,\,k)$ is expressed as

$$\sigma(l\,;\,i,\,j,k) = \sigma_c^{(l)}(E_l)\cdot K_{ijk}/\sum_{m,n,p} K_{mnp}$$
$$= \sigma_c^{(l)}(E_l)\cdot \mathfrak{P}_{ijk}.$$

$\sigma_c^{(l)}(E_l)$ is the cross section for the formation of a compound nucleus by the impinging particle l with an incident energy E_l and $\mathfrak{P}_{ijk}$ is the decay probability for having emission i, j and k from the compound nucleus. In the calculation of $\mathfrak{P}_{ijk}$, the constant a appeared in the level density formula $W(E) = c\cdot\exp(2\sqrt{aE})$ is chosen as 0.42 for ^{32}P according to the previously proposed relationship.[1] This value of a is not so much different from that of Blatt and Weisskopf.

The calculated $\mathfrak{P}_{ijk}$ for the reactions $^{23}\mathrm{Na}(^{14}\mathrm{N}\,;\,\alpha p)^{32}\mathrm{P}$ and $^{23}\mathrm{Na}(^{14}\mathrm{N}\,;\,\alpha 2p)^{31}\mathrm{Si}$ with the incident nitrogen ion energy of 26 Mev are as follows;

$$\mathfrak{P}_{\alpha p\gamma} \simeq 32.5\times10^{-3},\quad \mathfrak{P}_{\alpha pp} \simeq 1.22\times10^{-3}.$$

Then we have the ratio of the cross sections in both reactions as

$$\sigma(^{32}\mathrm{P})/\sigma(^{31}\mathrm{Si}) = \mathfrak{P}_{\alpha p\gamma}/\mathfrak{P}_{\alpha pp} \simeq 26.6,$$

and this is in fair agreement with the experimental ratio[2] 22 ± 5.

Using the experimental values $\sigma(^{32}\mathrm{P}) = 91\pm18$ mb, $\sigma(^{31}\mathrm{Si}) = 4.1\pm0.8$ mb and the calculated decay probabilities $\mathfrak{P}_{\alpha p\gamma}$ and $\mathfrak{P}_{\alpha pp}$, we find the value of about 3 barns for the compound nucleus formation cross section for incident 26 Mev nitrogen ions on sodium target. This value of $\sigma_c^{(N)}$ (26 Mev) seems to be unexpectedly large. The estimated value $\sigma_c^{(N)}$ (26 Mev) of about 400 mb, derived from the conventional theory, depends sensitively on the choice of the nuclear radius, but the discrepancy of about one-order of magnitude might remain unexplained by a modification of the nuclear radius only. In the light of experience obtained in the previous work[1] on the emission probabitities of $2p$ or 2α, the present results of $\mathfrak{P}_{\alpha p\gamma}$ and $\mathfrak{P}_{\alpha pp}$ seem to be the right order of magnitude. Thus we are obliged to seek the reason of the unexpected result in the estimation of the probability of the formation of the compound nucleus.

Breit et al.[3] have suggested the distortion effects in nuclei involved in the reaction and estimated the effect by means of the liquid drop model. As the consequence of this effect, the Coulomb barrier height is diminished so that the cross section for compound nucleus formation becomes larger. Applying the formulae derived by them to the present case, we find the ratio of the effective Coulomb barrier height to the usual one to be

$$B_{\mathrm{eff}}/B_c \simeq 41/44.$$

In the estimation of this value, many parameters are chosen appropriately according to their discussions. This effect, however, is not so effective for the purpose of the present investigation.

As another approach to the present problem, the effect of the diffused boundary of the nuclei is considered. Analysing the reaction $^{27}\mathrm{Al}(n\,;\,\alpha)^{24}\mathrm{Na}$, Kikuchi[4] has shown that the diffuseness of nuclear surface makes the penetration of the potential barrier easier. According to his analysis it will be found to be

$$B_{\mathrm{eff}}/B_c \simeq 2/3,$$

and this value results in the ratio of the transmission coefficient of about 10. There-

fore the effect of the diffused surface of nuclei seems to explain the anomalously large cross section for the compound nucleus formation $\sigma_c^{(N)}$.

Although some modification of the compound nucleus model may explain the behavior of the cross section for heavy ion induced nuclear reactions, the recent experimental result[5] of the angular distribution seems to be difficult to explain by the compound nucleus model. The experimental result, which shows a definite forward peak in the angular distribution of the emitted alpha particles in nitrogen ion induced nuclear reaction, suggests us the existence of a buckshot type of nuclear reactions. More complete analysis and consideration, in which the excitation cross sections shall be also calculated, are now in progress.

The author would like to express his gratitude to Professor K. Husimi for his guidance and encouragement. He is also indebted to Messrs. S. Yamasaki and T. Kammuri for valuable discussions.

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Elastic Scattering in High Energy and Impulse Approximation

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April 9, 1958

In 1950, Chew[1] first presented the impulse approximation as an useful method to treat high energy nucleon-nucleus scatterings. However, in those days, our knowledge on elementary scatterings was limited to the cross sections and decisive information on the phase shifts was hardly given. So from elementary scatterings we could not get any information on the interference terms in nucleon-nucleus scatterings. This made it difficult to put the impulse approximation to practical use.

Recently, the increasing knowledge on $n-p$ and $p-p$ scatterings made it possible to get an insight into elementary scatterings in the energy region up to 310 Mev phenomenologically and up to about 150 Mev theoretically. As a result, we can choose a few sets of phase shifts concerning elementary scatterings.

On the other hand, Wolfenstein et al.[4] analysed the spin matrix of elementary scatterings between particles of spin 1/2, and Wright[5] related it to phase shifts.

We can make use of these results in applying the impulse approximation to estimate high energy scatterings. Especially, as Chew pointed out, one can expect to get satisfactory results applying the approximation to high energy $n-d$ or $p-d$ scatterings.

Along this line, we calculate the cross

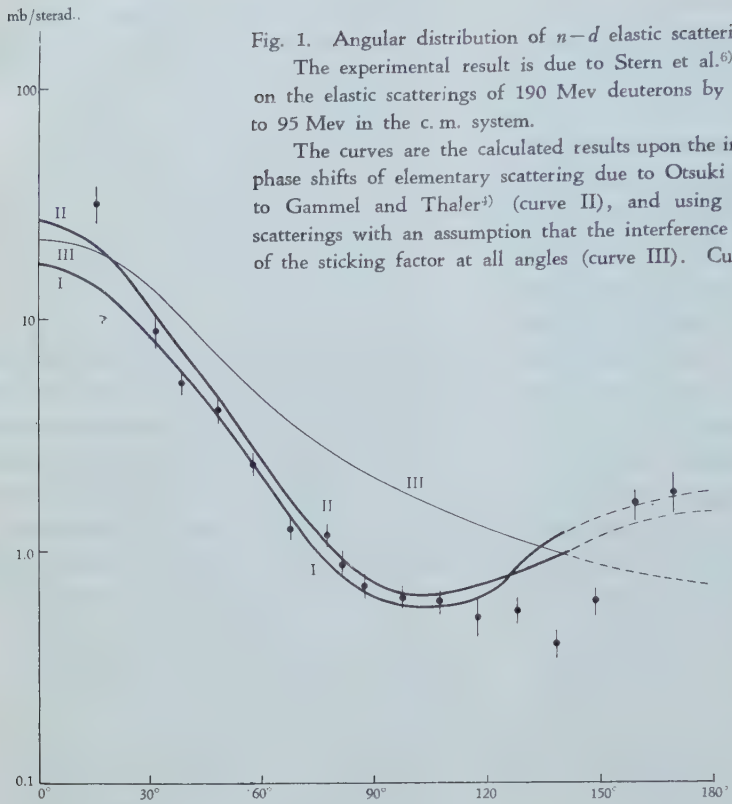

Fig. 1. Angular distribution of $n-d$ elastic scattering at 90 Mev.

The experimental result is due to Stern et al.[6] This experiment was done on the elastic scatterings of 190 Mev deuterons by protons. This is equivalent to 95 Mev in the c. m. system.

The curves are the calculated results upon the impulse approximation, using phase shifts of elementary scattering due to Otsuki et al.[5] (curve I), and due to Gammel and Thaler[4] (curve II), and using cross-sections of elementary scatterings with an assumption that the interference term is some constant times of the sticking factor at all angles (curve III). Curve III is due to Chew.[1]

section of the elastic scattering of 90 Mev neutrons by deuterons. As the deuteron ground state we use the wave function of the Hulthén type with $\alpha=\sqrt{\varepsilon M/\hbar^2}=0.232\cdot 10^{13}\,\mathrm{cm}^{-1}$ and $\beta=7\alpha$, for S-state. In this paper we neglect the contributions from D-state. For the phase shifts of elementary scattering, we use those due to Gammel and Thaler[2] which are determined phenomenologically and are characterized by a strong $L-S$ force, and those due to Otsuki, Tamagaki and Watari[3] which are suggested theoretically from meson theory and are characterized by the strong tensor potentials.

Experimental result of 190 Mev deuterons scattered by protons due to Stern et al.[6] is compared with our calculated results. This energy is equivalent to 95 Mev in the c. m. system. At such high energies as 90 Mev, we can disregard the difference between $n-d$ and $p-d$ scatterings except for very small angles. Also we may neglect the difference between 90 Mev and 95 Mev. Hence we calculate the cross section for the scatterings of 90 Mev neutron by deuteron and compare to the experimental result cited above.

The agreement between experimental and calculated results is satisfactory for both sets of phase shifts.[2),3)] (See curve I and curve II in Fig. 1.)

The sticking factor which decreases monotonically with increasing magnitude of momentum transfer implies that the forward scattering is favored. This result

was shown by Chew. (See curve III in Fig. 1.) Besides, the destructive interference terms between $n-n$ and $n-p$ scatterings play an important role, especially in large angles. Along with the nature of sticking factor this fact gives reasonable agreement with experimental values for angles of less than about 110°.

Authors wish to express their sincere thanks to Professor M. Kobayasi for his interest in this work and to Messrs. S. Hatano and M. Yasuno for discussions. They are indebted to Mr. R. Tamagaki and others for showing the results of their calculations on the phase shifts in nucleon problems prior to publication.

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R. H. Dalitz, Proc. Phys. Soc. **A65** (1952), 175.
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On the $|\Delta I|=1/2$ Rule for the Decay of Σ

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April 10, 1958

Isotopic spin selection rules in the decays of the strange particles have been investigated by several authors, and it has been strongly suggested that, in these decays, the interactions of the type of $|\Delta I|=1/2$ are dominant.[1] For example, the branching ratio between the two modes of Λ^0-decay is fairly explained by the rule $|\Delta I|=1/2$. Moreover, the experimental data for the ratio of the lifetimes of Σ^- and Σ^+, and the branching ratio between the two modes of Σ^+-decay are not inconsistent with this rule. However, this rule can never be the absolute one, since it cannot explain the data concerning the K-meson decays. Thus, as the first step to the future theory, it seems very useful to investigate to what extent this rule does hold. In this note, we shall suggest a possible method for the test of this rule for the case of the Σ-decays.

The selection rule $|\Delta I|=1/2$ means the assumption that, in the isotopic spin space, the decay interaction Hamiltonian transforms like one component of a spinor. Thus, the forms of the decay matrix elements with respect to the isotopic spin space are easily written down by the calculation of the Clebsch-Gordan coefficients:

a) $\Sigma^+\to p+\pi^0$:
$$(\sqrt{2}/3)\,[a_3+a_3'-a_1-a_1'] \quad (1a)$$

b) $\Sigma^+\to n+\pi^+$:
$$(1/3)\,[a_3+a_3'+2a_1+2a_1'] \quad (1b)$$

c) $\Sigma^-\to n+\pi^-$: a_3+a_3' (1c)

Here we have separately written the parity-conserving and reversing terms as a and a'. Suffix i of a_i means that it refers to the isotopic spin state $I=i/2$ of the final $(N+\pi)$ system. If we assume the spin of Σ being $1/2$, the angular distribution of the $(N+\pi)$ decay of the completely polarized Σ has the form

$$4\pi \mathrm{d}\sigma/\mathrm{d}\Omega = \sigma[1-\alpha\cos\theta]. \qquad (2)$$

σ and α can be expressed in terms of a's in (1). In the following, we assume the time-reversal invariance. Thus, we can relate the phases of a's with the $\pi-N$ scattering phase shifts δ's, and finally obtain the following expressions for the branching ratio ξ of Σ^+, the ratio η of the lifetimes of Σ^- and Σ^+, and the asymmetry parameters α's of the above three types of decays:

$$\xi \equiv \sigma_a/\sigma_\alpha = (2+2x^2-4Ax)/(1+4x^2+4Ax) \qquad (3)$$

$$\eta \equiv \tau(\Sigma^-)/\tau(\Sigma^+) = (1+2x^2)/3 \qquad (4)$$

$$\alpha_a = 2(B+Cx^2-Dx-Ex)/(1+x^2-2Ax) \qquad (5)$$

$$\alpha_b = 2(B+4Cx^2+2Dx+2Ex)/(1+4x^2+4Ax) \qquad (6)$$

$$\alpha_c = 2B. \qquad (7)$$

Here the suffices a, b, and c mean the three processes of (1), and

$$A = \sin\theta\sin\Phi\cos(\delta_3-\delta_1) + \cos\theta\cos\Phi\cos(\delta_{31}-\delta_{11})$$

$$B = \sin\theta\cos\theta\cos(\delta_3-\delta_{31})$$

$$C = \sin\Phi\cos\Phi\cos(\delta_1-\delta_{11})$$

$$D = \sin\theta\cos\Phi\cos(\delta_3-\delta_{11})$$

$$E = \cos\theta\sin\Phi\cos(\delta_1-\delta_{31})$$

and finally[2]

$$x^2 = [|a_1|^2+|a_1'|^2]/[|a_3|^2+|a_3'|^2]$$

$$\tan\theta = \pm|a_3|/|a_3'| \qquad (8)$$

$$\tan\Phi = \pm|a_1|/|a_1'|.$$

It should be noted that all these physical quantities are expressed in terms of three unknown parameters x, θ and Φ. This means that, if we use at least four independent experimental data, we can in principle test the $|\Delta I| = 1/2$ rule. For example, the data for the five quantities ξ, η, and α's provide the sufficient means for this test.

Now the experimental data for ξ and η are already known. The possibility of the direct measurements of α's is recently suggested by Lee and Yang.[3] However, it seems rather difficult at the present stage. What is measured directly is the product of α and P, P being degree of polarization of the "parent" Σ. Thus, it would be convenient to proceed as follows: Let us observe the "up-down" asymmetry $\beta = \alpha P$ in the decays a, b, and c of $\Sigma^\pm$, which are produced by the reactions $\pi^+ + p \to \Sigma^+ + K^+$ and $\pi^- + n \to \Sigma^- + K^0$, respectively. If the energy of the incident pion beam and the production angle of $\Sigma^\pm$ are same for both of two production processes, then the degrees of polarization of Σ^+ and Σ^- are also to be the same. Thus, the ratio of the β's for the decays of such $\Sigma^\pm$ provide directly the ratio of three α's. This gives two more independent restrictions in addition to the ones imposed by the values of ξ and η, and these four conditions on the three parameters x, θ and Φ enable us to test the $|\Delta I| = 1/2$ rule for this case.[4]

1) M. Kawaguchi, Phys. Rev. **107** (1957), 573. Also the list of the literatures on the strange particle decays is given there.
2) The parameters θ and Φ are same as those of ref. 1). However, x of the present note is $(1/\sqrt{2})$ times of the Kawaguchi's x.
3) T. D. Lee and C. N. Yang, Phys. Rev. **108** (1957), 1645.
4) We have here illustrated the method only for the case of spin 1/2. However, it is easily seen that our method is applicable independently of the spin value of Σ.

An Example of Isentropic Steady Flow in the Magnetohydrodynamics

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April 15, 1958

In this note, we deal with a problem of the hydromagnetic flow which is steady and isentropic, in which moreover the infinite conductivity of the fluid is assumed. Under these conditions, the basic equations can be written as follows[1]:

$$\mathrm{curl}[\boldsymbol{v}\times\boldsymbol{H}]=0, \tag{1}$$

$$(\boldsymbol{v}\ \mathrm{grad})\,\boldsymbol{v}+\mu_0/\rho[\boldsymbol{H}\times\mathrm{curl}\,\boldsymbol{H}] =-1/\rho\ \mathrm{grad}\,p, \tag{2}$$

$$\mathrm{div}\,\rho\boldsymbol{v}=0, \tag{3}$$

$$\mathrm{div}\,\boldsymbol{H}=0. \tag{4}$$

In what follows, we consider the special solution of them given by

$$\boldsymbol{H}=\pm\sqrt{\kappa}\,\rho\boldsymbol{v}, \tag{5}$$

In which κ is an arbitrary constant. It is the purpose of this paper to show that in this specialized case we have the conditions of the existence of discontinuity surface, one of which could not be found in the ordinary hydrodynamics and seems to be characteristic of the hydromagnetic flow. (We restrict ourselves to the two dimensional flow.)

Under the assumptions introduced above, the equations (1)$\sim$(4) reduce to

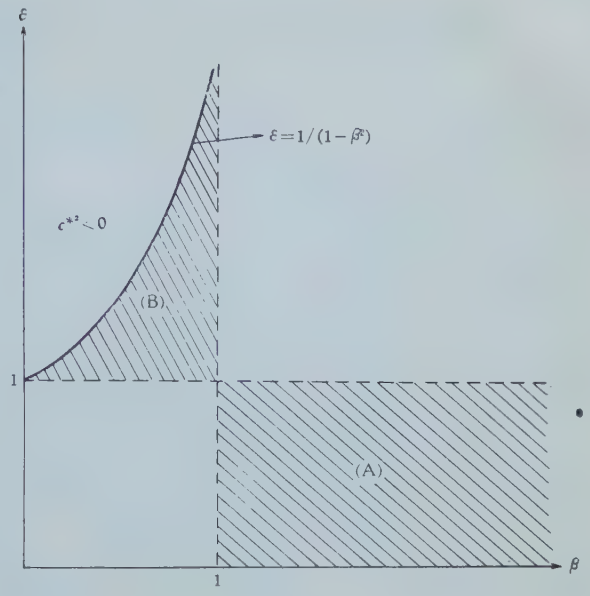

Fig. 1. The dashed regions correspond to (A) and (B).

$$\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_x+\bar{\mu}_0 v_y\left(\frac{\partial}{\partial x}(\rho v_y)-\frac{\partial}{\partial y}(\rho v_x)\right)=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}, \tag{6}$$

$$\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)v_y+\bar{\mu}_0 v_x\left(-\frac{\partial}{\partial x}(\rho v_y)+\frac{\partial}{\partial y}(\partial v_x)\right) =-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{7}$$

$$\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}\right)\rho+\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)=0, \tag{8}$$

in which $\bar{\mu}_0$ is equal to $\mu_0\kappa$.

Applying the usual method, we have the following equations of the characteristics in the (x, y) plane (the C-characteristics):

$$\Lambda=v_x y_\sigma-v_y x_\sigma=0, \tag{9}$$

$$(v_x^2-c^{*2})y_\sigma^2-2v_xv_yy_\sigma x_\sigma + (v_y-c^{*2})x_\sigma^2=0, \qquad (10)^*$$

where $d\sigma$ is the line element of the characteristics, c^* is defined by

$$c^{*2}=c_0^2(1-\varepsilon)+\varepsilon v^2, \qquad (11)$$

and we use the abbreviations :

$$\varepsilon=\bar{\rho}_0\rho, \quad v^2=v_x^2+v_y^2, \quad c_0=\sqrt{p'(\rho)}.$$

The equation (9) gives the stream line and the equation (10) is also the same as that in the ordinary hydrodynamics if the sound velocity c_0 is replaced by c^*. However, it should be noted that c^{*2} is not necessarily positive, but may be negative. From (10), one obtains the hyperbolicity condition :

$$c^{*2}(v^2-c^{*2})>0,$$

which is equivalent to the following two conditions :

(A) : $\varepsilon<1, \quad \beta>1$:

(B) : $1<\varepsilon<1/(1-\beta^2), \quad \beta<1,$

$(\beta=v/c_0)$.

The former condition (A) is similar to that of the ordinary sonic case, though it means that even in the extremely high speed flow any discontinuity surface does not appear if the magnetic field is sufficiently large. However, the latter, (B), is peculiar to the hydromagnetic flow, showing that the discontinuity surface can occur in the slow subsonic flow if the magnetic field is of adequate strength. (c. f. Fig. 1).

It seems to be worthwhile to investigate the equation of the characteristics in the hodograph plane (Γ-characteristics).

Along each stream line $\Lambda=0$, we have the characteristic relation :

$$\tfrac{1}{2}(v^2)_\sigma+P_\sigma/\rho=0,$$

or

$$v^2/2+i=\bar{v}^2/2, \qquad (12)$$

where i is the enthalpy, $\bar{v}$ is constant along a stream line. The relation (12) indicates that Bernouli's law for steady flow also holds in our case. However, this does not mean the existence of irrotational flow.

In fact, if v is constant everywhere, the equation (2) reduces to

$$\mathrm{curl}\,(1-\varepsilon)\boldsymbol{v}=0,$$

or

$$\mathrm{curl}\,(\boldsymbol{v}\mp\mu_0\sqrt{\kappa}\,\boldsymbol{H})=0 \qquad (13)$$

Combining (13), (12) and (3), one obtains the following Γ-characteristic equation, along the C-characteristics defined by (10),

$$c^{*2}c_0^2(dv_x^2+dv_y^2)=c^{*2}(v_x dv_x+v_y dv_y)^2 + \varepsilon c_0^2(v_y dv_x-v_x dv_y)^2. \qquad (14)$$

The equations (14) and (10) may serve to decide the flow patterns under various boundary conditions.

* These equations can be derived directly from the most general characteristic equation (2·3) in (I).[1]

1) T. Taniuti, Prog. Theor. Phys. **19** (1958), 69. This paper is denoted as I.

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